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Sum of the Cubes of Generalized Mersenne Numbers: the Sum Formula $\sum_{k=0}^{n} x^k W_{mk+j}^3$

Yüksel Soykan
Department of Mathematics, Art and Science Faculty, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey
e-mail: yuksel_soykan@hotmail.com

Abstract. *In this work, closed forms of the sum formulas* $\sum_{k=0}^{n} x^k W_{mk+j}^3$ *for generalized Mersenne numbers are presented. As special cases, we give sum formulas of Mersenne and Mersenne-Lucas numbers.*

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1. Introduction

A Mersenne number, denoted by M_n, and a Mersenne-Lucas number, denoted by H_n, are numbers of the form $M_n = 2^n - 1$ and $H_n = 2^n + 1$, respectively. The Mersenne sequence $\{M_n\}_{n\geq 0}$ and the Mersenne-Lucas sequence $\{H_n\}_{n\geq 0}$ can also be defined recursively by

$$M_n = 3M_{n-1} - 2M_{n-2}, \quad M_0 = 0, M_1 = 1$$

and

$$H_n = 3H_{n-1} - 2H_{n-2}, \quad H_0 = 2, H_1 = 3,$$

respectively. A generalized Mersenne sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = 3W_{n-1} - 2W_{n-2} \tag{1.1}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = \frac{3}{2}W_{-(n-1)} - \frac{1}{2}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n. For more information on generalized Mersenne numbers, see Soykan [47].

$\{M_n\}_{n\geq 0}$ is the sequence $A000225$ in the OEIS [30], whereas $\{H_n\}_{n\geq 0}$ is the id-number $A000051$ in OEIS. Note that Mersen-Lucas numbers are also called as Fermat numbers. In fact, there are two definitions of the Fermat numbers. The less common is a number of the form $2^n + 1$, the first few of which are $2, 3, 5, 9, 17, 33, \ldots$ (OEIS $A000051$). The much more commonly encountered Fermat numbers are a special case, given by the binomial number of the form $F_n = 2^{2^n} + 1$. The first few for $n = 0, 1, 2, \ldots$ are $3, 5, 17, 257, 65537, 4294967297, \ldots$ (OEIS $A000215$).

Mersenne sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to this sequence, see for example, [1,2,3,4,7,9,10,11,15,16,17,20,21,22,26,27,29,49,50].

In this work, we derive expressions for sums of second powers of generalized Mersenne numbers. We present some works on sum formulas of powers of the numbers in the following Table 1.

Table 1. A few special study on sum formulas of second, third and arbitrary powers.

Name of sequence	sums of second powers	sums of third powers	sums of powers
Generalized Fibonacci	[5,6,14,18,19,36,37,39,42,46]	[13,31,33,34,40,41,45,48,51]	[8,12,23]
Generalized Tribonacci	[25,32,38,43]		
Generalized Tetranacci	[24,28,35,44]		

2. The Sum Formula $\sum_{k=0}^{n} x^k W_{mk+j}^3$

The following theorem presents sum formulas of generalized Mersenne numbers.

THEOREM 1. *Let x be a real (or complex) number. For all integers m and j, for generalized Mersenne numbers we have the following sum formulas:*

(a): *If $(2^{3m}x^2 - xH_{3m} + 1)(2^{3m}x^2 - x2^m H_m + 1) \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_1}{(2^{3m}x^2 - xH_{3m} + 1)(2^{3m}x^2 - 2^m H_m x + 1)} \tag{2.1}$$

where

$$\Psi_1 = x^{n+1}2^{3m}(2^{3m}x^2 - 2^m xH_m + 1)W_{mn-m+j}^3 + x^{n+1}(2^{3m}x - H_{3m})(2^{3m}x^2 - 2^m xH_m + 1) W_{mn+j}^3 - 2^{3m}x(2^{3m}x^2 - 2^m xH_m + 1)W_{j-m}^3 + (2^{3m}x^2 - 2^m xH_m + 1)W_j^3 + 3x^n 2^{mn+m+j}x(2^{3m}x^2 - xH_{3m} + 1)(W_1^2 + 2W_0^2 - 3W_0W_1)W_{mn+m+j} + 3x^n 2^{mn+2m+j}x(2^{3m}x^2 - 2^m xH_m + 1)(W_1^2 + 2W_0^2 - 3W_0W_1)W_{mn-m+j} - 3x^n 2^{mn+j}x(2^{4m}x^2 H_m - (2^m xH_m - 1)H_{3m})(W_1^2 + 2W_0^2 - 3W_0W_1)W_{mn+j} - 3x2^{m+j}(2^{3m}x^2 - xH_{3m} + 1)(W_1^2 + 2W_0^2 - 3W_0W_1)W_{m+j} - 3x2^{2m+j}(2^{3m}x^2 - 2^m xH_m + 1)(W_1^2 + 2W_0^2 - 3W_0W_1)W_{j-m} + 3x2^j(2^{4m}x^2 H_m - H_{3m}(2^m xH_m - 1))(W_1^2 + 2W_0^2 - 3W_0W_1)W_j.$$

(b): *If $(2^{3m}x^2 - xH_{3m} + 1)(2^{3m}x^2 - x2^m H_m + 1) = u(x-a)(x-b)(x-c)(x-d) = 0$ for some $u, a, b, c, d \in \mathbb{C}$ with $u \neq 0$ and $a \neq b \neq c \neq d$, i.e., $x = a$ or $x = b$ or $x = c$ or $x = d$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_2}{\Lambda_1}$$

where

$\Psi_2 = 2^{3m}x^n(2^{3m}x^2(n+3) - x2^m(n+2)H_m + n+1)W^3_{mn-m+j} + (2^{6m}(n+4)x^3 - 2^{3m}(2^mH_m + H_{3m})(n+3)x^2 + 2^m(H_mH_{3m} + 2^{2m})(n+2)x - (n+1)H_{3m})x^nW^3_{mn+j} + 2^{3m}(-3\times2^{3m}x^2 + 2\times2^mxH_m - 1)W^3_{j-m} + (2\times2^{3m}x - 2^mH_m)W^3_j + 3\times2^{mn+m+j}(2^{3m}(n+3)x^2 - x(n+2)H_{3m} + n+1)(W_1^2 + 2W_0^2 - 3W_0W_1)x^nW_{mn+m+j} + 3\times2^{mn+2m+j}(2^{3m}(n+3)x^2 - x2^m(n+2)H_m + n+1)x^n(W_1^2 + 2W_0^2 - 3W_0W_1) W_{mn-m+j} + 3\times2^{mn+j}(-2^{4m}(n+3)x^2H_m + x2^m(n+2)H_{3m}H_m - (n+1)H_{3m})x^n(W_1^2 + 2W_0^2 - 3W_0W_1) W_{mn+j} + 3\times2^{m+j}(-3\times2^{3m}x^2 + 2xH_{3m} - 1)(W_1^2 + 2W_0^2 - 3W_0W_1)W_{m+j} + 3\times2^{2m+j}(-3\times2^{3m}x^2 + 2\times2^mxH_m - 1)(W_1^2 + 2W_0^2 - 3W_0W_1)W_{j-m} + 3\times2^j(3\times2^{4m}x^2H_m - 2\times2^mxH_mH_{3m} + H_{3m}) (W_1^2 + 2W_0^2 - 3W_0W_1)W_j$

and

$\Lambda_1 = (4\times2^{6m}x^3 - 3\times2^{3m}(2^mH_m + H_{3m})x^2 + 2\times2^m(2\times2^{2m} + H_mH_{3m})x - (2^mH_m + H_{3m}))$.

(c): *If* $(2^{3m}x^2 - xH_{3m} + 1)(2^{3m}x^2 - x2^mH_m + 1) = u(x-a)^2(x-b)(x-c) = 0$ *for some* $u, a, b, c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $x = a$ *or* $x = b$ *or* $x = c$, *then if* $x = b$ *or* $x = c$ *then*

$$\sum_{k=0}^{n} x^kW^3_{mk+j} = \frac{\Psi_3}{\Lambda_2}$$

where

$\Psi_3 = 2^{3m}x^n(2^{3m}x^2(n+3) - x2^m(n+2)H_m + n+1)W^3_{mn-m+j} + (2^{6m}(n+4)x^3 - 2^{3m}(2^mH_m + H_{3m})(n+3)x^2 + 2^m(H_mH_{3m} + 2^{2m})(n+2)x - (n+1)H_{3m})x^nW^3_{mn+j} + 2^{3m}(-3\times2^{3m}x^2 + 2\times2^mxH_m - 1)W^3_{j-m} + (2\times2^{3m}x - 2^mH_m)W^3_j + 3\times2^{mn+m+j}(2^{3m}(n+3)x^2 - x(n+2)H_{3m} + n+1)(W_1^2 + 2W_0^2 - 3W_0W_1)x^nW_{mn+m+j} + 3\times2^{mn+2m+j}(2^{3m}(n+3)x^2 - x2^m(n+2)H_m + n+1)x^n(W_1^2 + 2W_0^2 - 3W_0W_1) W_{mn-m+j} + 3\times2^{mn+j}(-2^{4m}(n+3)x^2H_m + x2^m(n+2)H_{3m}H_m - (n+1)H_{3m})x^n(W_1^2 + 2W_0^2 - 3W_0W_1) W_{mn+j} + 3\times2^{m+j}(-3\times2^{3m}x^2 + 2xH_{3m} - 1)(W_1^2 + 2W_0^2 - 3W_0W_1)W_{m+j} + 3\times2^{2m+j}(-3\times2^{3m}x^2 + 2\times2^mxH_m - 1)(W_1^2 + 2W_0^2 - 3W_0W_1)W_{j-m} + 3\times2^j(3\times2^{4m}x^2H_m - 2\times2^mxH_mH_{3m} + H_{3m}) (W_1^2 + 2W_0^2 - 3W_0W_1)W_j$

and

$\Lambda_2 = (4\times2^{6m}x^3 - 3\times2^{3m}(2^mH_m + H_{3m})x^2 + 2\times2^m(2\times2^{2m} + H_mH_{3m})x - (2^mH_m + H_{3m}))$

and if $x = a$ *then*

$$\sum_{k=0}^{n} x^kW^3_{mk+j} = \frac{\Psi_4}{2\times2^m(6\times2^{5m}x^2 - 3x2^{2m}(2^mH_m + H_{3m}) + 2\times2^{2m} + H_mH_{3m})}$$

where

$\Psi_4 = 2^{3m}(2^{3m}(n+3)(n+2)x^2 - 2^mx(n+2)(n+1)H_m + n(n+1))x^{n-1}W^3_{mn-m+j} + (2^{6m}(n+4)(n+3)x^3 - 2^{3m}(n+3)(n+2)(2^mH_m + H_{3m})x^2 + x2^m(n+2)(n+1)(H_mH_{3m} + 2^{2m}) - n(n+1)H_{3m})x^{n-1}W^3_{mn+j} + 2\times2^{4m}(H_m - 3\times2^{2m}x)W^3_{j-m} + 2\times2^{3m}W^3_j + 3\times2^{mn+m+j}(2^{3m}(n+3)(n+2)x^2 - x(n+2)(n+1)H_{3m} + n(n+1))(W_1^2 + 2W_0^2 - 3W_0W_1)x^{n-1}W_{mn+m+j} + 3x^{n-1}2^{mn+2m+j}(2^{3m}(n+3)(n+2)x^2 - x2^m(n+2)(n+1)H_m + n(n+1))(W_1^2 + 2W_0^2 - 3W_0W_1)W_{mn-m+j} + 3x^{n-1}2^{mn+j}(-x^22^{4m}(n+3)(n+2)H_m + x2^m(n+2)(n+1)H_{3m}H_m - n(n+1)H_{3m})(W_1^2 + 2W_0^2 - 3W_0W_1)W_{mn+j} + 6\times$

$2^{m+j}(H_{3m}-3\times 2^{3m}x)(W_1^2+2W_0^2-3W_0W_1)W_{m+j}+6\times 2^{3m+j}(H_m-3\times 2^{2m}x)(W_1^2+2W_0^2-3W_0W_1)W_{j-m}+6\times 2^{m+j}(3\times 2^{3m}x-H_{3m})H_m(W_1^2+2W_0^2-3W_0W_1)W_j.$

(d): *If* $(2^{3m}x^2-xH_{3m}+1)(2^{3m}x^2-x2^mH_m+1)=u(x-a)^3(x-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$, *i.e.,* $x=a$ *or* $x=b$, *then if* $x=b$ *then*

$$\sum_{k=0}^{n}x^kW_{mk+j}^3=\frac{\Psi_5}{\Lambda_3}$$

where

$\Psi_5=2^{3m}x^n(2^{3m}x^2(n+3)-x2^m(n+2)H_m+n+1)W_{mn-m+j}^3+(2^{6m}(n+4)x^3-2^{3m}(2^mH_m+H_{3m})(n+3)x^2+2^m(H_mH_{3m}+2^{2m})(n+2)x-(n+1)H_{3m})x^nW_{mn+j}^3+2^{3m}(-3\times 2^{3m}x^2+2\times 2^mxH_m-1)W_{j-m}^3+(2\times 2^{3m}x-2^mH_m)W_j^3+3\times 2^{mn+m+j}(2^{3m}(n+3)x^2-x(n+2)H_{3m}+n+1)(W_1^2+2W_0^2-3W_0W_1)x^nW_{mn+m+j}+3\times 2^{mn+2m+j}(2^{3m}(n+3)x^2-x2^m(n+2)H_m+n+1)x^n(W_1^2+2W_0^2-3W_0W_1)W_{mn-m+j}+3\times 2^{mn+j}(-2^{4m}(n+3)x^2H_m+x2^m(n+2)H_{3m}H_m-(n+1)H_{3m})x^n(W_1^2+2W_0^2-3W_0W_1)W_{mn+j}+3\times 2^{m+j}(-3\times 2^{3m}x^2+2xH_{3m}-1)(W_1^2+2W_0^2-3W_0W_1)W_{m+j}+3\times 2^{2m+j}(-3\times 2^{3m}x^2+2\times 2^mxH_m-1)(W_1^2+2W_0^2-3W_0W_1)W_{j-m}+3\times 2^j(3\times 2^{4m}x^2H_m-2\times 2^mxH_mH_{3m}+H_{3m})(W_1^2+2W_0^2-3W_0W_1)W_j$

and

$\Lambda_3=(4\times 2^{6m}x^3-3\times 2^{3m}(2^mH_m+H_{3m})x^2+2\times 2^m(2\times 2^{2m}+H_mH_{3m})x-(2^mH_m+H_{3m}))$

and if $x=a$ *then*

$$\sum_{k=0}^{n}x^kW_{mk+j}^3=\frac{\Psi_6}{6\times 2^{3m}(4(-s)^{3m}x-2^mH_m-H_{3m})}$$

where

$\Psi_6=2^{3m}(n+1)(2^{3m}(n+3)(n+2)x^2-x2^mn(n+2)H_m+n(n-1))x^{n-2}W_{mn-m+j}^3+(2^{6m}(n+3)(n+2)(n+4)x^3-2^{3m}(n+3)(n+2)(n+1)(2^mH_m+H_{3m})x^2+2^mn(n+2)(n+1)(H_mH_{3m}+2^{2m})x-n(n-1)(n+1)H_{3m})x^{n-2}W_{mn+j}^3-6\times 2^{6m}W_{j-m}^3+3\times 2^{mn+m+j}(n+1)(2^{3m}(n+3)(n+2)x^2-xn(n+2)H_{3m}+n(n-1))(W_1^2+2W_0^2-3W_0W_1)x^{n-2}W_{mn+m+j}+3\times 2^{mn+2m+j}(n+1)(2^{3m}(n+3)(n+2)x^2-x2^mn(n+2)H_m+n(n-1))(W_1^2+2W_0^2-3W_0W_1)x^{n-2}W_{mn-m+j}+3\times 2^{mn+j}(n+1)(-x^22^{4m}(n+3)(n+2)H_m+x2^mn(n+2)H_{3m}H_m-n(n-1)H_{3m})(W_1^2+2W_0^2-3W_0W_1)x^{n-2}W_{mn+j}-18\times 2^{4m+j}(W_1^2+2W_0^2-3W_0W_1)W_{m+j}-18\times 2^{5m+j}(W_1^2+2W_0^2-3W_0W_1)W_{j-m}+18\times 2^{4m+j}H_m(W_1^2+2W_0^2-3W_0W_1)W_j.$

(e): *If* $(2^{3m}x^2-xH_{3m}+1)(2^{3m}x^2-x2^mH_m+1)=u(x-a)^4=0$ *for some* $u,a\in\mathbb{C}, u\neq 0$ *i.e.,* $x=a$ *then*

$$\sum_{k=0}^{n}x^kW_{mk+j}^3=\frac{\Psi_7}{24\times 2^{6m}}$$

where

$\Psi_7=2^{3m}n(n+1)(2^{3m}(n+3)(n+2)x^2-x2^m(n-1)(n+2)H_m+(n-1)(n-2))x^{n-3}W_{mn-m+j}^3+(n+1)(x^32^{6m}(n+4)(n+3)(n+2)-x^22^{3m}n(n+3)(n+2)(2^mH_m+H_{3m})+x2^mn(n-1)(n+2)(H_mH_{3m}+$

$2^{2m}) - n(n-1)(n-2)H_{3m})x^{n-3}W_{mn+j}^{3} + 3\times 2^{mn+m+j}n(n+1)(x^{2}2^{3m}(n+3)(n+2) - x(n+2)(n-1)H_{3m} + (n-1)(n-2))(W_{1}^{2}+2W_{0}^{2}-3W_{0}W_{1})x^{n-3}W_{mn+m+j} + 3\times 2^{mn+2m+j}n(n+1)(x^{2}2^{3m}(n+3)(n+2) - x2^{m}(n+2)(n-1)H_{m} + (n-1)(n-2))(W_{1}^{2}+2W_{0}^{2}-3W_{0}W_{1})x^{n-3}W_{mn-m+j} + 3\times 2^{mn+j}n(n+1)(-x^{2}2^{4m}(n+3)(n+2)H_{m} + x2^{m}(n+2)(n-1)H_{3m}H_{m} - (n-1)(n-2)H_{3m})(W_{1}^{2}+2W_{0}^{2}-3W_{0}W_{1})x^{n-3}W_{mn+j}$.

Proof. *Take* $r = 3, s = -2$ and $H_n = H_n$ in Soykan [48, Theorem 2.1.]. □

Note that (2.1) can be written in the following form:

$$\sum_{k=1}^{n} x^{k}W_{mk+j}^{2} = \frac{\Psi_{3}}{(2^{3m}x^{2} - xH_{3m}+1)(2^{3m}x^{2} - x2^{m}H_{m}+1)}$$

where

$\Psi_{3} = x^{n+1}2^{3m}(2^{3m}x^{2} - 2^{m}xH_{m}+1)W_{mn-m+j}^{3} + x^{n+1}(2^{3m}x - H_{3m})(2^{3m}x^{2} - 2^{m}xH_{m}+1)W_{mn+j}^{3} - 2^{3m}x(2^{3m}x^{2} - 2^{m}xH_{m}+1)W_{j-m}^{3} + (H_{3m} - 2^{3m}x)(2^{3m}x^{2} - 2^{m}xH_{m}+1)xW_{j}^{3} + 3x^{n}2^{mn+m+j}x(2^{3m}x^{2} - xH_{3m}+1)(W_{1}^{2}+2W_{0}^{2}-3W_{0}W_{1})W_{mn+m+j} + 3x^{n}2^{mn+2m+j}x(2^{3m}x^{2} - 2^{m}xH_{m}+1)(W_{1}^{2}+2W_{0}^{2}-3W_{0}W_{1})W_{mn-m+j} - 3x^{n}2^{mn+j}x(2^{4m}x^{2}H_{m} - (2^{m}xH_{m}-1)H_{3m})(W_{1}^{2}+2W_{0}^{2}-3W_{0}W_{1})W_{mn+j} - 3x2^{m+j}(2^{3m}x^{2} - xH_{3m}+1)(W_{1}^{2}+2W_{0}^{2}-3W_{0}W_{1})W_{m+j} - 3x2^{2m+j}(2^{3m}x^{2} - 2^{m}xH_{m}+1)(W_{1}^{2}+2W_{0}^{2}-3W_{0}W_{1})W_{j-m} + 3x2^{j}(2^{4m}x^{2}H_{m} - H_{3m}(2^{m}xH_{m}-1))(W_{1}^{2}+2W_{0}^{2}-3W_{0}W_{1})W_{j}$.

As special cases of m and j in the last Theorem, we obtain the following proposition.

Proposition 2. *For generalized Mersenne numbers (the case* $r = 3, s = -2$*) we have the following sum formulas for* $n \geq 0$*:*

(a): $(m = 1,\ j = 0)$

If $(8x^{2} - 6x + 1)(8x^{2} - 9x + 1) = (4x-1)(2x-1)(8x-1)(x-1) \neq 0$*, i.e.,* $x \neq \frac{1}{8}$, $x \neq \frac{1}{4}$, $x \neq \frac{1}{2}$, $x \neq 1$*, then*

$$\sum_{k=0}^{n} x^{k}W_{k}^{3} = \frac{\Psi_{1}}{(4x-1)(2x-1)(8x-1)(x-1)}$$

where

$\Psi_{1} = x^{n+1}(8x-9)(4x-1)(2x-1)W_{n}^{3} + 8x^{n+1}(4x-1)(2x-1)W_{n-1}^{3} + 3\times 2^{n+1}x^{n+1}(8x-1)(x-1)(W_{1}^{2}+2W_{0}^{2}-3W_{0}W_{1})W_{n+1} - 9\times 2^{n}x^{n+1}(16x^{2} - 18x + 3)(W_{1}^{2}+2W_{0}^{2}-3W_{0}W_{1})W_{n} + 3\times 2^{n+2}x^{n+1}(4x-1)(2x-1)(W_{1}^{2}+2W_{0}^{2}-3W_{0}W_{1})W_{n-1} + (x(8x^{2}+12x+1)W_{1}^{3} + (-216x^{3} + 62x^{2} - 15x + 1)W_{0}^{3} - 18x^{2}(4x+3)W_{0}W_{1}^{2} + 36x^{2}(6x+1)W_{0}^{2}W_{1})$

and

if $(8x^{2} - 6x + 1)(8x^{2} - 9x + 1) = (4x-1)(2x-1)(8x-1)(x-1) = 0$*, i.e.,* $x = \frac{1}{8}$ *or* $x = \frac{1}{4}$ *or* $x = \frac{1}{2}$ *or* $x = 1$ *then*

$$\sum_{k=0}^{n} x^{k}W_{k}^{3} = \frac{\Psi_{2}}{256x^{3} - 360x^{2} + 140x - 15}$$

where

$\Psi_2 = x^n(n(8x-9)(4x-1)(2x-1)+256x^3-360x^2+124x-9)W_n^3+8x^n\left(n(4x-1)(2x-1)+24x^2-12x+1\right)W_{n-1}^3$
$3\times 2^{n+1}x^n(n(8x-1)(x-1)+24x^2-18x+1)(W_1^2+2W_0^2-3W_0W_1)W_{n+1}-9\times 2^n x^n(n(16x^2-18x+3)+48x^2-36x+3)(W_1^2+2W_0^2-3W_0W_1)W_n+3\times 2^{n+2}x^n(n(4x-1)(2x-1)+24x^2-12x+1)(W_1^2+2W_0^2-3W_0W_1)W_{n-1}+((24x^2+24x+1)W_1^3-(648x^2-124x+15)W_0^3-108x(2x+1)W_0W_1^2+72x(9x+1)W_0^2W_1)$.

(b): $(m=2,\ j=0)$

If $\left(64x^2-20x+1\right)\left(64x^2-65x+1\right)=(16x-1)(4x-1)(64x-1)(x-1)\neq 0$, *i.e.,* $x\neq\frac{1}{64}$, $x\neq\frac{1}{16}$, $x\neq\frac{1}{4}$, $x\neq 1$, *then*

$$\sum_{k=0}^{n}x^kW_{2k}^3=\frac{\Psi_1}{(16x-1)(4x-1)(64x-1)(x-1)}$$

where

$\Psi_1 = x^{n+1}(64x-65)(16x-1)(4x-1)W_{2n}^3+64x^{n+1}(16x-1)(4x-1)W_{2n-2}^3+3\times 2^{2n+2}x^{n+1}(64x-1)(x-1)(W_1^2+2W_0^2-3W_0W_1)W_{2n+2}-15\times 2^{2n}x^{n+1}(256x^2-260x+13)(W_1^2+2W_0^2-3W_0W_1)W_{2n}+3\times 2^{2n+4}x^{n+1}(16x-1)(4x-1)(W_1^2+2W_0^2-3W_0W_1)W_{2n-2}+(27x(64x^2+40x+1)W_1^3-(21\,952x^3+636x^2+93x-1)W_0^3-54x(224x^2+90x+1)W_1^2W_0+36x(784x^2+160x+1)W_0^2W_1)$

and

if $\left(64x^2-20x+1\right)\left(64x^2-65x+1\right)=0$, *i.e.,* $x=\frac{1}{64}$ *or* $x=\frac{1}{16}$ *or* $x=\frac{1}{4}$ *or* $x=1$ *then*

$$\sum_{k=0}^{n}x^kW_{2k}^3=\frac{\Psi_2}{16384x^3-16320x^2+2856x-85}$$

where

$\Psi_2 = x^n\left(n(64x-65)(16x-1)(4x-1)+16\,384x^3-16\,320x^2+2728x-65\right)W_{2n}^3+64x^n(n(16x-1)(4x-1)+192x^2-40x+1)W_{2n-2}^3+3\times 2^{2n+2}x^n(n(64x-1)(x-1)+192x^2-130x+1)(W_1^2+2W_0^2-3W_0W_1)W_{2n+2}-15\times 2^{2n}x^n(n(256x^2-260x+13)+768x^2-520x+13)(W_1^2+2W_0^2-3W_0W_1)W_{2n}+3\times 2^{2n+4}x^n(n(16x-1)(4x-1)+192x^2-40x+1)(W_1^2+2W_0^2-3W_0W_1)W_{2n-2}+3(9(192x^2+80x+1)W_1^3-(21\,952x^2+424x+31)W_0^3-18(672x^2+180x+1)W_0W_1^2+12(2352x^2+320x+1)W_0^2W_1)$.

(c): $(m=2,\ j=1)$

If $\left(64x^2-20x+1\right)\left(64x^2-65x+1\right)=(16x-1)(4x-1)(64x-1)(x-1)\neq 0$, *i.e.,* $x\neq\frac{1}{64}$, $x\neq\frac{1}{16}$, $x\neq\frac{1}{4}$, $x\neq 1$, *then*

$$\sum_{k=0}^{n}x^kW_{2k+1}^3=\frac{\Psi_1}{(16x-1)(4x-1)(64x-1)(x-1)}$$

where

$\Psi_1 = x^{n+1}(64x-65)(16x-1)(4x-1)W_{2n+1}^3+64x^{n+1}(16x-1)(4x-1)W_{2n-1}^3+3\times 2^{2n+3}x^{n+1}(64x-1)(x-1)(W_1^2+2W_0^2-3W_0W_1)W_{2n+3}-15\times 2^{2n+1}x^{n+1}(256x^2-260x+13)(W_1^2+2W_0^2-3W_0W_1)W_{2n+1}+3\times 2^{2n+5}x^{n+1}(16x-1)(4x-1)(W_1^2+2W_0^2-3W_0W_1)W_{2n-1}+((8x+1)(64x^2+250x+1)W_1^3-216x(64x^2+40x+1)W_0^3-18x(256x^2+640x+49)W_1^2W_0+108x(128x^2+180x+7)W_0^2W_1)$

and

if $\left(64x^2-20x+1\right)\left(64x^2-65x+1\right)=0$, *i.e.*, $x=\frac{1}{64}$ *or* $x=\frac{1}{16}$ *or* $x=\frac{1}{4}$ *or* $x=1$ *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^3=\frac{\Psi_2}{16384x^3-16320x^2+2856x-85}$$

where

$\Psi_2=x^n(n(64x-65)(16x-1)(4x-1)+16384x^3-16\,320x^2+2728x-65)W_{2n+1}^3+64x^n(n(16x-1)(4x-1)+192x^2-40x+1)W_{2n-1}^3+3\times 2^{2n+3}x^n(n(64x-1)(x-1)+192x^2-130x+1)(W_1^2+2W_0^2-3W_0W_1)W_{2n+3}-15\times 2^{2n+1}x^n(n(256x^2-260x+13)+768x^2-520x+13)(W_1^2+2W_0^2-3W_0W_1)W_{2n+1}+3\times 2^{2n+5}x^n(n(16x-1)(4x-1)+192x^2-40x+1)(W_1^2+2W_0^2-3W_0W_1)W_{2n-1}+6((256x^2+688x+43)W_1^3-36(192x^2+80x+1)W_0^3-3(768x^2+1280x+49)W_0W_1^2+18(384x^2+360x+7)W_1W_0^2)$.

(d): $(m=-1,\ j=0)$

If $(x-2)(x-4)(x-1)(x-8)\neq 0$, *i.e.*, $x\neq 1,\ x\neq 2,\ x\neq 4,\ x\neq 8$, *then*

$$\sum_{k=0}^{n} x^k W_{-k}^3=\frac{\Psi_1}{(x-2)(x-4)(x-1)(x-8)}$$

where

$\Psi_1=x^{n+1}(x-2)(x-4)W_{-n+1}^3+x^{n+1}(x-9)(x-2)(x-4)W_{-n}^3+6\times 2^{-n}x^{n+1}(x-2)(x-4)(W_1^2+2W_0^2-3W_0W_1)W_{-n+1}-18\times 2^{-n}x^{n+1}(x^2-9x+12)(W_1^2+2W_0^2-3W_0W_1)W_{-n}+12\times 2^{-n}x^{n+1}(x-1)(x-8)(W_1^2+2W_0^2-3W_0W_1)W_{-n-1}+(-x(x^2+12x+8)W_1^3+8(x^2+12x+8)W_0^3+18x(3x+4)W_1^2W_0-36x(x+6)W_0^2W_1)$

and

if $(x-2)(x-4)(x-1)(x-8)=0$, *i.e.*, $x=1$ *or* $x=2$ *or* $x=4$ *or* $x=8$ *then*

$$\sum_{k=0}^{n} x^k W_{-k}^3=\frac{\Psi_2}{(4x^3-45x^2+140x-120)}$$

where

$\Psi_2=x^n(n(x-2)(x-4)+3x^2-12x+8)W_{-n+1}^3+x^n(n(x-9)(x-2)(x-4)+4x^3-45x^2+124x-72)W_{-n}^3+6\times 2^{-n}x^n(n(x-2)(x-4)+3x^2-12x+8)(W_1^2+2W_0^2-3W_0W_1)W_{-n+1}-18\times 2^{-n}x^n(n(x^2-9x+12)+3x^2-18x+12)(W_1^2+2W_0^2-3W_0W_1)W_{-n}+12\times 2^{-n}x^n(n(x-1)(x-8)+3x^2-18x+8)(W_1^2+2W_0^2-3W_0W_1)W_{-n-1}+(-(3x^2+24x+8)W_1^3+16(x+6)W_0^3+36(3x+2)W_0W_1^2-72(x+3)W_0^2W_1)$.

(e): $(m=-2,\ j=0)$

If $(x-4)(x-16)(x-1)(x-64)\neq 0$, *i.e.*, $x\neq 1,\ x\neq 4,\ x\neq 16,\ x\neq 64$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^3=\frac{\Psi_1}{(x-4)(x-16)(x-1)(x-64)}$$

where

$\Psi_1=x^{n+1}(x-4)(x-16)W_{-2n+2}^3+x^{n+1}(x-65)(x-4)(x-16)W_{-2n}^3+12\times 2^{-2n}x^{n+1}(x-4)(x-16)(W_1^2+2W_0^2-3W_0W_1)W_{-2n+2}-60\times 2^{-2n}x^{n+1}(x^2-65x+208)(W_1^2+2W_0^2-3W_0W_1)W_{-2n}+48\times 2^{-2n}x^{n+1}(x-1)(x-64)(W_1^2+2W_0^2-3W_0W_1)W_{-2n-2}+(-27x(x^2+40x+64)W_1^3+8(x+8)(x^2+250x+64)W_0^3+54x(x^2+90x+224)W_1^2W_0-36x(x^2+160x+784)W_0^2W_1)$

and

if $(x-4)(x-16)(x-1)(x-64)=0$, *i.e.*, $x=1$ *or* $x=4$ *or* $x=16$ *or* $x=64$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^3 = \frac{\Psi_2}{(4x^3-255x^2+2856x-5440)}$$

where

$\Psi_2 = x^n(n(x-4)(x-16)+3x^2-40x+64)W_{-2n+2}^3 + x^n(n(x-65)(x-4)(x-16)+4x^3-255x^2+2728x-4160)W_{-2n}^3 + 12\times 2^{-2n}x^n(n(x-4)(x-16)+3x^2-40x+64)(W_1^2+2W_0^2-3W_0W_1)W_{-2n+2} - 60\times 2^{-2n}x^n(n(x^2-65x+208)+3x^2-130x+208)(W_1^2+2W_0^2-3W_0W_1)W_{-2n} + 48\times 2^{-2n}x^n(n(x-1)(x-64)+3x^2-130x+64)(W_1^2+2W_0^2-3W_0W_1)W_{-2n-2} + 3(-9(3x^2+80x+64)W_1^3 + 8(x^2+172x+688)W_0^3 + 18(3x^2+180x+224)W_1^2W_0 - 12(3x^2+320x+784)W_0^2W_1)$.

(f): $(m=-2,\ j=1)$

If $(x-4)(x-16)(x-1)(x-64)\neq 0$, *i.e.*, $x\neq 1$, $x\neq 4$, $x\neq 16$, $x\neq 64$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^3 = \frac{\Psi_1}{(x-4)(x-16)(x-1)(x-64)}$$

where

$\Psi_1 = x^{n+1}(x-4)(x-16)W_{-2n+3}^3 + x^{n+1}(x-65)(x-4)(x-16)W_{-2n+1}^3 + 24\times 2^{-2n}x^{n+1}(x-4)(x-16)(W_1^2+2W_0^2-3W_0W_1)W_{-2n+3} - 120\times 2^{-2n}x^{n+1}(x^2-65x+208)(W_1^2+2W_0^2-3W_0W_1)W_{-2n+1} + 96\times 2^{-2n}x^{n+1}(x-1)(x-64)(W_1^2+2W_0^2-3W_0W_1)W_{-2n-1} + (-(343x^3+636x^2+5952x-4096)W_1^3 + 216x(x^2+40x+64)W_0^3 + 18x(49x^2+640x+256)W_0W_1^2 - 108x(7x^2+180x+128)W_0^2W_1)$

and

if $(x-4)(x-16)(x-1)(x-64)=0$, *i.e.*, $x=1$ *or* $x=4$ *or* $x=16$ *or* $x=64$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^3 = \frac{\Psi_2}{(4x^3-255x^2+2856x-5440)}$$

where

$\Psi_2 = x^n(n(x-4)(x-16)+3x^2-40x+64)W_{-2n+3}^3 + x^n(n(x-65)(x-4)(x-16)+4x^3-255x^2+2728x-4160)W_{-2n+1}^3 + 24\times 2^{-2n}x^n(n(x-4)(x-16)+3x^2-40x+64)(W_1^2+2W_0^2-3W_0W_1)W_{-2n+3} - 120\times 2^{-2n}x^n(n(x^2-65x+208)+3x^2-130x+208)(W_1^2+2W_0^2-3W_0W_1)W_{-2n+1} + 96\times 2^{-2n}x^n(n(x-1)(x-64)+3x^2-130x+64)(W_1^2+2W_0^2-3W_0W_1)W_{-2n-1} + 3(-(343x^2+424x+1984)W_1^3 + 72(3x^2+80x+64)W_0^3 + 6(147x^2+1280x+256)W_0W_1^2 - 36(21x^2+360x+128)W_0^2W_1)$.

From the above proposition, we have the following corollary which gives sum formulas of Mersenne numbers (take $W_n=M_n$ with $M_0=0, M_1=1$).

COROLLARY 3. *For* $n\geq 0$, *Mersenne numbers have the following properties:*

(a): $(m=1,\ j=0)$

If $(8x^2-6x+1)(8x^2-9x+1) = (4x-1)(2x-1)(8x-1)(x-1) \neq 0$, *i.e.,* $x \neq \frac{1}{8}$, $x \neq \frac{1}{4}$, $x \neq \frac{1}{2}$, $x \neq 1$, *then*

$$\sum_{k=0}^{n} x^k M_k^3 = \frac{\Psi_1}{(4x-1)(2x-1)(8x-1)(x-1)}$$

where

$\Psi_1 = x^{n+1}(8x-9)(4x-1)(2x-1)M_n^3 + 8x^{n+1}(4x-1)(2x-1)M_{n-1}^3 + 3\times 2^{n+1}x^{n+1}(8x-1)(x-1)M_{n+1} - 9\times 2^n x^{n+1}(16x^2-18x+3)M_n + 3\times 2^{n+2}x^{n+1}(4x-1)(2x-1)M_{n-1} + x(8x^2+12x+1)$

and

if $(8x^2-6x+1)(8x^2-9x+1) = (4x-1)(2x-1)(8x-1)(x-1) = 0$, *i.e.,* $x = \frac{1}{8}$ *or* $x = \frac{1}{4}$ *or* $x = \frac{1}{2}$ *or* $x = 1$ *then*

$$\sum_{k=0}^{n} x^k M_k^3 = \frac{\Psi_2}{256x^3 - 360x^2 + 140x - 15}$$

where

$\Psi_2 = x^n \left(n(8x-9)(4x-1)(2x-1) + 256x^3 - 360x^2 + 124x - 9\right) M_n^3 + 8x^n \left(n(4x-1)(2x-1) + 24x^2 - 12x + 3\times 2^{n+1}x^n(n(8x-1)(x-1) + 24x^2 - 18x + 1)M_{n+1} - 9\times 2^n x^n(n(16x^2-18x+3) + 48x^2 - 36x + 3)M_n + 3\times 2^{n+2}x^n(n(4x-1)(2x-1) + 24x^2 - 12x + 1)M_{n-1} + (24x^2+24x+1).$

(b): $(m = 2,\ j = 0)$

If $(64x^2-20x+1)(64x^2-65x+1) = (16x-1)(4x-1)(64x-1)(x-1) \neq 0$, *i.e.,* $x \neq \frac{1}{64}$, $x \neq \frac{1}{16}$, $x \neq \frac{1}{4}$, $x \neq 1$, *then*

$$\sum_{k=0}^{n} x^k M_{2k}^3 = \frac{\Psi_1}{(16x-1)(4x-1)(64x-1)(x-1)}$$

where

$\Psi_1 = x^{n+1}(64x-65)(16x-1)(4x-1)M_{2n}^3 + 64x^{n+1}(16x-1)(4x-1)M_{2n-2}^3 + 3\times 2^{2n+2}x^{n+1}(64x-1)(x-1)M_{2n+2} - 15\times 2^{2n}x^{n+1}(256x^2-260x+13)M_{2n} + 3\times 2^{2n+4}x^{n+1}(16x-1)(4x-1)M_{2n-2} + 27x(64x^2+40x+1)$

and

if $(64x^2-20x+1)(64x^2-65x+1) = 0$, *i.e.,* $x = \frac{1}{64}$ *or* $x = \frac{1}{16}$ *or* $x = \frac{1}{4}$ *or* $x = 1$ *then*

$$\sum_{k=0}^{n} x^k M_{2k}^3 = \frac{\Psi_2}{16384x^3 - 16320x^2 + 2856x - 85}$$

where

$\Psi_2 = x^n(n(64x-65)(16x-1)(4x-1) + 16\,384x^3 - 16\,320x^2 + 2728x - 65)M_{2n}^3 + 64x^n(n(16x-1)(4x-1) + 192x^2 - 40x + 1)M_{2n-2}^3 + 3\times 2^{2n+2}x^n(n(64x-1)(x-1) + 192x^2 - 130x + 1)M_{2n+2} - 15\times 2^{2n}x^n(n(256x^2-260x+13) + 768x^2 - 520x + 13)M_{2n} + 3\times 2^{2n+4}x^n(n(16x-1)(4x-1) + 192x^2 - 40x + 1)M_{2n-2} + 27(192x^2+80x+1).$

(c): $(m = 2,\ j = 1)$

If $\left(64x^2 - 20x + 1\right)\left(64x^2 - 65x + 1\right) = (16x-1)(4x-1)(64x-1)(x-1) \neq 0$, *i.e.,* $x \neq \frac{1}{64}$, $x \neq \frac{1}{16}$, $x \neq \frac{1}{4}$, $x \neq 1$, *then*

$$\sum_{k=0}^{n} x^k M_{2k+1}^3 = \frac{\Psi_1}{(16x-1)(4x-1)(64x-1)(x-1)}$$

where

$\Psi_1 = x^{n+1}(64x-65)(16x-1)(4x-1)M_{2n+1}^3 + 64x^{n+1}(16x-1)(4x-1)M_{2n-1}^3 + 3\times 2^{2n+3}x^{n+1}(64x-1)(x-1)M_{2n+3} - 15\times 2^{2n+1}x^{n+1}(256x^2-260x+13)M_{2n+1} + 3\times 2^{2n+5}x^{n+1}(16x-1)(4x-1)M_{2n-1} + (8x+1)(64x^2+250x+1)$

and

if $\left(64x^2 - 20x + 1\right)\left(64x^2 - 65x + 1\right) = 0$, *i.e.,* $x = \frac{1}{64}$ *or* $x = \frac{1}{16}$ *or* $x = \frac{1}{4}$ *or* $x = 1$ *then*

$$\sum_{k=0}^{n} x^k M_{2k+1}^3 = \frac{\Psi_2}{16384x^3 - 16320x^2 + 2856x - 85}$$

where

$\Psi_2 = x^n(n(64x-65)(16x-1)(4x-1) + 16384x^3 - 16\,320x^2 + 2728x - 65)M_{2n+1}^3 + 64x^n(n(16x-1)(4x-1) + 192x^2 - 40x + 1)M_{2n-1}^3 + 3\times 2^{2n+3}x^n(n(64x-1)(x-1) + 192x^2 - 130x + 1)M_{2n+3} - 15\times 2^{2n+1}x^n(n(256x^2-260x+13) + 768x^2 - 520x + 13)M_{2n+1} + 3\times 2^{2n+5}x^n(n(16x-1)(4x-1) + 192x^2 - 40x + 1)M_{2n-1} + 6(256x^2 + 688x + 43)$.

(d): $(m = -1,\ j = 0)$

If $(x-2)(x-4)(x-1)(x-8) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 2$, $x \neq 4$, $x \neq 8$, *then*

$$\sum_{k=0}^{n} x^k M_{-k}^3 = \frac{\Psi_1}{(x-2)(x-4)(x-1)(x-8)}$$

where

$\Psi_1 = x^{n+1}(x-2)(x-4)M_{-n+1}^3 + x^{n+1}(x-9)(x-2)(x-4)M_{-n}^3 + 6\times 2^{-n}x^{n+1}(x-2)(x-4)M_{-n+1} - 18\times 2^{-n}x^{n+1}(x^2-9x+12)M_{-n} + 12\times 2^{-n}x^{n+1}(x-1)(x-8)M_{-n-1} - x(x^2+12x+8)$

and

if $(x-2)(x-4)(x-1)(x-8) = 0$, *i.e.,* $x = 1$ *or* $x = 2$ *or* $x = 4$ *or* $x = 8$ *then*

$$\sum_{k=0}^{n} x^k M_{-k}^3 = \frac{\Psi_2}{(4x^3 - 45x^2 + 140x - 120)}$$

where

$\Psi_2 = x^n(n(x-2)(x-4) + 3x^2 - 12x + 8)M_{-n+1}^3 + x^n(n(x-9)(x-2)(x-4) + 4x^3 - 45x^2 + 124x - 72)M_{-n}^3 + 6\times 2^{-n}x^n(n(x-2)(x-4) + 3x^2 - 12x + 8)M_{-n+1} - 18\times 2^{-n}x^n(n(x^2-9x+12) + 3x^2 - 18x + 12)(M_1^2 + 2M_0^2 - 3M_0M_1)M_{-n} + 12\times 2^{-n}x^n(n(x-1)(x-8) + 3x^2 - 18x + 8)(M_1^2 + 2M_0^2 - 3M_0M_1)M_{-n-1} - (3x^2 + 24x + 8)$.

(e): $(m=-2,\ j=0)$

If $(x-4)(x-16)(x-1)(x-64)\neq 0,\ i.e.,\ x\neq 1,\ x\neq 4,\ x\neq 16,\ x\neq 64,$ *then*

$$\sum_{k=0}^{n} x^k M^3_{-2k} = \frac{\Psi_1}{(x-4)(x-16)(x-1)(x-64)}$$

where

$\Psi_1 = x^{n+1}(x-4)(x-16)M^3_{-2n+2} + x^{n+1}(x-65)(x-4)(x-16)M^3_{-2n} + 12\times 2^{-2n}x^{n+1}(x-4)(x-16)M_{-2n+2} - 60\times 2^{-2n}x^{n+1}(x^2-65x+208)M_{-2n} + 48\times 2^{-2n}x^{n+1}(x-1)(x-64)(M_1^2+2M_0^2-3M_0M_1)M_{-2n-2} - 27x(x^2+40x+64)$

and

if $(x-4)(x-16)(x-1)(x-64)=0,\ i.e.,\ x=1$ *or* $x=4$ *or* $x=16$ *or* $x=64$ *then*

$$\sum_{k=0}^{n} x^k M^3_{-2k} = \frac{\Psi_2}{(4x^3-255x^2+2856x-5440)}$$

where

$\Psi_2 = x^n(n(x-4)(x-16)+3x^2-40x+64)M^3_{-2n+2} + x^n(n(x-65)(x-4)(x-16)+4x^3-255x^2+2728x-4160)M^3_{-2n} + 12\times 2^{-2n}x^n(n(x-4)(x-16)+3x^2-40x+64)M_{-2n+2} - 60\times 2^{-2n}x^n(n(x^2-65x+208)+3x^2-130x+208)(M_1^2+2M_0^2-3M_0M_1)M_{-2n} + 48\times 2^{-2n}x^n(n(x-1)(x-64)+3x^2-130x+64)M_{-2n-2} - 27(3x^2+80x+64).$

(f): $(m=-2,\ j=1)$

If $(x-4)(x-16)(x-1)(x-64)\neq 0,\ i.e.,\ x\neq 1,\ x\neq 4,\ x\neq 16,\ x\neq 64,$ *then*

$$\sum_{k=0}^{n} x^k M^3_{-2k+1} = \frac{\Psi_1}{(x-4)(x-16)(x-1)(x-64)}$$

where

$\Psi_1 = x^{n+1}(x-4)(x-16)M^3_{-2n+3} + x^{n+1}(x-65)(x-4)(x-16)M^3_{-2n+1} + 24\times 2^{-2n}x^{n+1}(x-4)(x-16)(M_1^2+2M_0^2-3M_0M_1)M_{-2n+3} - 120\times 2^{-2n}x^{n+1}(x^2-65x+208)(M_1^2+2M_0^2-3M_0M_1)M_{-2n+1} + 96\times 2^{-2n}x^{n+1}(x-1)(x-64)(M_1^2+2M_0^2-3M_0M_1)M_{-2n-1} - (343x^3+636x^2+5952x-4096)$

and

if $(x-4)(x-16)(x-1)(x-64)=0,\ i.e.,\ x=1$ *or* $x=4$ *or* $x=16$ *or* $x=64$ *then*

$$\sum_{k=0}^{n} x^k M^3_{-2k+1} = \frac{\Psi_2}{(4x^3-255x^2+2856x-5440)}$$

where

$\Psi_2 = x^n(n(x-4)(x-16)+3x^2-40x+64)M^3_{-2n+3} + x^n(n(x-65)(x-4)(x-16)+4x^3-255x^2+2728x-4160)M^3_{-2n+1} + 24\times 2^{-2n}x^n(n(x-4)(x-16)+3x^2-40x+64)M_{-2n+3} - 120\times 2^{-2n}x^n(n(x^2-65x+208)+3x^2-130x+208)M_{-2n+1} + 96\times 2^{-2n}x^n(n(x-1)(x-64)+3x^2-130x+64)(M_1^2+2M_0^2-3M_0M_1)M_{-2n-1} - 3(343x^2+424x+1984).$

Taking $W_n = H_n$ with $H_0 = 2, H_1 = 3$ in the last proposition, we have the following corollary which presents sum formulas of Mersenne-Lucas numbers.

Corollary 4. *For $n \geq 0$, Mersenne-Lucas numbers have the following properties:*

(a): $(m = 1,\ j = 0)$

If $\left(8x^2 - 6x + 1\right)\left(8x^2 - 9x + 1\right) = (4x-1)(2x-1)(8x-1)(x-1) \neq 0$, *i.e.*, $x \neq \frac{1}{8}$, $x \neq \frac{1}{4}$, $x \neq \frac{1}{2}$, $x \neq 1$, *then*

$$\sum_{k=0}^{n} x^k H_k^3 = \frac{\Psi_1}{(4x-1)(2x-1)(8x-1)(x-1)}$$

where

$\Psi_1 = x^{n+1}(8x-9)(4x-1)(2x-1)H_n^3 + 8x^{n+1}(4x-1)(2x-1)H_{n-1}^3 - 3 \times 2^{n+1}x^{n+1}(8x-1)(x-1)H_{n+1} + 9 \times 2^n x^{n+1}(16x^2-18x+3)H_n - 3 \times 2^{n+2}x^{n+1}(4x-1)(2x-1)H_{n-1} - (216x^3-280x^2+93x-8)$

and

if $\left(8x^2 - 6x + 1\right)\left(8x^2 - 9x + 1\right) = (4x-1)(2x-1)(8x-1)(x-1) = 0$, *i.e.*, $x = \frac{1}{8}$ *or* $x = \frac{1}{4}$ *or* $x = \frac{1}{2}$ *or* $x = 1$ *then*

$$\sum_{k=0}^{n} x^k H_k^3 = \frac{\Psi_2}{256x^3 - 360x^2 + 140x - 15}$$

where

$\Psi_2 = x^n(n(8x-9)(4x-1)(2x-1) + 256x^3 - 360x^2 + 124x - 9)H_n^3 + 8x^n(n(4x-1)(2x-1) + 24x^2 - 12x + 1)H_{n-1}^3 - 3 \times 2^{n+1}x^n(n(8x-1)(x-1) + 24x^2 - 18x + 1)H_{n+1} + 9 \times 2^n x^n(n(16x^2-18x+3) + 48x^2 - 36x + 3)H_n - 3 \times 2^{n+2}x^n(n(4x-1)(2x-1) + 24x^2 - 12x + 1)H_{n-1} - (648x^2 - 560x + 93)$.

(b): $(m = 2,\ j = 0)$

If $\left(64x^2 - 20x + 1\right)\left(64x^2 - 65x + 1\right) = (16x-1)(4x-1)(64x-1)(x-1) \neq 0$, *i.e.*, $x \neq \frac{1}{64}$, $x \neq \frac{1}{16}$, $x \neq \frac{1}{4}$, $x \neq 1$, *then*

$$\sum_{k=0}^{n} x^k H_{2k}^3 = \frac{\Psi_1}{(16x-1)(4x-1)(64x-1)(x-1)}$$

where

$\Psi_1 = x^{n+1}(64x-65)(16x-1)(4x-1)H_{2n}^3 + 64x^{n+1}(16x-1)(4x-1)H_{2n-2}^3 - 3 \times 2^{2n+2}x^{n+1}(64x-1)(x-1)H_{2n+2} + 15 \times 2^{2n}x^{n+1}(256x^2 - 260x + 13)H_{2n} - 3 \times 2^{2n+4}x^{n+1}(16x-1)(4x-1)H_{2n-2} + (-8000x^3 + 5712x^2 - 555x + 8)$

and

if $\left(64x^2 - 20x + 1\right)\left(64x^2 - 65x + 1\right) = 0$, *i.e.*, $x = \frac{1}{64}$ *or* $x = \frac{1}{16}$ *or* $x = \frac{1}{4}$ *or* $x = 1$ *then*

$$\sum_{k=0}^{n} x^k H_{2k}^3 = \frac{\Psi_2}{16384x^3 - 16320x^2 + 2856x - 85}$$

where

$\Psi_2 = x^n(n(64x-65)(16x-1)(4x-1)+16\,384x^3-16\,320x^2+2728x-65)H_{2n}^3+64x^n(n(16x-1)(4x-1)+192x^2-40x+1)H_{2n-2}^3-3\times 2^{2n+2}x^n(n(64x-1)(x-1)+192x^2-130x+1)H_{2n+2}+15\times 2^{2n}x^n(n(256x^2-260x+13)+768x^2-520x+13)H_{2n}-3\times 2^{2n+4}x^n(n(16x-1)(4x-1)+192x^2-40x+1)H_{2n-2}-3(8000x^2-3808x+185).$

(c): $(m=2,\ j=1)$

If $\left(64x^2-20x+1\right)\left(64x^2-65x+1\right)=(16x-1)(4x-1)(64x-1)(x-1)\neq 0,$ *i.e.,* $x\neq\frac{1}{64}$, $x\neq\frac{1}{16}$, $x\neq\frac{1}{4}$, $x\neq 1$, *then*

$$\sum_{k=0}^{n} x^k H_{2k+1}^3 = \frac{\Psi_1}{(16x-1)(4x-1)(64x-1)(x-1)}$$

where

$\Psi_1 = x^{n+1}(64x-65)(16x-1)(4x-1)H_{2n+1}^3+64x^{n+1}(16x-1)(4x-1)H_{2n-1}^3-3\times 2^{2n+3}x^{n+1}(64x-1)(x-1)H_{2n+3}+15\times 2^{2n+1}x^{n+1}(256x^2-260x+13)H_{2n+1}-3\times 2^{2n+5}x^{n+1}(16x-1)(4x-1)H_{2n-1}-27(8x-1)(64x^2-50x+1)$

and

if $\left(64x^2-20x+1\right)\left(64x^2-65x+1\right)=0,$ *i.e.,* $x=\frac{1}{64}$ *or* $x=\frac{1}{16}$ *or* $x=\frac{1}{4}$ *or* $x=1$ *then*

$$\sum_{k=0}^{n} x^k H_{2k+1}^3 = \frac{\Psi_2}{16384x^3-16320x^2+2856x-85}$$

where

$\Psi_2 = x^n(n(64x-65)(16x-1)(4x-1)+16384x^3-16\,320x^2+2728x-65)H_{2n+1}^3+64x^n(n(16x-1)(4x-1)+192x^2-40x+1)H_{2n-1}^3-3\times 2^{2n+3}x^n(n(64x-1)(x-1)+192x^2-130x+1)H_{2n+3}+15\times 2^{2n+1}x^n(n(256x^2-260x+13)+768x^2-520x+13)H_{2n+1}-3\times 2^{2n+5}x^n(n(16x-1)(4x-1)+192x^2-40x+1)H_{2n-1}-54(768x^2-464x+29).$

(d): $(m=-1,\ j=0)$

If $(x-2)(x-4)(x-1)(x-8)\neq 0,$ *i.e.,* $x\neq 1,\ x\neq 2,\ x\neq 4,\ x\neq 8,$ *then*

$$\sum_{k=0}^{n} x^k H_{-k}^3 = \frac{\Psi_1}{(x-2)(x-4)(x-1)(x-8)}$$

where

$\Psi_1 = x^{n+1}(x-2)(x-4)H_{-n+1}^3+x^{n+1}(x-9)(x-2)(x-4)H_{-n}^3-6\times 2^{-n}x^{n+1}(x-2)(x-4)H_{-n+1}+18\times 2^{-n}x^{n+1}(x^2-9x+12)H_{-n}-12\times 2^{-n}x^{n+1}(x-1)(x-8)H_{-n-1}-(27x^3-280x^2+744x-512)$

and

if $(x-2)(x-4)(x-1)(x-8)=0,$ *i.e.,* $x=1$ *or* $x=2$ *or* $x=4$ *or* $x=8$ *then*

$$\sum_{k=0}^{n} x^k H_{-k}^3 = \frac{\Psi_2}{(4x^3-45x^2+140x-120)}$$

where

$\Psi_2 = x^n(n(x-2)(x-4)+3x^2-12x+8)H^3_{-n+1} + x^n(n(x-9)(x-2)(x-4)+4x^3-45x^2+124x-72)H^3_{-n} - 6\times 2^{-n}x^n(n(x-2)(x-4)+3x^2-12x+8)H_{-n+1} + 18\times 2^{-n}x^n(n(x^2-9x+12)+3x^2-18x+12)H_{-n} - 12\times 2^{-n}x^n(n(x-1)(x-8)+3x^2-18x+8)H_{-n-1} - (81x^2-560x+744).$

(e): $(m=-2,\ j=0)$

If $(x-4)(x-16)(x-1)(x-64) \neq 0,$ *i.e.,* $x\neq 1,\ x\neq 4,\ x\neq 16,\ x\neq 64,$ *then*

$$\sum_{k=0}^{n} x^k H^3_{-2k} = \frac{\Psi_1}{(x-4)(x-16)(x-1)(x-64)}$$

where

$\Psi_1 = x^{n+1}(x-4)(x-16)H^3_{-2n+2} + x^{n+1}(x-65)(x-4)(x-16)H^3_{-2n} - 12\times 2^{-2n}x^{n+1}(x-4)(x-16)H_{-2n+2} + 60\times 2^{-2n}x^{n+1}(x^2-65x+208)H_{-2n} - 48\times 2^{-2n}x^{n+1}(x-1)(x-64)H_{-2n-2} - (125x^3-5712x^2+35520x-32768)$

and

if $(x-4)(x-16)(x-1)(x-64)=0,$ *i.e.,* $x=1$ *or* $x=4$ *or* $x=16$ *or* $x=64$ *then*

$$\sum_{k=0}^{n} x^k H^3_{-2k} = \frac{\Psi_2}{(4x^3-255x^2+2856x-5440)}$$

where

$\Psi_2 = x^n(n(x-4)(x-16)+3x^2-40x+64)H^3_{-2n+2} + x^n(n(x-65)(x-4)(x-16)+4x^3-255x^2+2728x-4160)H^3_{-2n} - 12\times 2^{-2n}x^n(n(x-4)(x-16)+3x^2-40x+64)H_{-2n+2} + 60\times 2^{-2n}x^n(n(x^2-65x+208)+3x^2-130x+208)H_{-2n} - 48\times 2^{-2n}x^n(n(x-1)(x-64)+3x^2-130x+64)H_{-2n-2} - 3(125x^2-3808x+11840).$

(f): $(m=-2,\ j=1)$

If $(x-4)(x-16)(x-1)(x-64) \neq 0,$ *i.e.,* $x\neq 1,\ x\neq 4,\ x\neq 16,\ x\neq 64,$ *then*

$$\sum_{k=0}^{n} x^k H^3_{-2k+1} = \frac{\Psi_1}{(x-4)(x-16)(x-1)(x-64)}$$

where

$\Psi_1 = x^{n+1}(x-4)(x-16)H^3_{-2n+3} + x^{n+1}(x-65)(x-4)(x-16)H^3_{-2n+1} - 24\times 2^{-2n}x^{n+1}(x-4)(x-16)H_{-2n+3} + 120\times 2^{-2n}x^{n+1}(x^2-65x+208)H_{-2n+1} - 96\times 2^{-2n}x^{n+1}(x-1)(x-64)H_{-2n-1} - 27(27x^3-964x^2+4928x-4096)$

and

if $(x-4)(x-16)(x-1)(x-64)=0,$ *i.e.,* $x=1$ *or* $x=4$ *or* $x=16$ *or* $x=64$ *then*

$$\sum_{k=0}^{n} x^k H^3_{-2k+1} = \frac{\Psi_2}{(4x^3-255x^2+2856x-5440)}$$

where

$\Psi_2 = x^n(n(x-4)(x-16)+3x^2-40x+64)H^3_{-2n+3} + x^n(n(x-65)(x-4)(x-16)+4x^3-255x^2+2728x-4160)H^3_{-2n+1} - 24\times 2^{-2n}x^n(n(x-4)(x-16)+3x^2-40x+64)H_{-2n+3} + 120\times 2^{-2n}x^n$

$(n(x^2-65x+208)+3x^2-130x+208)H_{-2n+1}-96\times 2^{-2n}x^n(n(x-1)(x-64)+3x^2-130x+64)H_{-2n-1}-27(81x^2-1928x+4928)$.

Taking $x=1$ in the last two corollaries we get the following corollary.

COROLLARY 5. *For $n\geq 0$, Mersenne numbers and Mersenne-Lucas numbers have the following properties:*

(1)

(a): $\sum_{k=0}^{n}M_k^3=\frac{1}{21}(-(3n-11)M_n^3+8(3n+13)M_{n-1}^3+42\times 2^nM_{n+1}-9\times 2^n(n+15)M_n+12\times 2^n(3n+13)M_{n-1}+49)$.

(b): $\sum_{k=0}^{n}M_{2k}^3=\frac{1}{315}(-(5n-303)M_{2n}^3+64(5n+17)M_{2n-2}^3+21\times 2^{2n+2}M_{2n+2}-15\times 2^{2n}(n+29)M_{2n}+3\times 2^{2n+4}(5n+17)M_{2n-2}+819)$.

(c): $\sum_{k=0}^{n}M_{2k+1}^3=\frac{1}{315}(-(5n-303)M_{2n+1}^3+64(5n+17)M_{2n-1}^3+168\times 2^{2n}M_{2n+3}-30\times 2^{2n}(n+29)M_{2n+1}+96\times 2^{2n}(5n+17)M_{2n-1}+658)$.

(d): $\sum_{k=0}^{n}M_{-k}^3=\frac{1}{21}(-(3n-1)M_{-n+1}^3+(24n-11)M_{-n}^3-6\times 2^{-n}(3n-1)M_{-n+1}+18\times 2^{-n}(4n-3)M_{-n}+84\times 2^{-n}M_{-n-1}+35)$.

(e): $\sum_{k=0}^{n}M_{-2k}^3=\frac{1}{315}(-(5n+3)M_{-2n+2}^3+(320n+187)M_{-2n}^3-12\times 2^{-2n}(5n+3)M_{-2n+2}+60\times 2^{-2n}(16n+9)M_{-2n}+336\times 2^{-2n}M_{-2n-2}+441)$.

(f): $\sum_{k=0}^{n}M_{-2k+1}^3=\frac{1}{315}(-(5n+3)M_{-2n+3}^3+(320n+187)M_{-2n+1}^3-24\times 2^{-2n}(5n+3)M_{-2n+3}+120\times 2^{-2n}(16n+9)M_{-2n+1}+672\times 2^{-2n}M_{-2n-1}+917)$.

(2)

(a): $\sum_{k=0}^{n}H_k^3=\frac{1}{21}(-(3n-11)H_n^3+8(3n+13)H_{n-1}^3-42\times 2^nH_{n+1}+9\times 2^n(n+15)H_n-12\times 2^n(3n+13)H_{n-1}-181)$.

(b): $\sum_{k=0}^{n}H_{2k}^3=\frac{1}{315}(-(5n-303)H_{2n}^3+64(5n+17)H_{2n-2}^3-84\times 2^{2n}H_{2n+2}+15\times 2^{2n}(n+29)H_{2n}-48\times 2^{2n}(5n+17)H_{2n-2}-1459)$.

(c): $\sum_{k=0}^{n}H_{2k+1}^3=\frac{1}{315}(-(5n-303)H_{2n+1}^3+64(5n+17)H_{2n-1}^3-168\times 2^{2n}H_{2n+3}+30\times 2^{2n}(n+29)H_{2n+1}-96\times 2^{2n}(5n+17)H_{2n-1}-1998)$.

(d): $\sum_{k=0}^{n}H_{-k}^3=\frac{1}{21}(-(3n-1)H_{-n+1}^3+(24n-11)H_{-n}^3+6\times 2^{-n}(3n-1)H_{-n+1}-18\times 2^{-n}(4n-3)H_{-n}-84\times 2^{-n}H_{-n-1}+265)$.

(e): $\sum_{k=0}^{n}H_{-2k}^3=\frac{1}{315}(-(5n+3)H_{-2n+2}^3+(320n+187)H_{-2n}^3+12\times 2^{-2n}(5n+3)H_{-2n+2}-60\times 2^{-2n}(16n+9)H_{-2n}-336\times 2^{-2n}H_{-2n-2}+2719)$.

(f): $\sum_{k=0}^{n}H_{-2k+1}^3=\frac{1}{315}(-(5n+3)H_{-2n+3}^3+(320n+187)H_{-2n+1}^3+24\times 2^{-2n}(5n+3)H_{-2n+3}-120\times 2^{-2n}(16n+9)H_{-2n+1}-672\times 2^{-2n}H_{-2n-1}+9243)$.

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