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Compactness of any Countable Product of Compact Metric Spaces in Product Topology without UsingTychonoff's Theorem

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Abstract:
Background: *For infinite products of compact spaces, Tychonoff's theorem asserts that their product is compact, in the product topology. Tychonoff's theorem is shown to be equivalent to the axiom of choice. In this paper, we show that any countable product of compact metric spaces iscompact, without using Tychonoff's theorem. The proof needs only basic and standard facts of compact metric spaces and the Bolzano-Weierstrass property. Moreover, the component spaces need not be assumed to be copies of the same compact metric space, and each component space can be an arbitrary nonempty compact metric space independently.*
Materials and Methods: *Total boundedness together with completeness of a metric space implies its compactness. Completeness of a product of complete spaces is easily inferred from the completeness of each component. Total boundedness therefore suffices to prove the compactness of a countably (infinitely) many nonempty compact component spaces. The countable infiniteness is needed in the proof to exhibit a standard metric that gives rise to the product topology.Any such metric topology for the product arises as exhibited, andthey are all equivalent to the product topology. The requirement of summability ofthe sequences restricts the scope of the result to countably infinite products.*
Results: *The product space obtained by taking the product of any sequence of nonempty compact metric spaces in the product topology is shown to be compact, using only the basic and standard facts of compact metric spaces.**Conclusion:** Compactness of the product of a countably infinitely many nonempty compact metric spaces can be proved within Cantor's set theory, without using the axiom of choice and Tychonoff's theorem.*

Key Word: *Metric Spaces; Bolzano-Weierstrass Property; Compactness; Completeness; Total Boundedness.*

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I. Introduction

Tychonoff's theorem states that any infinite product of compact spaces is compact in the product topology. The only nonessential assumption in the general statement is that the component spaces are copies of the same compact space. It is interesting to explore the possibility of proving compactness without using Tychonoff's theorem, for infinite products. This is possible, but with some restrictions. It is assumed that the product consists of only countably many nonempty compact metric spaces. For metric spaces, compactness equivalent to the Bolzano-Weierstrass property, whereby requiring that any sequence containing infinitely many distinct elements has a limit point. By completeness, the limit point of any such sequence is required to belong to the given space. Metrizability of a countable product of nonempty compact metric spaces is easy to establish. The completeness of the product of complete metric spaces can be inferred by the componentwise completeness, for Cauchy sequences in metric spaces. It follows that the total boundedness implies compactness, for a complete metric space. Now, total boundednesscan be proved for the product of a countably infinitely many nonempty compact metric spaces by observing that any ε-net effectively induces an $\frac{\varepsilon}{2}$-net on a product of a fixed finite number of compact metric spaces, which permits a finite $\frac{\varepsilon}{2}$-net cover, enabling us to recover a finite ε-net cover of the originally given product space consisting of countably infinitely many nonempty compact metric spaces. In this regards, the main fact is the demonstration of a metric that is easy to define on the product space, but restricting its applicability to sequences of spaces, meaning countably infinitely many nonempty compact metric spaces.The metric topology thus obtained is equivalent to the product topology.

II. Material And Methods

Let $\mathbb{N}$ be the set of positive integers. The section starts with the following basic proposition:

Proposition 1 *Let $(X,\ d)$ be a complete metric space, with anonempty set of elements X and a metric d for its topology. If X is totally bounded, then it is compact.*

Proof By the assumption of total boundedness, for every positive integer *n*, there is a finite subset$A_n \subseteq X$, such that $X = \bigcup_{y \in A_n} B\left(y,\ \frac{1}{2n}\right)$. Our objective is to prove that the Bolzano-Weierstrass property holds for *X*. Let$\{x_i : i \in \mathbb{N}\}$ be a sequence, containing infinitely many distinct elements. A sequence is conveniently identified with a set, whose elements are ordered by the natural number subscripts. For each positive integer *n*, by the finiteness of the setA_n, there is an element$y_n \in A_n$, such that $B\left(y_n,\ \frac{1}{2n}\right)$contains the elements x_i of the given sequence, forinfinitely many indexes $i \in \mathbb{N}$. A refined sequence of basic open sets $B\left(y_n,\ \frac{1}{2n}\right)$, with$y_n \in A_n$, such that $B\left(y_{n+1},\ \frac{1}{2(n+1)}\right)$contains infinitely many elementsx_i from the subsequence that is already included in $B\left(y_n,\ \frac{1}{2n}\right)$, can be extracted, by repeating the preceding observation inductively.For every $m \in \mathbb{N}$, $B\left(y_{m+n},\ \frac{1}{2(m+n)}\right) \cap B\left(y_n,\ \frac{1}{2n}\right)$ is nonempty−in fact,$B\left(y_{m+n},\ \frac{1}{2(m+n)}\right)$contains$x_i$for infinitelymany indexes $i \in \mathbb{N}$,that are already contained in $B\left(y_n,\ \frac{1}{2n}\right)$ −and therefore, by the triangle inequality, $d(y_{m+n}, y_n) < \frac{1}{2(m+n)} + \frac{1}{2n} < \frac{1}{n}$. Thus, the sequence$\{y_n : n \in \mathbb{N}\}$is a Cauchy sequence in *X*, converging to an element $z \in X$, by the completeness of *X*. It is easy to check thatthe element $z \in X$ is a limit point of the sequence$\{x_i : i \in \mathbb{N}\}$, hence proving the Bolzano-Weierstrass property.

▢

An important property the proof of Proposition 1 interludes is that any totally bounded metric space admits a countable dense subset $S = \bigcup_{n=1}^{\infty} A_n$: if $x \in X$ is any point, then for every $\varepsilon > 0$, $B(x,\ \varepsilon) \cap S$ is nonempty.The following proposition is a useful fact about compact metric spaces:

Proposition 2 *Let$(X\ ,\ d)$ be a nonempty compact metric space, with nonempty set X and metric d. Then, there is a pair of elements x and y, both in X, such that* $d(x\ ,\ y) = sup\{d(s,\ t) : s, t \in X\} < \infty$.

Proof If *X* is finite, the contention is obvious. Now, it is assumed that *X* is an infinite set. By the continuity of the metric, with respect to itself, and by the compactness of *X*, for every element $x \in X$, there is an element$y_x \in X$, such that $d(x\ ,\ y_x) = \sup\{d(x,\ z) : z \in X\} = M_x < \infty$. Thus, $d(s\ ,\ t) \le 2M_x$, for every pair of elements *s* and *t*, both in *X*.Let$\{(s_i\ , t_i) \in X \times X : i \in \mathbb{N}\}$ be a sequence such that $d(s_i, t_i) \le d(s_{i+1}, t_{i+1})$, for $i \in \mathbb{N}$, and$\lim_{i \to \infty} d(s_i,\ t_i) = M = \sup\{d(s,\ t) : s, t \in X\}$. By the compactness of the finite product $X \times X$, there is an element$(x, y) \in X \times X$, which is a limit point of the sequence $\{(s_i\ , t_i) \in X \times X : i \in \mathbb{N}\}$in $X \times X$. Since the distances are nondecreasing, it follows that $d(x, y) = M$.

▢

Let$(X_i\ , d_i)$ be a sequence of nonempty compact metric spaces, with nonempty sets$X_i$and metricsd_i, for $i \in \mathbb{N}$. Let$l_i$and$M_i$be positive real numbers, such that$\sum_{i=1}^{\infty} l_i = L < \infty$ and$d_i(s, t) \le M_i$, for every$(s, t) \in X_i \times X_i$, for $i \in \mathbb{N}$. Let$\mathbf{X} = \prod_{i=1}^{\infty} X_i$, and, for$\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \mathbf{X}$ and$\mathbf{y} = (y_1, y_2, y_3, \ldots) \in \mathbf{X}$, let $D(\mathbf{x}\ ,\ \mathbf{y}) = \sum_{i=1}^{\infty} \frac{l_i d_i(x_i\ , y_i)}{1 + M_i}$. By the positiveness of$l_i$, it can be easily seen that $D(\mathbf{x}\ ,\ \mathbf{y}) = 0$, if and only if $d_i(x_i\ , y_i) = 0$, for every $i \in \mathbb{N}$, and the symmetry and the triangle inequality follow immediately, ensuring that $(\mathbf{X}, D)$ is a metric space. Likewise, completeness is also easy to establish: if$\mathbf{x}_j = \left(x_{j,\ 1}\ ,\ x_{j,\ 2}\ ,\ x_{j,\ 3}\ , \ldots\right) \in \mathbf{X}$, for $j \in \mathbb{N}$, is a Cauchy sequence in $(\mathbf{X}, D)$, then so isthe sequence $x_{j,\ i}$, for $j \in \mathbb{N}$, for each component index $i \in \mathbb{N}$, and since the Cauchy sequence$x_{j,\ i}$, for $j \in \mathbb{N}$, has a unique limit pointz_i in the complete metric spaceX_i, for each component, it follows that that there is a unique limit point $\mathbf{z} = (z_1, z_2, z_3, \ldots)$of the sequence$\mathbf{x}_j$, $j \in \mathbb{N}$. Thus, $(\mathbf{X}, D)$ is a complete metric space.

Every nonempty metric space is also topologically equivalent to another metric space, whose metric is bounded. Let(X, d)be a metric space, where X is nonempty. For $x, y, z \in X$, let d'be obtained by clamping d above the threshold 1, to the threshold value 1, namely$d'(x, y) = d(x, y)$, whenever $d(x, y) \le 1$, and$d'(x, y) = 1$, whenever $d(x, y) > 1$. Symmetry is easily seen to hold, and$d'(x, y) = 0$, exactly when $d(x, y) = 0$. As to the triangle inequality, there are two possibilities to consider. If $d'(x, y) = 1$ or $d'(y, z) = 1$, then, clearly, $d'(x, z) \le 1 + d'(y, z) = d'(x, y) + d'(y, z)$or $d'(x, z) \le d'(x, y) + 1 = d'(x, y) + d'(y, z)$. If both $d'(x, y) < 1$ and $d'(y, z) < 1$, then $d'(x, y) = d(x, y)$and $d'(y, z) = d(y, z)$, and hence $d'(x, z) \le d(x, z) \le d(x, y) + d(y, z) = d'(x, y) + d'(y, z)$. Thus, $d'$is a metric, and, locally, $d'$and d generate the same open ε-neighborhoods of radius $\varepsilon < 1$, meaning that they are topologically equivalent. In the preceding paragraph, we may take$M_i = 1$, for every $i \in \mathbb{N}$, without loss of generality.

It is a more important fact that the metric D induces the product topology on$\mathbf{X} = \prod_{i=1}^{\infty} X_i$. In order to show that this is indeed so, we need to show that the basic open sets in the product topology remain open with respect to the metric space topology, and conversely, every basic open set in the metric topology is also an open set in the product topology. This is the content of the next proposition.

Proposition 3 *Let (X_i, d_i) be a nonempty metric space, with a bounded metric d_i, for $i \in \mathbb{N}$, and $(\mathbf{X}, D)$ be as in the notation just discussed. The topology of $(\mathbf{X}, D)$ is equivalent to the product topology of $\mathbf{X} = \prod_{i=1}^{\infty} X_i$.*
Proof Let $\mathbf{U} = \prod_{i=1}^{\infty} U_i$ be a nonempty open subset of $\mathbf{X}$ in the product topology, and let $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \mathbf{U}$. By the condition of the product topology, there are only finitely many indexes $i \in \mathbb{N}$, for which $U_i \neq X_i$. Let $n \in \mathbb{N}$ be such that $U_{n+j} = X_{n+j}$, for every $j \in N$. Let $\varepsilon > 0$ be such that whenever $\frac{l_i d_i(x_i, y_i)}{1+M_i} < \varepsilon$, it holds that $y_i \in U_i$, for every index i, where $1 \leq i \leq n$. Now, if $D(\mathbf{x}, \mathbf{y}) < \varepsilon$, then $\mathbf{y} = (y_1, y_2, y_3, \ldots) \in \mathbf{U}$, and $\mathbf{x}$ is an interior point of $\mathbf{U}$, with respect to the metric topology. Conversely, let $B(\mathbf{x}, \varepsilon)$ be the basic open subset of $\mathbf{X}$, with respect to the metric topology, centered at $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \mathbf{X}$, of radius ε, for some $\varepsilon > 0$. Let $n \in \mathbb{N}$ be such that $\sum_{i=n+1}^{\infty} l_i < \frac{\varepsilon}{2}$. Then, the open set $\mathbf{U} = \prod_{i=1}^{\infty} U_i$, with respect to the product topology, obtained by constraining the first n components by the condition $\sum_{i=1}^{n} \frac{l_i d_i(x_i, y_i)}{1+M_i} < \frac{\varepsilon}{2}$, and by letting the components of indexes $(n+j)$ be X_{n+j}, for $j \in N$, is included in $B(\mathbf{x}, \varepsilon)$, and $\mathbf{x}$ is an interior point of $B(\mathbf{x}, \varepsilon)$, with respect to the product topology.▫

III. Result

The following is the main result.

Theorem1 (Sagar) *Let (X_i, d_i), for $i \in \mathbb{N}$, and $(\mathbf{X}, D)$ be as in the notation discussed in the preceding section. If (X_i, d_i) is a compact metric space for each $i \in \mathbb{N}$, then the metric space $(\mathbf{X}, D)$ is totally bounded (hence becomes compact, when combined with the fact that it is a complete metric space).*
Proof Let $B(\mathbf{x}, \varepsilon)$ be the basic open subset of $\mathbf{X}$, with respect to the metric topology, centered at $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \mathbf{X}$, of radius ε, for some $\varepsilon > 0$. Let $n \in \mathbb{N}$ be such that $\sum_{i=n+1}^{\infty} l_i < \frac{\varepsilon}{2}$. Then, the open set $\mathbf{U} = \prod_{i=1}^{\infty} U_i$, with respect to the product topology, obtained by constraining $\prod_{i=1}^{n} U_i$ by the condition $\sum_{i=1}^{n} \frac{l_i d_i(x_i, y_i)}{1+M_i} < \frac{\varepsilon}{2}$, with $U_{n+j} = X_{n+j}$, for $j \in N$, is included in $B(\mathbf{x}, \varepsilon)$. For the open cover $\mathcal{V}(\varepsilon) = \{B(\mathbf{x}, \varepsilon) : \mathbf{x} \in \mathbf{X}\}$, by the fact that the finite product $\prod_{i=1}^{n} X_i$ is compact, as can be proved using only basic topological facts, without using Tychonoff's theorem, there is a finite net $A_n(\varepsilon) \subseteq \prod_{i=1}^{n} X_i$, such that every element in $\prod_{i=1}^{n} X_i$ is at a distance less than $\frac{\varepsilon}{2}$ from or to some element in $A_n(\varepsilon)$. For every $(x_1, x_2, \ldots, x_n) \in A_n(\varepsilon)$, an element $(x_1, x_2, \ldots, x_n, \xi_{n+1}, \xi_{n+2}, \ldots) \in \mathbf{X}$ is formed, by selecting $\xi_{n+j} \in X_{n+j}$, for $j \in N$, arbitrarily, for inclusion in the set $\mathbf{A}(\varepsilon) \subseteq \mathbf{X} = \prod_{i=1}^{\infty} X_i$. The nonemptiness of X_i, for $i \in \mathbb{N}$, suffices, for this purpose, and this assignment does not require the use of the axiom of choice. If insisted upon, it can be stated as part of the assumption that $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3, \ldots) \in \mathbf{X}$ is an *a priori* specified fixed element, from which the remaining components $\xi_{n+j} \in X_{n+j}$, for $j \in N$, can be taken. The one-to-one correspondence between $A_n(\varepsilon)$ and $\mathbf{A}(\varepsilon)$ ensures the finiteness of the latter, thereby proving the total boundedness of $(\mathbf{X}, D)$.
▫

Example 1 Let $X = \{0, 1\}$ and $d(x, y) = |x - y|$, for $x, y \in X$. We take (X_i, d_i) to be (X, d), for $i \in \mathbb{N}$. The set $\mathbf{X} = \prod_{i=1}^{\infty} X_i$ can be identified with the interval $I = [0, 1]$, by a function ɨ, mapping $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \mathbf{X}$ to the real number defined by ɨ$(\mathbf{x}) = \sum_{i=1}^{\infty} 2^{-i} x_i$. For the metric $D_{\mathbf{X}}$ of $\mathbf{X}$, for $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \mathbf{X}$ and $\mathbf{y} = (y_1, y_2, y_3, \ldots) \in \mathbf{X}$, let $D_{\mathbf{X}}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} 2^{-i} |x_i - y_i|$. By the triangle inequality applied to the series, the following holds: $\left|\sum_{i=1}^{\infty} 2^{-i} x_i - \sum_{i=1}^{\infty} 2^{-i} y_i\right| = \left|\sum_{i=1}^{\infty} 2^{-i}(x_i - y_i)\right| \leq \sum_{i=1}^{\infty} 2^{-i} |x_i - y_i|$. Let d_I be the standard absolute difference metric on *I*. Then, $d_I($ɨ$(\mathbf{x})$, ɨ$(\mathbf{y})) \leq D_{\mathbf{X}}(\mathbf{x}, \mathbf{y})$, for $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, and the function ɨ becomes continuous. If ϕ is a continuous function with domain *I*, then so is the function $\hat{\phi} = \phi \circ$ ɨ, with domain $\mathbf{X}$. ▫

However, the function ɨ in Example 1 is not one-to-one, as can be seen by observing that the sequence $(0, 1, 1, \ldots)$ and $(1, 0, 0, \ldots)$ are both mapped to the real number $\frac{1}{2}$ by the function ɨ. More generally, let ɨ$(\mathbf{x})$ = ɨ$(\mathbf{y})$, for two sequences $\mathbf{x} \neq \mathbf{y}$, and let n be the least positive integer such that $x_n \neq y_n$, and for the sake of definiteness, let $x_n = 1$ and $y_n = 0$. The condition ɨ$(\mathbf{x})$ = ɨ$(\mathbf{y})$ implies that $\sum_{i=1}^{\infty} 2^{-i} x_i = \sum_{i=1}^{\infty} 2^{-i} y_i$, from which we find that $2^{-n} + \sum_{i=n+1}^{\infty} 2^{-i} x_i = \sum_{i=n+1}^{\infty} 2^{-i} y_i$ and that $2^{-n} = \sum_{i=n+1}^{\infty} 2^{-i}(y_i - x_i)$. Now, if $y_i < x_i$, for any index $i \geq n+1$, then $\sum_{i=n+1}^{\infty} 2^{-i}(y_i - x_i) < 2^{-n}$. Thus, if ɨ$(\mathbf{x})$ = ɨ$(\mathbf{y})$, for two sequences $\mathbf{x} \neq$

y, then one of them is eventually all 1s, and the other is eventually all 0s, with the component bits just before the beginning of the eventuality also occurring complementarily. Now, to get a continuous function ϕ defined on the interval *I*, from a continuous function $\hat{\phi}$ defined on **X**, we have to follow an indirect approach, wherein the quotient space **W** of **X** with respect to an equivalence relation is formed. In this example, the equivalence relation must preserve the continuity of the function ɨ, and hence let $\mathbf{x} \equiv \mathbf{y}(\bmod ɨ)$, exactly when $ɨ(\mathbf{x}) = ɨ(\mathbf{y})$. The quotient space **W** consists of the equivalence classes, which are singleton sets for the sequence of all components 1 each, for the sequence of all components 0 each, for the sequences not ending in all 1s with at least one 0 or not ending in all 0s with at least one 1, and two-point sets consisting of a sequence ending with all 1s, with at least one component 0, and another sequence ending with all 0s, with at least one component 1. Let $\pi_{\mathbf{X}/\equiv(\bmod ɨ)}$ be the quotient map from **X** to **W** that assigns to a sequence $\mathbf{x} \in \mathbf{X}$ the equivalence class $\pi_{\mathbf{X}/\equiv(\bmod ɨ)}(\mathbf{x})$ in **W**, such that $\pi_{\mathbf{X}/\equiv(\bmod ɨ)}(\mathbf{x})$ contains **x**. Now, the topology of **W** is obtained by taking the largest collection of subsets of **W** to be open subsets, such that the quotient map $\pi_{\mathbf{X}/\equiv(\bmod ɨ)}$ remains continuous, namely a subset $\mathbf{U} \subseteq \mathbf{W}$ is open if and only if the set $\pi_{\mathbf{X}/\equiv(\bmod ɨ)}{}^{-1}(\mathbf{U})$ is open in **X**Thus, in the quotient space, if a subset $\mathbf{U} \subseteq \mathbf{W}$ containing $\pi_{\mathbf{X}/\equiv(\bmod ɨ)}(\mathbf{x})$, for some $\mathbf{x} \in \mathbf{X}$, is open, then **x** is an interior point of $\pi_{\mathbf{X}/\equiv(\bmod ɨ)}{}^{-1}(\mathbf{U})$ in **X**. For a two-point set $\{\mathbf{x}, \mathbf{y}\} \in \mathbf{W}$, where $\mathbf{x}, \ \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \neq \mathbf{y}$, but, of course, $ɨ(\mathbf{x}) = ɨ(\mathbf{y})$, the subset $\pi_{\mathbf{X}/\equiv(\bmod ɨ)}\big(B_{\mathbf{X}}(\mathbf{x}, \ \varepsilon)\big)$ of **W** must contain $\{\mathbf{x}, \mathbf{y}\}$, for any neighborhood $B_{\mathbf{X}}(\mathbf{x}, \ \varepsilon)$ of **x** in **X**, not containing **y**. In the next few propositions, the topological spaces **W** and *I* are shown to be homeomorphic.

Proposition 4 *Let* **X** *and* **W** *be as in the notation of the preceding discussion. Then,* **W** *is compact.*
Proof Being the image of the compact space **X** under the continuous quotient map $\pi_{\mathbf{X}/\equiv(\bmod ɨ)}$.

▢

Let ɉ be the function from **W** into ***I***, defined by $ɉ(\mathbf{w}) = ɨ(\mathbf{x})$, for any $\mathbf{x} \in \mathbf{w}$ and $\mathbf{w} \in \mathbf{W}$. It is easy to check that ɉ is well-defined, since ɨ remains constant on $\mathbf{w} \in \mathbf{W}$. Moreover, ɉ is one-to-one from **W** onto *I*. For any $c \in I$, the set $ɨ^{-1}(\{c\})$ is precisely the equivalence class $ɉ^{-1}(c)$. For an open subset $U \subseteq I$, $ɨ^{-1}(U)$ is open in X, but $ɨ^{-1}(U) = \bigcup_{c \in U} ɉ^{-1}(c)$. We have to show that $\bigcup_{c \in U}\{ɉ^{-1}(c)\}$ is open in **W**. Some subtle clarification about the notation is as follows: the set $\bigcup_{c \in U} ɉ^{-1}(c)$ is the union of singleton or two-point sets and a subset of **X**, but the set $\bigcup_{c \in U}\{ɉ^{-1}(c)\}$ is the collection of singleton or two-point sets and a subset of **W**. Now, $\pi_{\mathbf{X}/\equiv(\bmod ɨ)}{}^{-1}(\bigcup_{c \in U}\{ɉ^{-1}(c)\}) = \bigcup_{c \in U} \pi_{\mathbf{X}/\equiv(\bmod ɨ)}{}^{-1}\{ɉ^{-1}(c)\} = \bigcup_{c \in U} ɉ^{-1}(c) = ɨ^{-1}(U)$, which is open in *X*, and hence, $ɉ^{-1}(U) = \bigcup_{c \in U}\{ɉ^{-1}(c)\}$ is open in **W**, following the quotient topology.

Proposition 5 *Let X and Y be nonempty topological spaces, and $f : X \to Y$ be a continuous function from X onto Y. If X is compact, then so is Y, and if Y is Hausdorff and f is one-to-one, then X is also Hausdorff.*
Proof The first statement is a standard fact. For the second statement, let $x_1, x_2 \in X$, such that $x_1 \neq x_2$. We have to show that there are open subsets U_1 and U_2 of *X*, such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since *f* is one-to-one, by the assumption in the statement $y_1 \neq y_2$, and hence, there are open subsets V_1 and V_2 of *X*, such that $y_1 \in V_1$, $y_2 \in V_2$ and $V_1 \cap V_2 = \emptyset$. Now, let $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$. Here, and in some contexts of the preceding proofs, we have to observe that, for any function, or even a single-valued mapping, $f\big(f^{-1}(R)\big) \subseteq R$, for any subset *R* of the set into which f maps, and that $S \subseteq f^{-1}\big(f(S)\big)$, for any subset *S* of the domain of *f*. Now, $x_1 \in U_1$ and $x_2 \in U_2$, and if $z \in U_1 \cap U_2$, then $f(z) \in V_1 \cap V_2$, contrary to the choice that $V_1 \cap V_2 = \emptyset$, proving that *X* is Hausdorff.

▢

Proposition 6 *Let X and Y be nonempty spaces and $f : X \to Y$ be a continuous one-to-one correspondence from X onto Y. If X compact and Y is Hausdorff, then f is a homeomorphism.*
Proof Since *f* is a continuous and one-to-one, it follows that *X* is also Hausdorff, and since *f* is a continuous and $Y = f(X)$, it follows that *Y* is also compact. It is required to check only that f is an open map. For any open subset $U \subseteq X$, the set $X \setminus U$ is closed and, hence compact, in X, which is a compact Hausdorff space. Now, $Y \setminus f(U) = f(X \setminus U)$ is a compact subset of *Y*, by the continuity of f. Thus, $Y \setminus f(U)$ is closed in *Y*, which is also a compact Hausdorff space, and hence, $f(U)$ is an open subset of *Y*.

▢

Proposition 6 needs the compactness of the space X, for its contention to hold: if X is equipped with the discrete topology, then X is a complete metric space, which may not be compact, and the mapping in Proposition 6 may fail to be a homeomorphism, even when *Y* is compact.

Proposition 7 *Let I,* **W** *and* ɉ *be as in the notation of the preceding discussion. Then,* ɉ *is a homeomorphism of* **W** *onto I.*
Proof Follows from the discussion of the paragraph preceding Proposition 5.
▫

The following result generalizes the concepts discussed in the Example 1 and thereafter:

Theorem 2 *Let X and Y be nonempty spaces and* $ɨ : X \to Y$ *be a continuous function from X onto Y. Let* $\equiv (\,mod\, ɨ\,)$ *be the equivalence relation defined by the condition* $x_1 \equiv x_2(\,mod\, ɨ\,)$ *exactly when* $ɨ(x_1) = ɨ(x_2)$*, for* $x_1\,, x_2 \in X$*. Let W be the quotient space* $X \,/ \equiv (\,mod\, ɨ\,)$*, obtained by collecting the equivalence classes with respect to* $\equiv (\,mod\, ɨ\,)$*, equipped with the quotient topology, and let* $\pi_{\mathbf{X}\,/\,\equiv\,(\,\mathrm{mod}\,\,ɨ\,)}$ *be the quotient map from X onto W. Let* $ɉ : W \to Y$ *be defined by* $ɉ(w) = ɨ(x)$*, for any* $x \in w$ *and* $w \in W$*. Then,*

- **(a)** ɉ *is a well-defined, one-to-one and continuous function from* W *onto* Y*;*
- **(b)** *if X is compact, then so are W and Y;*
- **(c)** *if Y is Hausdorff, then so is W; and*
- **(d)** *if both X is compact and Y is Hausdorff, then* ɉ *is a homeomorphism of W onto Y.*

Proof The function ɨ is exactly $ɉ \circ \pi_{\mathbf{X}\,/\,\equiv\,(\,\mathrm{mod}\,\,ɨ\,)}$, hence ɉ *is* continuous[6]. The remaining properties follow immediately, by applying Propositions 5 and 6.
▫

Example 2 Let $0 < a < 1$, $X = [0, 1]$ and $d_X(x, y) = |x - y|$, for $x, y \in X$. We take $(X_i\,,\, d_{X_i})$ to be (X, d_X), for $i \in \mathbb{N}$. Let $\mathbf{X} = \prod_{i=1}^{\infty} X_i$, with its metric $D_{\mathbf{X}}(\mathbf{x},\ \mathbf{y}) = \sum_{i=1}^{\infty} a^i d_{X_i}(x_i, y_i)$, for $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \mathbf{X}$ and $\mathbf{y} = (y_1, y_2, y_3, \ldots) \in \mathbf{Y}$. Let $I = \left[0,\ \frac{a}{1-a}\right]$, with its metric $d_I(x, y) = |x - y|$, for $x, y \in I$. Let ɨ be the real valued function defined by $ɨ(\mathbf{x}) = \sum_{i=1}^{\infty} a^i x_i$, for $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \mathbf{X}$. Since $0 \le x_i \le 1$, for $i \in \mathbb{N}$, it follows that $ɨ(\mathbf{x}) \in I$, for $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \mathbf{X}$. For any $y \in I$, if $y < \frac{a}{1-a}$, then there exists $n \in \mathbb{N}$, such that $\sum_{i=1}^{n} a^i > y$, and for definiteness, let n be the least positive integer, for which $\sum_{i=1}^{n} a^i > y$. Now, $\sum_{i=1}^{n-1} a^i \le y$, and the sequence $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \mathbf{X}$, defined by $x_i = 1$, for $1 \le i \le n - 1$, $x_n = a^{-n}(y - \sum_{i=1}^{n-1} a^i)$ and $x_{n+i} = 0$, for $i \in \mathbb{N}$, is such that $ɨ(\mathbf{x}) = y$, and hence the function ɨ is a surjection onto the interval *I*. It is easy to check that $d_I\big(ɨ(\mathbf{x}), ɨ(\mathbf{y})\big) \le D_{\mathbf{X}}(\mathbf{x},\ \mathbf{y})$, hence the continuity of the function ɨ. But the function ɨ is not one-to-one, as can be found by observing that $\frac{a}{2} + \frac{a^2}{2} = \frac{a(1+a)}{2}$. The two sequences, with one obtained by taking the first component to be $\frac{(1+a)}{2}$ and all others to be 0, and with another obtained by taking the first two component to be $\frac{1}{2}$ each, are both mapped to the same element $\frac{a}{2} + \frac{a^2}{2}$, by the function ɨ. By Theorem 2, the quotient space $\mathbf{W} = \mathbf{X} \,/ \equiv (\mathrm{mod}\ ɨ\,)$ is homeomorphic to the interval *I*, with respect to the quotient topology of **W**. ▫

Theorem 3 *Let X and Y be nonempty spaces, with X compact and Y Hausdorff. Let* $ɨ : X \to Y$ *be a continuous function from X onto Y. Let* $\equiv (\,mod\, ɨ\,)$ *be the equivalence relation defined by the condition* $x_1 \equiv x_2(\,mod\, ɨ\,)$ *exactly when* $ɨ(x_1) = ɨ(x_2)$*, for* $x_1\,, x_2 \in X$*. Let W be the quotient space* $X \,/ \equiv (\,mod\, ɨ\,)$*, obtained by collecting the equivalence classes with respect to* $\equiv (\,mod\, ɨ\,)$*, equipped with the quotient topology, and let* $\pi_{\mathbf{X}\,/\,\equiv\,(\,\mathrm{mod}\,\,ɨ\,)}$ *be the quotient map from X onto W. Let* $ɉ : W \to Y$ *be defined by* $ɉ(w) = ɨ(x)$*, for any* $x \in w$ *and* $w \in W$*. Let* $\phi_{X,\ Z} : X \to Z$ *be a continuous function, such that* $\phi_{X,\ Z}(x_1) = \phi_{X,\ Z}(x_2)$*, whenever* $ɨ(x_1) = ɨ(x_2)$*, for* $x_1\,, x_2 \in X$*. Then, the functions* $\phi_{W,\ Z} : W \to Z$ *and* $\phi_{Y,\ Z} : Y \to Z$*, defined by the conditions* $\phi_{W,\ Z}(w) = \phi_{X,\ Z}(x)$*, for any* $x \in w$ *and* $w \in W$*, and* $\phi_{Y,\ Z} = \phi_{W,\ Z} \circ ɉ^{-1}$*, are both continuous.*
Proof The continuity of $\phi_{W,\ Z}$ follows by observing that[6] $\phi_{X,\ Z} = \phi_{W,\ Z} \circ \pi_{\mathbf{X}\,/\,\equiv\,(\,\mathrm{mod}\,\,ɨ\,)}$, and the continuity of $\phi_{Y,\ Z}$ follows by observing that ɉ is a homeomorphism from W onto Y.
▫

IV. Discussion

Walter Rudin gave an example of diagonal sampling to show the sequential compactness of sequences with component values from a compact set[5]. In this regards, we have exhibited a metric on the countable product of nonempty compact metric spaces. The metric topology and the product topology are shown to be equivalent. The example of a bounded metric is taken from an exercise given in the book by Serge Lang[3]. The quotient topology is taken from the book by Wilson A. Sutherland[6].

V. Conclusion

Compactness of the product of a countably infinitely many nonempty compact metric spaces can be proved within Cantor's set theory, without using the axiom of choice and Tychonoff's theorem.

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