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Mathematical Modeling of the dynamics of Prey-Predator with Scavenger in a closed habitat.

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Abstract
In this study we develop a mathematical model which describes the dynamics of prey- predator interaction with scavenger. We develop the model based on Holling type II functional response. We solved the equilibrium points and their existence. The positivity and of the solution of the model are also determined. Conditions for local and global stability analysis are studied both analytically and numerically. The study also addresses the effect of extinction of a population and mechanism that three species coexist. As a result the mechanism that three species become coexist if there is large number of prey population compute with small number of predator and average number of scavenger population. The scavenger species also has a great role in stabilizing as well as for coexistence of three species. Numerical simulations are carried out to illustrate the analytical findings. Finally the biological implication of analytical and numerical are discussed critically

Key words: *Prey-Predator, Scavenger, Lyapunov function, Local stability, Global stability*

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I. Introduction

Mathematical models are often used to examine the dynamics of complex interacting populations. In mathematical ecology a widely known and applied model is the classical Lotka-Volterra model. This model was formulated in 1926 by Vito Volterra and around the same time Alfred Lotka independently studied similar equations. The Lotka-Volterra model, also known as predator-prey model, is a cornerstone in the field of mathematical ecology and a significant amount of literature devoted to studying variants of these equations has been established. It consists of two coupled nonlinear differential equations and illustrates the interactions of one predator and one prey populations. These nonlinear differential equations are:

$$\begin{cases} \frac{dx}{dt} = x(a - by) \\ \frac{dy}{dt} = y(-c + dx) \end{cases} \text{------------------ (1)}$$

Where y(t) and x(t) represent, respectively, the predator and prey population as functions. of time. The parameters $a, b, c, d > 0$, are interpreted as follows:

- a represents the natural growth rate of the prey in the absence of predators,
- b represents the effect of predation on the prey,
- c represents the death rate of the predator due to lack of food,
- d represents the efficiency and propagation rate of the predator in the presence of prey

Without loss of generality, taking parameters $a = 1, b = 1$ and $d = 1$, with fixed point (c, 1) all trajectories in positive space are closed curves which are given by the family of equations, $\ln(y) - y + cln(x) = c$. Here the predator population will grow according to the amount of prey. If the prey population is large, the predators will have more food to support a larger population. However, when the predator population grows too large, the prey begins to die out. This will result in a decrease in the predators. This trend continues as time goes on, implying a stable coexistence of the two populations.

The modified two dimensional Lotka-Volterra predator-prey model also uses a nonlinear of equations that includes logistic growth of two species, a carrying capacity of the prey, and a predatory factor. The modified Lotka-Volterra predator-prey model is given by

$$\begin{cases} \frac{dx}{dt} = x(1 - bx - y) \\ \frac{dy}{dt} = y(-c + x) \end{cases} \text{----------------- (2)}$$

Where b is the carrying capacity of the prey and c is the death rate of the predator.
In the modified model, we find that the populations start as a pair of oscillating curves, but over time the amplitude of the oscillations decrease with each period of time until the curves atten out. This implies that the populations experienced a stable coexistence that saturates after sometime such that the populations will remain constant.
The Lotka-Volterra model indeed may be the simplest possible predator-prey model. It has been criticized as being unrealistic mainly for its structural instability and the assumption of the unlimited growth of the prey population x(t) in the absence of a predator Nevertheless, it is a useful tool containing the basic properties of the real predator-prey dynamics, and serves as a robust basis from which it is possible to develop more sophisticated and realistic models.
The model (2) can be naturally generalized for the multi-species case. The generalization of the Lotka-Volterra model (1), for the multi-species case retains the basic features of real ecological systems and, allows us to obtain valuable results that are easy to be interpreted. For the three-species predator-prey interaction two possibilities arise the two prey-one predator systems

$$\begin{cases} \frac{dx_1}{dt} = x_1(a_1 - b_1 y) \\ \frac{dx_2}{dt} = x_2(a_1 - b_2 y) \\ \frac{dy}{dt} = y(-c + d_1 x_1 + d_2 x_2) \end{cases} \text{---------------} (3)$$

Here x_1, x_2 are prey population and y(t) is predator population. The interaction of kiwi-rabbits-stoats is an example. The populations of both species are far from reaching their media capacity as a result of predation and artificial controlling of the rabbit population. Furthermore, both the kiwi and the rabbits have the same predators: stoats, cats, minks, or any others. Hence, the kiwi-rabbit-stoat interaction can be adequately described by the equations (3).

And two predator-one prey systems,

$$\begin{cases} \frac{dx}{dt} = x(a_1 - b_1 y_1 - b_2 y_2) \\ \frac{dy_1}{dt} = y_1(-c_1 + d_1 x) \\ \frac{dy_2}{dt} = y_2(-c_2 + d_1 x_1 + d_2 x) \end{cases} \text{---------------} (4)$$

Where as y_1, y_2 are predator population and x (t) is prey population. For example, lion-tiger-antelope, describe the situation when two predator species depend on a common prey and furthermore, these two predator species do not interact directly, they do not get directly, do not predate one the other, and do not depend one on the other as a food source. Such situations are not unusual, and can arise in many different cases.

We consider a three species, the prey, predator and scavenger where the scavenger is a predator of the prey and scavenges the carcasses of the predator. There is the case where the scavenger has no negative effects on the population that it scavenges. Possible A triple of such species are hyena/lion/antelope, where the hyena scavenges lion carcasses and preys upon antelope. But, in our study the scavenger affect the population it scavenges and also eaten by predator. An example of such a triple can be a lion, zebra and hyena, where the lion is considered as predator, zebra the prey and the hyena represents scavenger. To understand more let us see the diagram below.

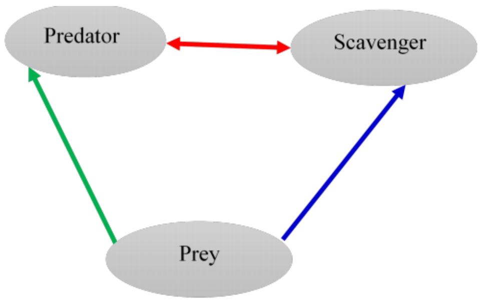

Figure 1: Schematic diagram for the dynamics of the prey, predator and scavenger in an ecosystem

As we observe from the above diagram both species predator and scavenger use a variety of different resource from two tropic level, so we called as generalist Interaction among these three species can be observed by learning the population growth investigate these growth we develop the mathematical model which represent the dynamics of the prey, predator and scavenger. In the model we use functional response Holling type II to make the system more real. Then we determine equilibrium points, positivity and boundedness of the solution and also both local as well as global stability of the system.

II. Model Formulation

We consider the following predator–prey model with scavenger interaction.

a) Assumptions and Parameters of the Model

The following assumptions are made in order to construct the model:

- ❖ The prey will grow logistically in the absence of predators and scavenger population. The logistic growth model is illustrated by the term $AX(1-\frac{B}{A}X)$
- ❖ the rate of predation upon the prey is proportional to the encounters of predators and prey or the effect on prey population due to interaction with some of predator populations. This assumption is represented by the term $\frac{BXY}{A_1+X}$
- ❖ the rate of predation upon the prey is proportional to the encounters of scavenger and prey or interaction between prey and scavenger populations. This assumption is represented by the term $\frac{DXY}{A_1+X}$
- ❖ Predators will die out exponentially in the absence of prey and scavengers or natural death rates of predator populations. This assumption is represented by the term EY.
- ❖ the predator population increases, due to predation upon prey or benefit of predator population from prey population. The terms $\frac{FXY}{A_1+X}$ is represents this assumption.
- ❖ the predator population increases, due to predation upon scavenger that is there is a predator population which eat scavenger population. The terms $\frac{GXY}{A_1+X}$ is represents this assumption.
- ❖ without predators and prey the scavengers will also almost goes to extinct. The terms that represent this assumption are HZ and HZ^2
- ❖ the term HZ represent natural death rate of the scavenger and HZ^2ensures that the interaction within scavenger species itself for the same resource.
- ❖ The scavenger population benefits from prey and predator that die naturally. The terms representing this assumptions are $\frac{JXY}{A_1+X}$ and $\frac{KXY}{A_1+X}$ respectively.
- ❖ predators prey and scavengers will come across each other randomly in the environment.
- ❖ The populations live in closed environment

Parameters of the model

Table 1: parameters of the model

Parameters	interpretation
A	natural growth rate of X
A_1	half saturation constants for Y and Z
A_2	half saturation constants for Z
A_3	half saturation constants for Y
B	interaction between prey population itself
C	effect on x due to predation of Y
D	rate changes on the X population in due to presence of S
E	natural death rate of Y
F	benefit to Y from X
G	benefit` to Y from Z
H	natural death rate of Z
I	interaction between S population itself
J	benefit to Z from X
K	benefit to Z from X

Description of Model

Three species predator-prey models, where the third species are scavenger are considered in this paper. This model is constructed by assuming that there are only three species in The first species, called prey, acting as the prey for the second and the third species. The second species called predator, feeds on the first and third species and can extinct with the absence of them. The third species, namely scavenger eat prey and the carcasses of predator. Consequently, the scavenger predations reduces the prey population and also affect the predator growth indirectly. Based on these assumptions, the mathematical model representing those three population dynamics is governed by non linear system ordinary differential equation.

$$\begin{cases}\frac{dX}{dt}=AX-BX^2-\frac{CXY}{A_1+X}-\frac{DXY}{A_1+X}\\ \frac{dY}{dt}=\frac{FXY}{A_1+X}+\frac{GXY}{A_1+Z}-EY\\ \frac{dZ}{dt}=\frac{JXY}{A_1+X}-\frac{KXY}{A_1+Y}-HZ-HZ^2\end{cases} \quad \text{---------------------- (6)}$$

Where X(t), Y(t), Z(t) are the prey, the predator and scavenger populations respectively. All the parameters in the equation (6) are positive.

Dimensionless form of the model

A model can be transformed into a dimensionless form. That is rewriting the system in terms of dimensionless quantities. One of the advantages of a system in the dimensionless form is that the number of parameters is reduced to a minimum and it makes the analysis easier. Furthermore parameters can be better compared with each other, in terms of small and large and thus one gets more insight into the system. It is also possible to make a comparison between different systems. The dynamics of the new model will be the same as in the original system. The equation (6) can be rewritten as:

$$\begin{cases}\frac{dX}{dt}=AX(1-\frac{B}{A}X)-\frac{CXY}{A_1(1+\frac{X}{A_1})}-\frac{DXY}{A_1(1+\frac{X}{A_1})}\\ \frac{dY}{dt}=-EY+\frac{FXY}{A_1(1+\frac{X}{A_1})}+\frac{GXY}{A_1(1+\frac{Z}{A_1})}\\ \frac{dZ}{dt}=-HZ-HZ^2+\frac{GXY}{A_1(1+\frac{X}{A_1})}+\frac{KXY}{A_1(1+\frac{X}{A_1})}\end{cases} \quad \text{-------------------- (7)}$$

Letting, $\tau=At, x=\frac{X}{A_1}, y=\frac{Y}{A_{2,}}\ z=\frac{Z}{A_3}$ and substituting (7), dimensionless form of system becomes

$$\begin{cases}\frac{dx}{dt}=x(1-\alpha_1 x)-\frac{\beta_1 xy}{(1+x)}-\frac{\theta_1 xz}{(1+x)}=f(x,y,z)\\ \frac{dy}{dt}=\frac{\alpha_2 xy}{(1+x)}+\frac{\beta_2 xy}{(1+z)}-\theta_2 y=g(x,y,z)\\ \frac{dz}{dt}=\frac{\alpha_3 xz}{(1+x)}+\frac{\beta_3 xz}{(1+y)}-\theta_3 z-\theta_4 z^2=h(x,y,z)\end{cases} \quad \text{------------------------ (8)}$$

Where $\alpha_1=\frac{A_1\ B}{A}, \beta_1=\frac{A_2\ C}{AA_1}, \ \theta_1=\frac{A_3 D}{AA_1}, \alpha_2=\frac{F}{A}, \beta_2=\frac{G}{A}, \theta_2=\frac{E}{A}, \ \alpha_3=\frac{J}{A}, \ \beta_3=\frac{K}{A}, \ \theta_3=\frac{H}{A}, \ \theta_4=\frac{IA_3}{A}$ and τ is substituted for t for simplicity.

i) Positivity of solution

Theorem: All solutions of the system (8) are positive

Proof: the equation (8) can be written as

$$x(t)=x(0)\exp\left(\int_0^t\left(\left(1-\alpha_1 x(s)\right)-\frac{\beta_1 y(s)}{(1+x(s))}-\frac{\theta_1 y(s)}{(1+x(s))}\right)ds\right)$$

$$y(t)=y(0)\exp\left(\int_0^t\left(\frac{\alpha_2 x(s)}{(1+x(s))}+\frac{\beta_2 x(s)}{(1+z(s))}-\theta_2\right)ds\right)$$

$$z(t)=z(0)\exp\left(\int_0^t\left(\frac{\alpha_3 x(s)}{(1+x(s))}+\frac{\beta_3 x(s)}{(1+x(s))}-\theta_3-\theta_4 z(s)\right)ds\right)$$

For $x(0)>0, y(0)>0, z(0)>0$ all solutions remain within the first quadrant of the x-y-z plane starting from an interior point of it.

Hence, $\mathbb{R}_+^3=\{(x,y,z): x,y,z\geq 0\}$ is invariant set

ii) Boundedness of Solution

Theorem: All solution of (8) are uniformly bounded if the initial conditions $x(0), y(0), z(0)>0$.

Proof: from first equation (8)

$\frac{dx}{dt}=x(1-\alpha_1 x)-\frac{\beta_1 xy}{(1+x)}-\frac{\theta_1 xz}{(1+x)}\leq x(1-\alpha_1 x)\Rightarrow x(t)\leq\frac{1}{\frac{e^{-t}}{x_0}+\alpha_1}$ For all $t\geq 0$,which implies $x(t)<1$ for sufficiently large t.

Let x (t), y(t), z(t) be any positive solution of (8), with positive initial conditions and define that

$w=\alpha_3\,\alpha_2\,x(t)+\beta_1\,\alpha_3 y(t)+\theta_1\,\alpha_2\,z(t)$ Then $\frac{dw}{dt}=\alpha_3\,\alpha_2\frac{dx}{dt}+\beta_1\,\alpha_3\frac{dy}{dt}+\ \theta_1\,\alpha_2\frac{dz}{dt}$

Therefore $\frac{dw}{dt}=\alpha_3\,\alpha_2(x(1-\alpha_1 x))+\beta_1\,\alpha_3(\frac{\beta_2 z_0 y}{(1+z_0)}-\theta_2 y)+\ \theta_1\,\alpha_2(\frac{\beta_3 y_0 z}{(1+y_0)}-\theta_3 z-\theta_4 z^2)$

Now choosing, $\alpha_1=1, \ \beta_2=\frac{1+z_0}{z_0}, \beta_3=\frac{1+y_0}{y_0}$ then

$\frac{dw}{dt}\leq\alpha_3\,\alpha_2(x(1-x))+\beta_1\,\alpha_3 y+\ \theta_1\,\alpha_2 z$ Moreover, $\frac{dw}{dt}\leq\alpha_3\,\alpha_2-\gamma(x+y+z)$

Where $\gamma=\min\{\alpha_3\,\alpha_2, \ \beta_1\,\alpha_3, \theta_1\,\alpha_2\}$ this implies that $\frac{dw}{dt}+\gamma w\leq\alpha_3\,\alpha_2$ now using method first order ordinary differential equation and applying inequality, we get

$0 < w < \frac{\alpha_3\,\alpha_2(1-e^{-\gamma t})}{\gamma} + w(x_0, y_0, z_0)e^{-\gamma t}$ for $t \to \infty, w \to \frac{\alpha_3\,\alpha_2}{\gamma}$

Therefore there exist $\eta_1, \eta_2, \eta_3 > 0$ such that $\mathbb{R}^3_+ = \{(x, y, z): 0 \le x \le \eta_1, 0 \le y \le \eta_2, 0 \le z \le \eta_3\}$ hence, our system is bounded.

iii) Equilibrium points and their existence

The equilibriums points of the system (8) can be found that setting all the equation to zero and solving the system for x(t),y(t) and z(t). To find equilibriums points of the system, we solve the following system equations simultaneously

$$\begin{cases} \frac{dx}{dt} = x(1-\alpha_1 x) - \frac{\beta_1 xy}{(1+x)} - \frac{\theta_1 xz}{(1+x)} = f(x,y,z) = 0 \\ \frac{dy}{dt} = \frac{\alpha_2 xy}{(1+x)} + \frac{\beta_2 xy}{(1+z)} - \theta_2 y = g(x,y,z) = 0 \\ \frac{dz}{dt} = \frac{\alpha_3 xz}{(1+x)} + \frac{\beta_3 xz}{(1+y)} - \theta_3 z - \theta_4 z^2 = h(x,y,z) = 0 \end{cases} \quad \text{----------------------------- (9)}$$

That is equivalent to solving eight equilibriums points of system (9), namely,

$E_0(0,0,0)$, $E_1(x^*, 0,0), E_2(0, y^*, 0), E_3(0,0,z^*), E_4(x^*, y^*, 0), E_5(x^*, 0, z^*), E_6(0, y^*, z^*), E_7(x^*, y^*, z^*)$

With $\{E_0(0,0,0), E_1\left(\frac{1}{\alpha}, 0,0\right), E_2(0,0,0), E_3\left(0,0,\frac{-\theta_3}{\theta_4}\right), E_4\left(\frac{\theta_2}{\alpha_2-\theta_2}, \frac{\alpha_2((\alpha_2-\theta_2)-\alpha_1\theta_2}{\beta_1(\alpha_2-\theta_2)^2}, 0\right), E_5(x^*, 0, z^*),$

$$E_6\left(0, \frac{\theta(\beta-\theta)+\theta\theta}{(\beta_3-\theta_3)(\beta_2-\theta_2)-\theta_2\theta_4}, \frac{\theta_2}{\beta_2-\theta_2}\right), E_7(x^*, y^*, z^*)\}$$

Where $E_5(x^*, 0, z^*)$ *and* $E_7(x^*, y^*, z^*)$ determined using numerically.

Local stability analysis of equilibrium points

The local stability of each equilibrium points are studied using Jacobian matrix and finding Eigen value at each equilibrium points. We continue analysis of Jacobian matrix J of three dimensional systems by

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{bmatrix}, \quad J(x,y,z) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{bmatrix}$$

$$J(x,y,z) = \begin{bmatrix} 1-2\alpha x - \frac{y\beta_1}{(1+x)^2} - \frac{y\theta_1}{(1+x)^2} & -\frac{x\beta_1}{1+x} & -\frac{x\theta_1}{1+x} \\ \frac{y\alpha_2}{(1+x)^2} & -\frac{x\alpha_2}{1+x} - \frac{z\beta_2}{1+z} - \theta_2 & \frac{z\beta_3}{(1+z)^2} \\ \frac{z\alpha_3}{(1+x)^2} & \frac{z\beta_3}{(1+y)^2} & \frac{x\alpha_3}{1+x} - \frac{y\beta_3}{1+y} - 2\theta_4 z - \theta_3 \end{bmatrix} \text{--------(10)}$$

i) Local stability of $E_0(0,0,0)$

The Jacobian of matrix at $E_0(0,0,0)$ of (10) is

$J(0,0,0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\theta_2 & 0 \\ 0 & 0 & -\theta_3 \end{bmatrix}$, since $J(0,0,0)$ is diagonal matrix, with eigenvalues: $1, -\theta_2, and\ -\theta_3$. this shows that our system is unstable at (0,0,0) because the sign of eigen values are different.

ii) Local stability at $E_1\left(\frac{1}{\alpha}, 0, 0\right)$

$J\left(\frac{1}{\alpha}, 0,0\right) = \begin{bmatrix} -1 & -\frac{\beta_1}{1+\alpha_1} & -\frac{\theta_1}{1+\alpha_1} \\ 0 & \frac{\alpha_2}{1+\alpha_1} - \theta_2 & 0 \\ 0 & 0 & \frac{\alpha_3}{1+\alpha_1} - \theta_3 \end{bmatrix}$, since $J\left(\frac{1}{\alpha}, 0,0\right)$ is upper triangular matrix, the eigen values are: -1, $\frac{\alpha_2}{1+\alpha_1} - \theta_2\ and\ \frac{\alpha_3}{1+\alpha_1} - \theta_3$. Now the system to be stable at $E_1\left(\frac{1}{\alpha}, 0,0\right)$, the two conditions must satisfy:

a) $\frac{\alpha_2}{1+\alpha_1} < \theta_2$,

b) $\frac{\alpha_3}{1+\alpha_1} < \theta_3$.

If these two conditions holds true then our system is local asymptotically stable at $E_1\left(\frac{1}{\alpha}, 0,0\right)$.

iii) Local stability at $E_2(0,0,0)$

Which is the same as i) above?

iv) Local stability at $E_3\left(0,0,\frac{-\theta_3}{\theta_4}\right)$?

v) Local stability at $E_4\left(\frac{\theta_2}{\alpha_2-\theta_2}, \frac{\alpha_2((\alpha_2-\theta_2)-\alpha_1\theta_2}{\beta_1(\alpha_2-\theta_2)^2}, 0\right)$

$$J(E_4)=\begin{bmatrix}\frac{\alpha_2\,\theta_2-\alpha_1\,\alpha_2\theta_2-\alpha_1{\theta_2}^2-{\theta_2}^2}{\alpha_2\,(\alpha_2-\theta_2)} & -\frac{\beta_1\theta_2}{\alpha_2} & -\frac{\theta_1\theta_2}{\alpha_2}\\ \frac{\alpha_2-\theta_2-\beta_1\theta_2}{\beta_1} & 0 & 0\\ 0 & 0 & \frac{\alpha_2\,\beta_3(\alpha_2-\theta_2)-\alpha_1\,\alpha_2\theta_2\beta_3-\alpha_1{\theta_2}^2-{\theta_2}^2}{\beta_1(\alpha_2-\theta_2)^2+\alpha_2\,(\alpha_2-\theta_2)-\alpha_1\,\alpha_2\theta_2}+\frac{\theta_2\alpha_3}{\alpha_2}-\theta_3\end{bmatrix}$$

The characteristic equation can be

$\lambda^3-(a_{11}+a_{33})\lambda^2+(a_{11}\,a_{33}-a_{12}a_{21})\lambda+a_{12}a_{21}a_{33}=0$ Where, $a_{11}=\frac{\alpha_2\,\theta_2-\alpha_1\,\alpha_2\theta_2-\alpha_1{\theta_2}^2-{\theta_2}^2}{\alpha_2\,(\alpha_2-\theta_2)}$, $a_{12}=-\frac{\beta_1\theta_2}{\alpha_2}$, $a_{13}=-\frac{\theta_1\theta_2}{\alpha_2}$, $a_{33}=\frac{\alpha_2\,\beta_3(\alpha_2-\theta_2)-\alpha_1\,\alpha_2\theta_2\beta_3-\alpha_1{\theta_2}^2-{\theta_2}^2}{\beta_1(\alpha_2-\theta_2)^2+\alpha_2\,(\alpha_2-\theta_2)-\alpha_1\,\alpha_2\theta_2}+\frac{\theta_2\alpha_3}{\alpha_2}-\theta_3$

thus it is form of $a_3\lambda^3+a_2\lambda^2+a_1\lambda+a_0=0$. Since $a_3=1$ which is positive by Routh-Hurwitz criteria, $\lambda's$ negative if $a_2>0, a_0>0, a_2a_1-a_0>0$. Each of these conditions are considered next as follows:

(a). $a_2>0,\Rightarrow-(a_{11}+a_{33})>0$ this can be satisfied if $a_{11}<0\ or\ a_{33}<0$

i) if $a_{11}<0\ \Rightarrow\alpha_2\,\theta_2-\alpha_1\,\alpha_2\theta_2-\alpha_1{\theta_2}^2-{\theta_2}^2<0\Rightarrow\alpha_2\,\theta_2(1-\alpha_1)-{\theta_2}^2(\alpha_1-1)<0$ this will hold if $\alpha_1<1$. In terms of original parameters it represents that $F<A$. This implies that, the benefits of predator populations from interaction with prey population is less than natural growth rates of prey population in the absence of predator and scavenger populations.

ii) $a_{33}<0\ \Rightarrow\Big({\alpha_2}^2\beta_3(\alpha_2-1)+(2\beta_1\alpha_2\theta_2-\beta_1{\alpha_2}^2)(\theta_3-\theta_2\alpha_3)+\alpha_1\,\alpha_2\theta_2(\alpha_3-\theta_2)(\theta_3-\beta_2\alpha_2)+$
α2θ22β1θ2−β3+α2θ2α3α2−θ2+θ2θ3α2−β1θ2<0

Therefore, $a_{33}<0\ \ if\ \alpha_2<1,\theta_3<\theta_2\alpha_3\,,\alpha_3<\theta_2,\beta_1\theta_2<\beta_3,\alpha_2<\theta_2\ and\ \alpha_2<\beta_1\theta_2$

c) $a_0>0\Rightarrow a_{12}a_{21}a_{33}>0$ this is satisfied if, for $a_{33}<0\ and\ \ a_{21}>0$

Clearly, $a_{12}=-\frac{\beta_1\theta_2}{\alpha_2}$, this is negative.

d) $a_2a_1-a_0>0\Rightarrow\ a_{11}<0, a_{12}<0, a_{33}<0\ \ and\ \ a_{11}a_{33}-a_{12}a_{21}<0$ (∇)

Therefore, $E_4\left(\frac{\theta_2}{\alpha_2-\theta_2},\frac{\alpha_2((\alpha_2-\theta_2)-\alpha_1\theta_2}{\beta_1(\alpha_2-\theta_2)^2},0\right)$ is locally asymptotically stable if conditions satisfied.

vi) **Local stability at E_5**

$$J(E_5)=\begin{bmatrix}\frac{(1+x^*)^2-2\alpha_1\,x^*(1+x^*)^2-\theta_1z^*}{(1+x^*)^2} & -\frac{\beta_1x^*}{1+x^*} & -\frac{\theta_1x^*}{1+x^*}\\ 0 & \frac{\alpha_2\,x^*(1+z^*)+\beta_2z^*(1+x^*)-\theta_2(1+x^*)(1+z^*)}{(1+x^*)(1+z^*)} & \frac{\beta_3z^*}{(1+z^*)^2}\\ \frac{\alpha_3z^*}{(1+x^*)^2} & \beta_3z^* & \frac{\alpha_3z^*-2\theta_4z^*\,(1+x^*)-\theta_3(1+x^*)}{1+x^*}\end{bmatrix}$$

The Eigen values of $J(E_5)$ are obtained by solving

$$det\begin{bmatrix}a_{11}-\lambda & -a_{12} & -a_{13}\\ 0 & a_{22}-\lambda & -a_{23}\\ -a_{31} & -a_{32} & a_{33}-\lambda\end{bmatrix}=0$$

Where;$a_{11}=\frac{(1+x^*)^2-2\alpha_1\,x^*(1+x^*)^2-\theta_1z^*}{(1+x^*)^2}$, $a_{12}=\frac{\beta_1x^*}{1+x^*}$, $a_{13}=\frac{\theta_1x^*}{1+x^*}$ $a_{22}=\frac{\alpha_2\,x^*(1+z^*)+\beta_2z^*(1+x^*)-\theta_2(1+x^*)(1+z^*)}{(1+x^*)(1+z^*)}$, $a_{31}=\frac{\alpha_3z^*}{(1+x^*)^2}$, $a_{32}=\beta_3z^*$, $a_{33}=\frac{\alpha_3z^*-2\theta_4z^*\,(1+x^*)-\theta_3(1+x^*)}{1+x^*}$

The characteristic equation becomes

$\lambda^3-(a_{11}+a_{22})\lambda^2+(a_{11}\,a_{22}+a_{11}\,a_{33}+a_{22}a_{33}-a_{13}\,a_{31})\lambda+a_{12}a_{23}a_{31}+a_{22}a_{13}\,a_{31}-a_{11}a_{22}a_{33}=0$

This is form $a_3\,\lambda^3+a_2\,\lambda^2+a_1\,\lambda^1+a_0=0$

By Routh-Hurwitz criteria the λ's are negative if $a_2>0, a_0>0, a_2\,a_1-a_0>0$. And we considered each of these conditions as follows:

i) $a_2>0\Rightarrow-(a_{11}+a_{22}+a_{33})>0\ or\ a_{11}+a_{22}+a_{33}<0$

ii) $a_0>0\Rightarrow\ \ a_{12}a_{23}a_{31}+a_{22}a_{13}\ a_{31}>a_{11}a_{22}a_{33}$

iii) $a_2\,a_1-a_0>0\Rightarrow\ -(a_{11}+a_{22})(a_{11}\,a_{22}+a_{11}\,a_{33}+a_{22}a_{33}-a_{13}\ a_{31})+\ a_{11}a_{22}a_{33}-a_{12}a_{23}a_{31}-a_{22}a_{13}\ a_{31}>0$ ($\nabla\nabla$)

Therefore, E_5 is locally asymptotically stable if conditions (i), (ii) and (iii) holds true.

vii) **Local stability at E_7**

Similar to $\boldsymbol{E_5}$ after solving Jacobian matrix of system (10) at equilibrium point $E_7(x^*,y^*,z^*)$, the eigenvalues are found by solving

$$det\begin{bmatrix}a_{11}-\lambda & -a_{12} & -a_{13}\\ a_{21} & a_{22}-\lambda & -a_{23}\\ -a_{31} & -a_{32} & a_{33}-\lambda\end{bmatrix}=0$$

Where; $a_{11} = \frac{(1+x^*)^2 - 2\alpha_1 x^*(1+x^*)^2 - \theta_1 z^*}{(1+x^*)^2}, a_{12} = \frac{\beta_1 x^*}{1+x^*}, \quad a_{13} = \frac{\theta_1 x^*}{1+x^*}, a_{21} = \frac{\alpha_2 y^*}{(1+x^*)^2}, \quad a_{22} = \frac{\alpha_2 x^*(1+z^*) + \beta_2 z^*(1+x^*) - \theta_2(1+x^*)(1+z^*)}{(1+x^*)(1+z^*)}, a_{31} = \frac{\alpha_3 z^*}{(1+x^*)^2}, a_{32} = \beta_3 z^*, a_{33} = \frac{\alpha_3 z^* - 2\theta_4 z^*(1+x^*) - \theta_3(1+x^*)}{1+x^*}$

The characteristic equation becomes

$\lambda^3 - (a_{11} + a_{22})\lambda^2 + (a_{11} a_{22} + a_{11} a_{33} + a_{22} a_{33} - a_{13} a_{31})\lambda + a_{12} a_{23} a_{31} + a_{22} a_{13} a_{31} - a_{11} a_{22} a_{33} = 0$

This is form $a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda^1 + a_0 = 0$

By Routh-Hurwitz criteria the λ's are negative if $a_2 > 0, a_0 > 0, a_2 a_1 - a_0 > 0$. And we considered each of these conditions as follows:

i) $a_2 > 0 \Rightarrow -(a_{11} + a_{22} + a_{33}) > 0 \ \ or \ a_{11} + a_{22} + a_{33} < 0$

ii) $a_0 > 0 \Rightarrow \ a_{12} a_{23} a_{31} + a_{22} a_{13} a_{31} > a_{11} a_{22} a_{33}$

iii) $a_2 a_1 - a_0 > 0 \Rightarrow -(a_{11} + a_{22})(a_{11} a_{22} + a_{11} a_{33} + a_{22} a_{33} - a_{13} a_{31}) + a_{11} a_{22} a_{33} - a_{12} a_{23} a_{31} - a_{22} a_{13} a_{31} > 0$

Therefore, E_7 is locally asymptotically stable if conditions (i), (ii) and (iii) holds true.

III. Global Stability Analysis of the Equilibrium Points

Theorem: the equilibrium point $E_4(x^*, y^*, 0)$ is globally asymptotically stable.

Proof : Let us consider the following Liapunovs function

$$V(x,y,z) = x - x^* - x^* \ln {x}/{x^*} + y - y^* - y^* \ln {y}/{y^*} + \frac{z^2}{2}$$

Now the time derivative of V, along the solution of (8/9) can be written as

$$\frac{dV}{dt} = \frac{x - x^*}{x}\frac{dx}{dt} + \frac{y - y^*}{y}\frac{dy}{dt} + z\frac{dz}{dt}$$

$$= (x - x^*)\left(1 - \alpha x - \frac{\beta_1 y}{1+x}\right) + (y - y^*)\left(\frac{\alpha_2 x}{1+x} - \theta_2\right)$$

This is also simplified to

$\frac{dV}{dt} = -\alpha_1 (x - x^*)^2 - (\beta_1 - \alpha_2)\frac{(x-x^*)(y-y^*)}{(1+x)(1+x^*)} - \beta_1 \frac{(x-x^*)(xy - x^* y^*)}{(1+x)(1+x^*)}$ which is negative definite when $\beta_1 > \alpha_2$

Therefore, with these conditions, $E_4(x^*, y^*, 0)$ is globally asymptotically stable.

IV. Numerical Simulation

4.1 Dynamics of prey and predator populations

If the scavenger population dies out, the remaining model reverts back to the classical prey predator model. The only difference is functional response included between them.

The model which describes their dynamics are

$\frac{dx}{dt} = x(1 - \alpha_1 x) - \frac{\beta_1 xy}{(1+x)}$------------------------------- (11)

$\frac{dy}{dt} = \frac{\alpha_2 xy}{(1+x)} - \theta_2 y$

By using $\alpha_1 = 0.1, \alpha_2 = 0.25, \beta_1 = 0.3, \theta_2 = 0.2$ we can plot the above system of equation (11) as follows

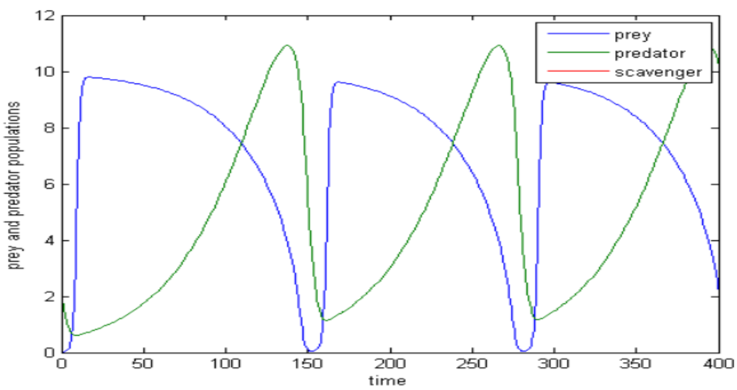

Figure1: Dynamics of prey and predator without scavenger populations

In the initial due to less number of prey species the predator species fails downward and prey species dominated for a low rate. The prey abundances are not strongly positively correlated then as one prey species

becomes scarce, the predator can continue to feed and increase its population size. As we observe from the above figure when the predator population decreases the prey population conversely increased and vice versa.
4.2 Dynamics of predator and scavenger populations
In the absence of prey population the system of differential equation which describes the dynamics between predator and scavenger populations is given by

$$\frac{dy}{dt} = \frac{\beta_2 yz}{(1+z)} - \theta_2 y - - - - - - - - -(12)$$
$$\frac{dz}{dt} = \frac{\beta_3 xz}{(1+y)} - \theta_3 z - \theta_4 z^2$$

One thing that differentiates this paper from other is the benefits of predator species from scavenger species are included. Now we see their dynamics without prey species by using

$$\beta_2 = 0.25 \quad \beta_3 = 0.6 \quad \theta_2 = 0.2 \quad \theta_3 = 0.3 \quad and\ \theta_4 = 0{,}1$$

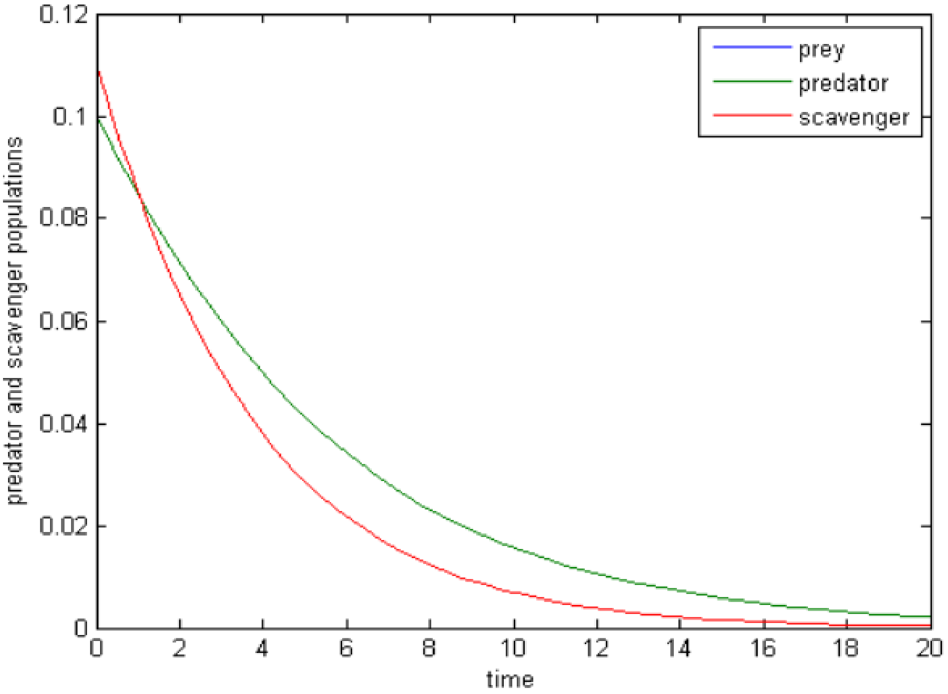

Figure 2:Dynamics of predator and scavenger with out prey

As we observe from the graph without prey population predator as well as scavenger population can exist only for a few time. The scavenger benefit more from interaction between prey and predator, but here the only its source is natural death rates of predator. The predator population benefits from the interaction with scavenger population but not enough for them. In the case of initial both species almost the same and the scavenger species dominates the predator species. But, after a low rates the predator species dominates the scavenger species always, even-though their population decreases from time to time. Furthermore after a long period of time both species almost goes to extinct. From this we conclude that predator and scavenger populations cannot live without prey population for long time.

4.3 Dynamics of Prey and Scavengers

Similarly, the non-linear differential equation system which shows the dynamics between prey and scavenger populations without predator population is represented as follows,

$$\frac{dx}{dt} = x(1-\alpha_1 x) - \frac{\theta_1 xz}{(1+x)}$$
$$\frac{dz}{dt} = \frac{\alpha_3 xz}{(1+x)} - \theta_3 z - \theta_4 z^2$$

Using parameters, $\alpha_1 = 0.1$, $\alpha_3 = 0.01$, $\theta_1 = 0.1$, $\theta_3 = 0.3$, *and* $\theta_4 = 0.1$ to show dynamics of two species:

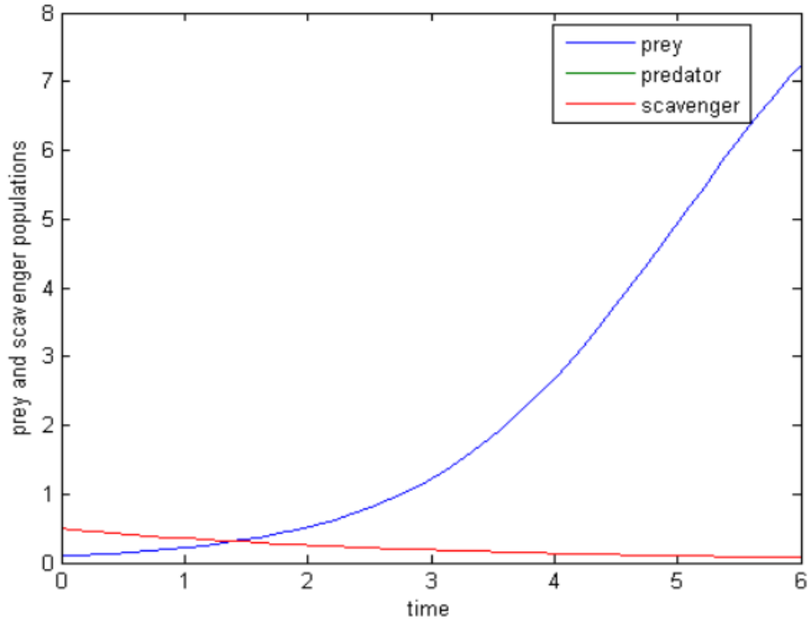

Figure 3:Dynamics of prey and scavenger without predator

4.5. Coexistence of three species

One of the most important question in mathematical biology is concerns the long term survival or coexistence of all the species in multi-species community.

$$\frac{dx}{dt} = x(1-\alpha_1 x) - \frac{\beta_1 xy}{(1+x)} - \frac{\theta_1 xz}{(1+x)}$$

$$\frac{dy}{dt} = \frac{\alpha_2 xy}{(1+x)} + \frac{\beta_2 xy}{(1+z)} - \theta_2 y$$

$$\frac{dz}{dt} = \frac{\alpha_3 xz}{(1+x)} + \frac{\beta_3 xz}{(1+y)} - \theta_3 z - \theta_4 z^2$$

So, to show coexistence of prey, predator and scavenger population we shall use parameters $\alpha_1 = 0.1,\ \alpha_2 = 0.25, \alpha_3 = 0.01,\ \beta_1 = 0.3\, \beta_2 = 0.35,\ \beta_3 = 0.003\,, \theta_1 = 0.04, \theta_2 = 0.21, \theta_3 = 0.035, \theta_4 = 0.1$ and initial point $(x_0, y_0\,, z_0) = (15.2,\ 4.6,\ 10.4)$

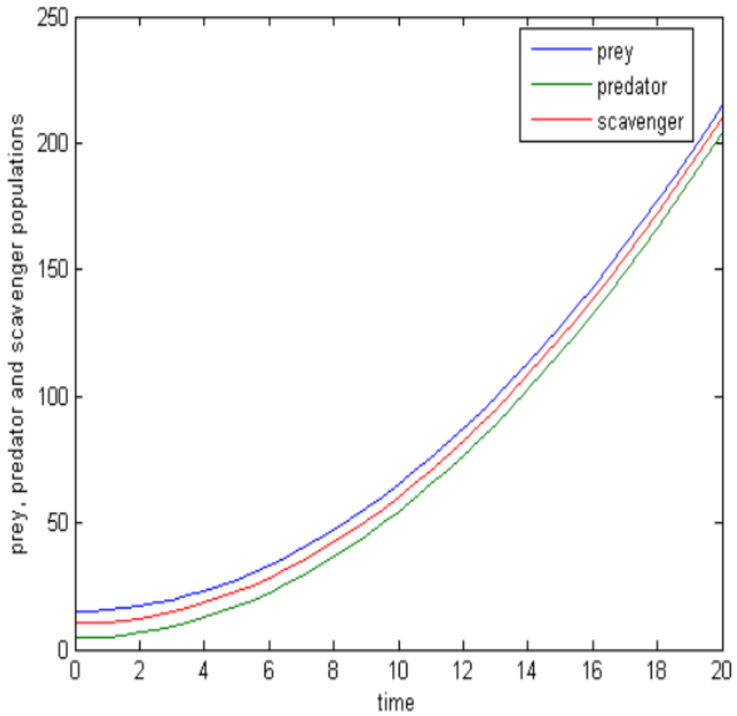

Figure 4: coexistence of three species

The mechanism that prey, predator and scavenger species coexist is if low predation occurs in an ecosystem. Furthermore, three species become coexist if large number of prey population computes with less number of predator and average number of scavenger populations. As their figure shows all three species increases constantly. Even-though, both predator and scavenger depends on prey species due to large number of prey species their number rise up constantly. Additionally, the main things that maintain their coexistence is both predator and scavenger species cannot get all of the prey species. The scavenger species benefits from the large number prey species as well as from their dead bodies results to increase constantly from time to time. The predator species also gets enough food from other two and their population flows the prey and scavenger species.

EQUILBRIUM POINTS

In previous section we solved four equilibrium point analysis. Now for these and left two were determine the equilibrium points numerically. We start by setting value for parameter $\alpha_1 = 0.1,\ \alpha_2 = 0.25, \alpha_3 = 0.01,\ \beta_1 = 0.3\ \beta_2 = 0.4,\ \beta_3 = 0.6\ , \theta_1 = 0.5, \theta_2 = 0.2, \theta_3 = 0.3, \theta_4 = 0.01.$

Then the equilibriums become,

i) Trivial equilibrium point(0,0,0)

ii) $E_1 = \left(\frac{1}{\alpha}, 0,0\right) = (10,0,0)$

iii) $E_2 = \left(\frac{\theta_2}{\alpha_2-\theta_2}, \frac{\alpha_2((\alpha_2-\theta_2)-\alpha_1\theta_2}{\beta_1(\alpha_2-\theta_2)^2}, 0\right) = (4,10,0)$

iv) $E_3 = \left(0, \frac{\theta(\beta-\theta)+\theta\theta}{(\beta_3-\theta_3)(\beta_2-\theta_2)-\theta_2\theta_4}, \frac{\theta_2}{\beta_2-\theta_2}\right) = (0,2,1)$

v) $E_4 = (x^*, 0, z^*)$

In previous section we made difficulty to solve this equilibrium point analytically. Now easily by numerical $E_4 = (x^*, 0, z^*) = (1.9854, 0, 7.2185)$

vi) Positivity of $E_5 = (x^*, y^*, z^*)$, similar to E_4 it is difficult to solve analytically, but by using MATLAB computer program

$$E_5 = (x^*, y^*, z^*) = (0.0321, 1.8874, 0,9251)$$

Equilibrium	Existence condition	Stability condition
E_o	Always exist	saddle
E_1	Always exist	If $\theta_2 < \frac{\alpha_2}{1+\alpha_1}$ and $\theta_3 < \frac{\alpha_3}{1+\alpha_1}$
E_2	If $\theta_2 < \alpha_2$ and $\alpha_1\theta_2 < \alpha_2 - \theta_2$	if(∇) holds
E_3	If $\theta_2 < \beta_2,\ \theta_3 < \beta_3$ and $\theta_2\,\theta_4 < (\beta_3 - \theta_3)(\beta_2 - \theta_2)$	if ($\nabla\nabla$)holds

V. Result and discussion

$p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\,\lambda + a_3 = 0$ be characteristic equation for matrix defined in (10). Then the following statements are true.

i) If every roots of the characteristic equation is less than one, then the equilibrium point of the system (9) is locally asymptotically stable and equilibrium is called sink

ii) If at least one of the root has absolute value greater than one, then the equilibrium point of system (9) is unstable and the equilibrium is called saddle.

iii) If every root of system has absolute value greater than one, then the system is unstable and the equilibrium is called source.

Using this lemma we check the stability analysis numerically at give equilibrium points;

Local stability of E_O

we shall use parameters $\alpha_1 = 0.1,\ \alpha_2 = 0.25, \alpha_3 = 0.01,\ \beta_1 = 0.3\ \beta_2 = 0.4,\ \beta_3 = 0.06\ , \theta_1 = 0.5, \theta_2 = 0.2, \theta_3 = 0.3, \theta_4 = 0.01.$ and $E_o(0.2, 0.1, 0.5)$ and substituting in the Jacobean matrix (10) this set of parameter values the Eigen value becomes $\lambda_1 = 1, \lambda_2 = -0.2, \lambda_3 = -0.3$. This implies the equilibrium $E_o(0.2, 0.1, 0.5)$ is unstable. Now by using above parameters we show their graph as follows.

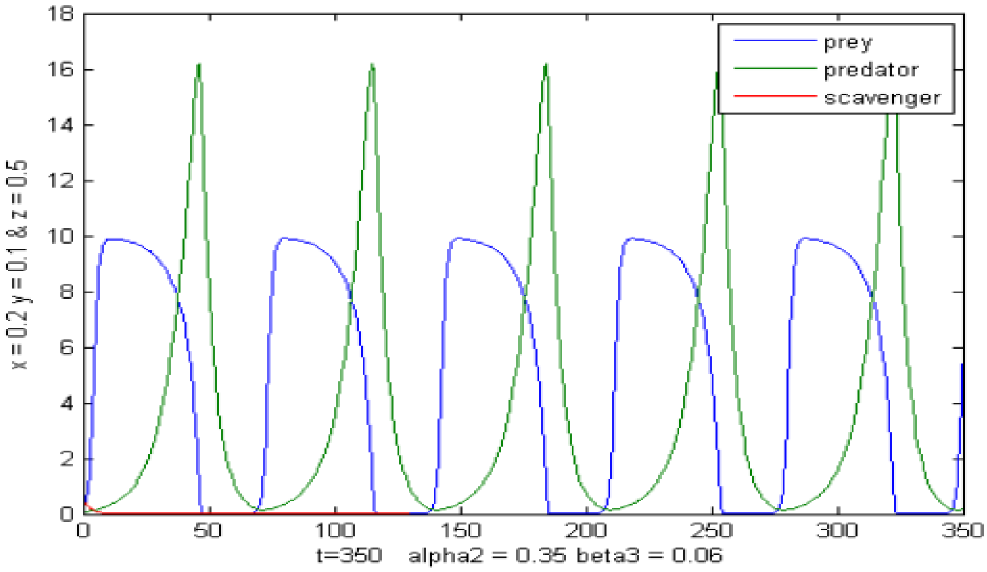

Figure 5:Time serious plot for unstable equilibrium point E_o

local stability of E_1

Local stability of E_1 we shall use $\beta_3 = 0.06$, $\theta_2 = 0.25, \theta_3 = 0.3$ and other as before we get eigen value $\lambda_1 = -1, \lambda_2 = -0.023, \lambda_3 = -0.29$ and $E_1(9.7,0.5,0.6)$. hence with this parameter E_1 is asymptotically stable and their graph is shown as

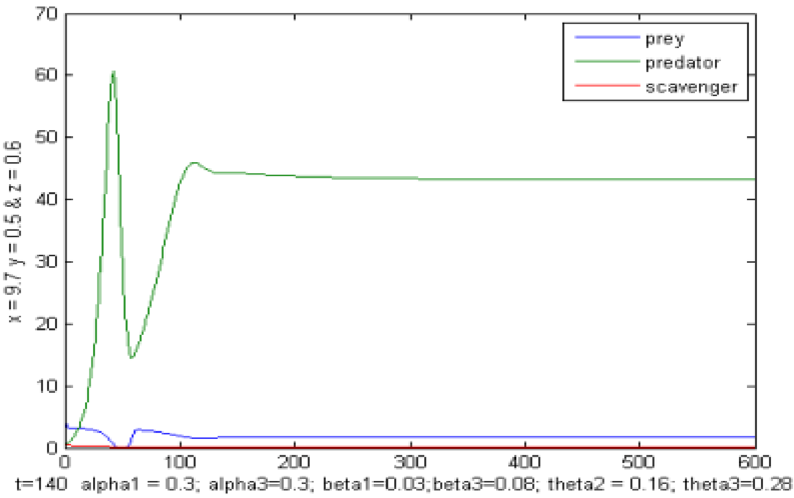

Figure 6:time serious plot for stable equilibrium point

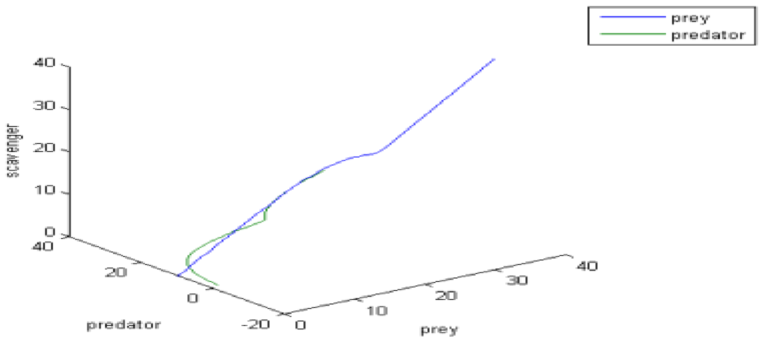

Figure 7:Phase portrait at equilibrium point E_1

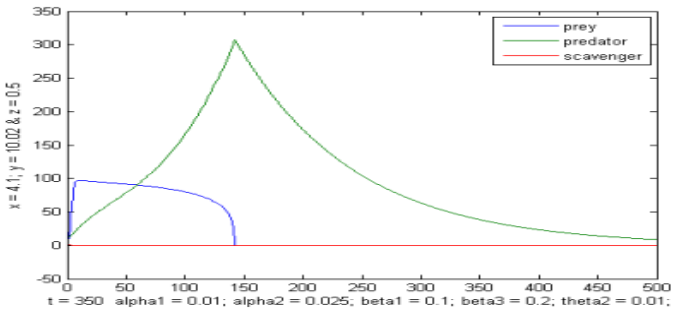

Figure 8: Time serious plot for stability at equilibrium point E_2

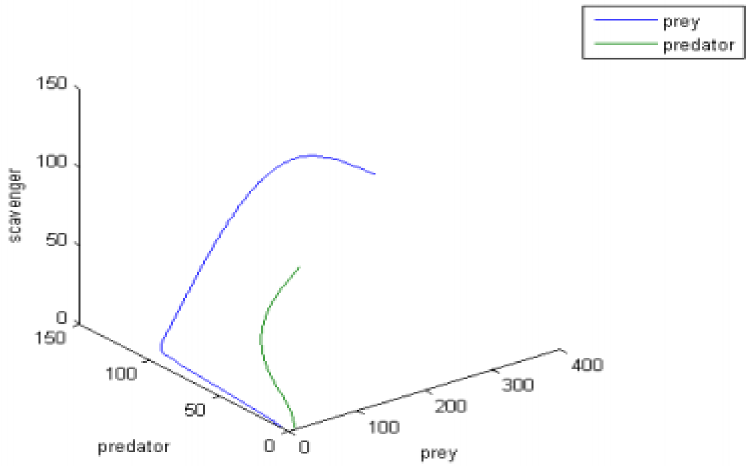

Figure 9:Phase portrait at equilibrium E_2

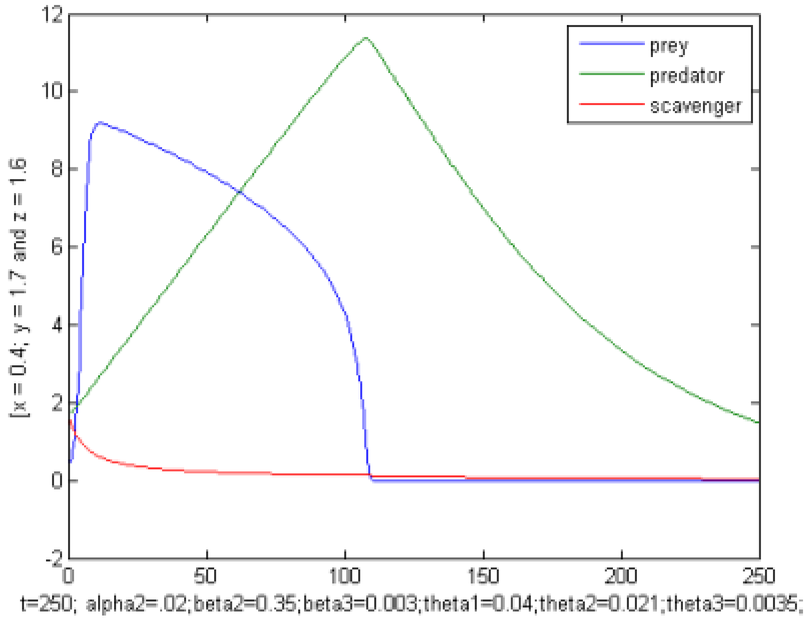

Figure 10:Time serious plot for stability equilibrium point E_3

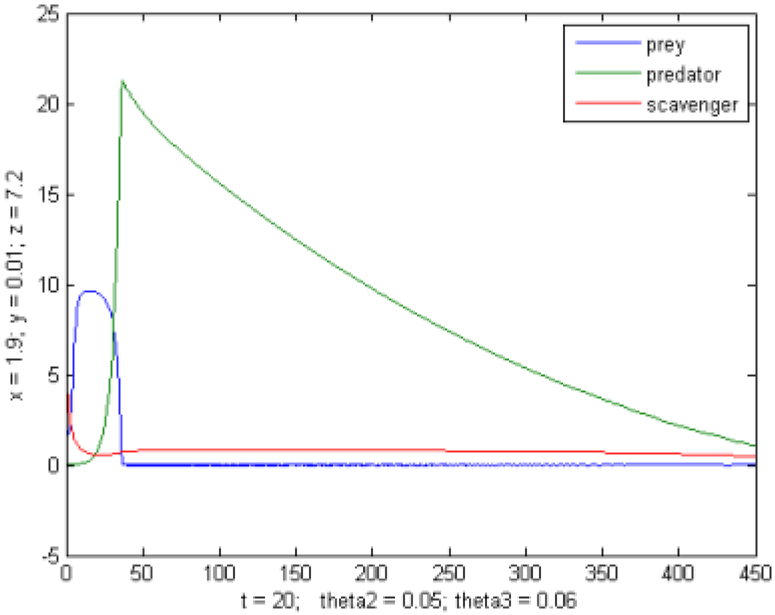

Figure 11:Time serious plot for stability equilibrium point E_4

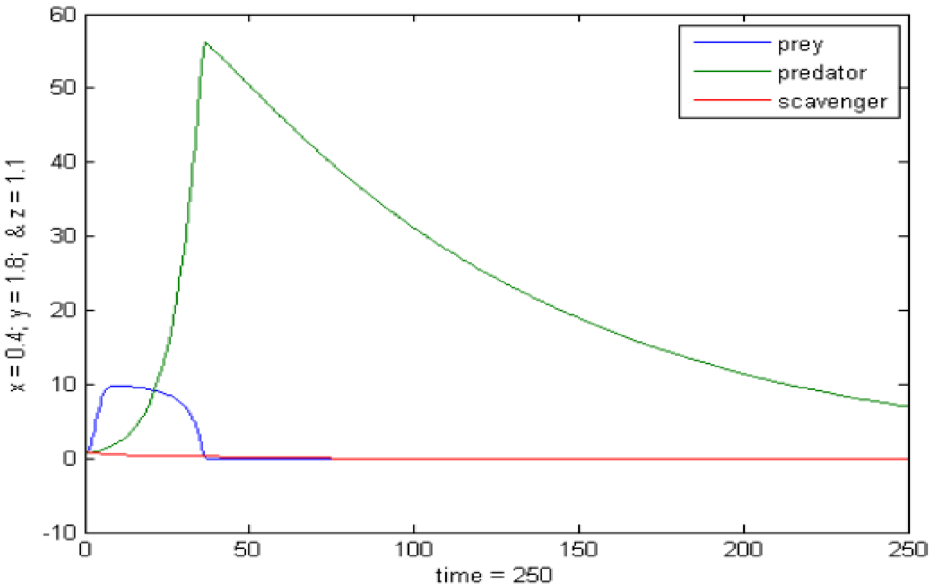

Figure 12:Time serious plot for stability equilibrium point E_5

VI. Conclusion

Our Lotka Volterra predator prey model with a scavenger demonstrates the possible population trends when a predator, a prey and a scavenger population interact. We have found that the predator and the prey can coexist in the absence of the scavenger, and the scavenger and the prey can coexist in the absence of the predator. However, the scavenger and the predator cannot coexist without the prey. Biologically this is reasonable, because without the prey, the predator will have no food and will die of. The scavenger will then lose all sources of food and will too die out. We have also found that the three populations can coexist in two ways: they will oscillate between stable populations over time, or the populations will oscillate until they saturate and remain constant over time.

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