IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 17, Issue 4 Ser. II (Jul. – Aug. 2021), PP 14-28
www.iosrjournals.org

The Differential Geometry of Curves

Kande Dickson Kinyua[1], Kuria Joseph Gikonyo[1, 2]
[1]*Department of Mathematics, Physics and Computing, Moi University, Eldoret, Kenya*
[2]*Department of Mathematics, Karatina University, Karatina, Kenya*

Abstract: *Differential geometry of curves studies the properties of curves and higher-dimensional curved spaces using tools from calculus and linear algebra. This study has two aspects: the classical differential geometry which started with the beginnings of calculus and the global differential geometry which is the study of the influence of the local properties on the behavior of the entire curve. The local properties involves the properties which depend only on the behavior of the curve in the neighborhood of a point. The methods which have shown themselves to be adequate in the study of such properties are the methods of differential calculus. Due to this, the curves considered in differential geometry will be defined by functions which can be differentiated a certain number of times. The other aspect is the so-called global differential geometry which study the influence of the local properties on the behavior of the entire curve or surface. This paper aims to give an advanced introduction to the theory of curves, and those that are curved in general.*

Keywords: *Arc-length, Curvature, Curves, Differential Geometry, Parametrized, Planes, Torsion*

Date of Submission: 24-07-2021 Date of Acceptance: 09-08-2021

I. Introduction

Differential geometry of curves involves the study of the properties of curves and higher-dimensional curved spaces using tools from calculus and linear algebra, [1, 2]. This paper aimed to study two aspects of curves: the classical differential geometry which started with the beginnings of calculus and the global differential geometry which studies the influence of the local properties on the behavior of the entire curve. This first part presents local properties of plane curves, where by a local property one means properties that are defined in a neighborhood of a point on the curve. The second part analyzed global properties which concern attributes about the curve taken as a whole, [1-4].

II. Plane Curves: Local Properties

Local properties of a curve are related to a point on the curve based on information contained just in a neighborhood of that point. These local properties involve derivatives of the parametric equations. For comparison with calculus, the derivative $f'(a)$ of a function f at a point a is a local property of the function as only the knowledge of $f(t)$ for $x \in (a-\varepsilon,\ a+\varepsilon)$ is needed to define $f'(a)$, [1, 4, 5] .

2.1 Parametrizations

Given any plane in R^2 , there exists several curves at specified time *t* corresponding to coordinates *x* and *y* of the position of a point traveling along the curve; which can be represented by two functions $x(t)$ and $y(t)$. By use of vector notation to locate a point on the curve, we write $\vec{r}(t) = (x(t),\ y(t))$ for the above pair of coordinates and call $\vec{r}(t)$ a vector function into R^2 . In this study, t does not have to refer to time, it is simply called the parameter of the vector function, [1, 4-6].

Definition 2.1. Let $\vec{u}$ be a vector in R^n with coordinates $\vec{u} = (u_1, u_2, \ldots\ldots, u_n)$ in the standard basis. The (Euclidean) length of $\vec{u}$ is given by

$$\left\|\vec{u}\right\| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{u_1^2 + u_2^2 + \ldots\ldots + u_n^2} \qquad 1$$

If p and q are two points in R^n with coordinates given by vectors $\vec{u}$ and $\vec{v}$, then the Euclidean distance between p and q is $\left\|\vec{v}-\vec{u}\right\|$, [1, 4, 7].

Definition 2.2. Let $\vec{r}(t)$ be a vector function from a subset of R into R^n. We say that the limit of $\vec{r}(t)$ as t approaches a is a vector $\vec{w}$, which is written, $\lim_{t \to a} \vec{r}(t) = \vec{w}$, if for all ε > 0 there exists a δ > 0 such that $0 < |t - a| < \delta$ implies $\left\| \vec{r}(t) - \vec{w} \right\| < \varepsilon$, [1, 4, 7].

Definition 2.3. Let I be an open interval of R, let $a \in I$, and let $\vec{r}(t) : I \to R^2$ be a vector function. We say that $\vec{r}(t)$ is continuous at a if the limit as t approaches a of $\vec{r}(t)$ exists; which is written as, [1, 4, 7],

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a) \tag{2}$$

Proposition 2.1. Let $\vec{r}$ be a vector function from a subset of R into R^n that is defined over an interval containing a, though perhaps not at a itself. Suppose in coordinates we have $\vec{r}(t) = (x(t), y(t))$ wherever $\vec{r}$ is defined. If $\vec{w}(t) = (w_1, w_2)$, then $\lim_{t \to a} \vec{r}(t) = \vec{w}$ if and only if $\lim_{t \to a} \vec{r}(t) = w_1$ and $\lim_{t \to a} \vec{r}(t) = w_2$, [1, 4, 7].

Corollary 2.1. Let I be an open interval of R, let $a \in I$, and consider a vector function $\vec{r}(t) : I \to R^2$ with $\vec{r}(t) = (x(t), y(t))$. Then $\vec{r}(t)$ is continuous at $t = a$ if and only if $x(t)$ and $y(t)$ are continuous at $t = a$.

Definition 2.3 and Corollary 2.1 provide a mathematical base for introduction to a curve in physical concept, [1, 4, 7].

Definition 2.4. Let I be an interval of R. A parametrized curve in the plane is a continuous function $\vec{r}(t) : I \to R^2$. If we write $\vec{r}(t) = (x(t), y(t))$, then the functions $x : I \to R$ and $y : I \to R$ are called the coordinate functions or parametric equations of the parametrized curve. We call the locus of $\vec{r}(t)$ the image of $\vec{r}(t)$ as a subset of R^2, [1, 4, 7].

Definition 2.5. Given a parametrized curve $\vec{r}(t) : I \to R^2$ and any continuous functions $f(t)$ from an interval J onto the interval I, we can produce a new vector function $\vec{\alpha}(t) : J \to R^2$ defined by $\vec{\alpha}(t) = \vec{r}(t) \circ \vec{f}(t)$. The image of $\vec{\alpha}(t)$ is again the set *C*, and $\vec{\alpha}(t) = \vec{r}(t) \circ \vec{f}(t)$ is called a reparametrization of $\vec{r}(t)$. If $f(t)$ is not onto I, then the image of $\vec{\alpha}(t)$ may be a proper subset of C. In this case, we usually do not call $\vec{\alpha}(t)$ a reparametrization as it does not trace out the same locus of $\vec{r}(t)$, [1, 4, 7].

2.2 Position, Velocity, and Acceleration

The vector function $\vec{r}(t) = (x(t), y(t))$ is interpreted as the location along a curve at time t in reference to some fixed frame. The frame refer to a basis attached to a fixed origin. The point $O = (0, 0)$ along with the basis $\{\vec{i}, \vec{j}\}$, where $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$, form the standard reference frame. This vector function $\vec{r}(t)$ is called the position vector which can be written as, [1, 4, 6, 7],

$$\vec{r}(t) = (x(t), y(t)) = x(t)\vec{i} + y(t)\vec{j} \tag{3}$$

Using Newton's approach from the points $\vec{r}(t_0)$ and $\vec{r}(t_1)$ on the curve, the direction of motion is described by the vector $\vec{r}(t_1) - \vec{r}(t_0)$, while the rate of change in position is

$$\frac{\left(\vec{r}(t_1) - \vec{r}(t_0)\right)}{t_1 - t_0}. \tag{4}$$

Further, the instantaneous rate of change of the curve at the point t_0, is calculated by the following limit:

$$\vec{r}\,'(t_0) = \lim_{h \to 0} \left(\frac{\vec{r}(t_0 + h) - \vec{r}(t_0)}{h} \right). \qquad 5$$

The limit in Equation (5) exists if and only if $x(t)$ and $y(t)$ are both differentiable at t_0, and if this limit exists, then,

$$\vec{r}(t) = \left(x'(t), y'(t) \right) \qquad 6$$

Definition 2.6. Let $\vec{r}(t) : I \to R^2$ be a vector function where $\vec{r}(t) = \left(x(t), y(t) \right)$. Then, $\vec{r}(t)$ is differentiable at $t = t_0$ if $x(t)$ and $y(t)$ are both differentiable at t_0. If *J* is the common domain to $x'(t)$ and $y'(t)$, then we define the derivative of the vector function $\vec{r}(t)$ as the new vector function $\vec{r}\,'(t) : J \to R^2$ defined by

$$\vec{r}(t) = \left(x'(t), y'(t) \right). \qquad 7$$

If $x(t)$ and $y(t)$ are both differentiable functions on their domain, the derivative vector function $\vec{r}(t) = \left(x'(t), y'(t) \right)$ is called the velocity vector. Mathematically, this is just another vector function that trace out another curve when placed in the standard reference frame. However, the second derivative of $\vec{r}(t)$, which is called the acceleration vector is, [1, 4, 8];

$$\vec{r}\,''(t) = \left(x''(t), y''(t) \right) \qquad 8$$

Definition 2.7. Let I be an interval of R, and let $\vec{r}(t) : I \to R^2$ be a vector function, then, $\vec{r}$ is said to be of class C^r on *I*, or write $\vec{r} \in C^r(I)$, if the rth derivative of $\vec{r}$ exists and is continuous on I. We denote by C^∞ the class of functions that have derivatives of all orders on I. In addition, we also denote by C^ω the subclass of C^∞ functions that are real analytic on *I*, i.e., functions $\vec{r}$ such that, at each point $a \in I$, $\vec{r}$ is equal to its Taylor series centered at a over some open interval $(a - \varepsilon,\ a + \varepsilon)$, [1, 4, 6, 7].

Proposition 2.2. Let $\vec{u}(t)$ and $\vec{v}(t)$ be vector functions defined and differentiable over an interval $I \subset R$. Then the following hold, if, [1, 4, 8];

(a) $\vec{r}(t) = c\vec{u}(t)$, where $c \subset R$, then $\vec{r}\,'(t) = c\vec{u}\,'(t)$

(b) $\vec{r}(t) = c(t)\vec{u}(t)$, where $c : I \to R$ is a real function, then $\vec{r}\,'(t) = c'(t)\vec{u}(t) + c(t)\vec{u}\,'(t)$

(c) $\vec{r}(t) = \vec{u}(t) + \vec{v}(t)$, then $\vec{r}\,'(t) = \vec{u}(t) + \vec{v}\,'(t)$

(d) $\vec{r}(t) = \vec{u}\left(f(t) \right)$ is a vector function and $f : J \to I$ is a real function into *I*, then $\vec{r}\,'(t) = f'(t)\vec{u}\,'\left(f(t) \right)$.

(e) $f(t) = \vec{u}(t) \cdot \vec{v}(t)$ is the dot product between $\vec{u}(t)$ and $\vec{v}(t)$, then $f'(t) = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$.

Let C be the arc traced out by a vector function $\vec{r}(t)$ for t ranging between a and b. Then one approximates the length of the arc $l(C)$ with the Riemann sum

$$l(C) \approx \sum_{i=1}^{n} \left\| \vec{r}(t_i) - \vec{r}(t_{i-1}) \right\| \approx \sum_{i=1}^{n} \left\| \vec{r}(t_i) \right\| \Delta t \qquad 9$$

Taking the limit of this Riemann sum, we obtain the following formula for the arc length of *C*:

$$l = \int_a^b \sqrt{\left(x'(t) \right)^2 + \left(y'(t) \right)^2}\, dt \qquad 10$$

In light of Equation (9), we define the arc length function $s:[a,b]\to R$ related to $\vec{r}(t)$ as the arc length along *C* over the interval $[a,t]$. Thus,

$$s(t)=\int_a^t \left\|\vec{r}\,'(u)\right\| du=\int_a^b \sqrt{\left(x'(u)\right)^2+\left(y'(u)\right)^2}\,dt \quad 11$$

By the Fundamental Theorem of Calculus, we then have

$$s(t)=\left\|\vec{r}\,'(t)\right\|=\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2} \quad 12$$

which, still following the vocabulary from the trajectory of a moving particle, we call the speed function of $\vec{r}(t)$.

The speed of a vector function $\vec{r}(t)$ at a point is not a geometrical quantity since it can change under any reparametrization, which does not change the locus of the curve. If $t_0=f(u_0)$ where f is a differentiable function at u_0, and we have, [1, 4, 8];

$$\vec{r}(t_0)=\vec{r}\left(f(u_0)\right) \quad 13$$

and

$$\left\|\frac{d\left(\vec{r}\left(f(u_0)\right)\right)}{du}\right\|=\left\|f'(u_0)\vec{r}\,'\left(f(u_0)\right)\right\| \quad 14$$

$$=\left|f'(u_0)\right|\left\|\vec{r}\,'(t_0)\right\|=\left|f'(u_0)\right|s'(t_0)$$

It's worthwhile to note, the direction of $\vec{r}(t)$, by which we mean the unit vector associated with $\vec{r}(t)$, does not change under reparametrization. More precisely, if $\alpha=\vec{r}\circ f$, at any parameter value t where $f(t)\neq 0$ and $\vec{r}(t)\neq 0$, we have

$$\frac{\alpha'(t)}{\left\|\alpha'(t)\right\|}=\frac{f'(t)}{\left\|f'(t)\right\|}\frac{\vec{r}\,'\left(f(t)\right)}{\left\|\vec{r}\,'\left(f(t)\right)\right\|} \quad 15$$

where $f'(t)/\left\|f'(t)\right\|$ is +1 or −1.

For the right-hand side to be well defined, then, $f'(t)\neq 0$, and therefore, for a parametrized curve $\vec{r}(t):I\to R^2$ and a real function f from an interval *J* onto *I*, we call $\alpha=\vec{r}\circ f$ a regular reparametrization if for all $t\in J$, provided $f'(t)$ is well defined and $f'(t)\neq 0$. Equation (1.14) shows that the unit vector $\vec{r}\,'\left(f(t)\right)/\left\|\vec{r}\,'\left(f(t)\right)\right\|$ is invariant under a regular reparametrization up to a possible change in sign, [1, 4, 8].

Definition 2.8. Let $\vec{r}(t):I\to R^2$ be a plane parametric curve; a point $t_0\in I$ is called a critical point of $\vec{r}(t)$ if $\vec{r}(t)$ is not differentiable at t_0 or if $\vec{r}\,'(t)=\vec{0}$. If t_0 is a critical point, then $\vec{r}(t_0)$ is called a critical value. A point $t=t_0$ that is not critical is called a regular point. A parametrized curve $\vec{r}(t):I\to R^2$ is called regular if it is of class C^1 and $\vec{r}\,'(t)\neq\vec{0}$ for all $t\in I$. lastly, if t is a regular point of $\vec{r}(t)$, we define the unit tangent vector $\vec{T}(t)$ as, [1, 4, 8];

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\left\|\vec{r}\,'(t)\right\|} \qquad 16$$

At any point of a curve where $\vec{T}(t)$ is defined, the velocity vector is written as,

$$\vec{r}\,'(t) = s'(t)\vec{T}\,'(t) \qquad 17$$

which is an expression of the velocity vector as the product of its magnitude and direction and is mostly simplified by reparametrizing by arc length.

2.3 Curvature

Let $\vec{r}(t) : I \to R^2$ be a twice-differentiable parametrization of a curve C. By taking one more derivative of equation (16), we obtain the decomposition, [1, 2, 4, 8];

$$\vec{r}\,''(t) = s''(t)\vec{T}(t) + s'(t)\vec{T}\,'(t) \qquad 18$$

Since $\vec{T}(t)$ is a unit vector, then, $\vec{T}(t)\cdot\vec{T}(t) = 1$ and $\vec{T}(t)\cdot\vec{T}\,'(t) = 0$. Thus, $\vec{T}(t)$ is perpendicular to $\vec{T}\,'(t)$.

Definition 2.9. Let $\vec{r}(t) : I \to R^2$ be a regular parametrized curve and $\vec{T} = (T_1, T_2)$ the tangent vector at a regular value $\vec{r}(t)$. The unit normal vector to $\vec{r}(t)$ is defined by $\vec{N}(t) = (-T_2(t), T_1(t))$. Since $\vec{T}(t)$ is perpendicular to $\vec{T}\,'(t)$, the vector function $\vec{T}\,'(t)$ is a multiple of $\vec{N}(t)$ for all t, [1, 2, 4, 8].

Definition 2.10. Let $\vec{r}(t) : I \to R^2$ be a regular twice-differentiable parametric curve. The curvature function $\kappa(t)$ is the unique real-valued function defined by

$$\vec{T}(t) = s'(t)\kappa(t)\vec{N}(t) \qquad 19$$

Using equations (17) and (18) the value of $\kappa(t)$ for an *xy*-plane is given by, [1, 2, 4, 8];

$$\kappa(t) = \frac{x'(t)\,y''(t) - x''(t)\,y'(t)}{\left(\sqrt{(x'(t))^2 + (y'(t))^2}\right)^3} \qquad 20$$

Theorem 2.1. Let $\vec{r}(t) : I \to R^2$ be a regular plane curve that is of class C^2. Let $F : R^2 \to R^2$ be a rigid motion of the plane given by $F(\vec{u}) = A(\vec{u}) + \vec{C}$, where A is a rotation matrix and $\vec{C}$ is any vector in R^2. Consider the regular parametrized curve $\vec{\alpha}(t) = \vec{r}(t) \circ \vec{f}(t)$. Then $\vec{\alpha}(t)$ is a regular parametrized curve that is of class C^2, and the curvature function $\overline{\kappa}(t)$ of $\vec{\alpha}(t)$ is equal to the curvature function $\kappa(t)$ of $\vec{\alpha}(t)$, [1, 4, 8].

Theorem 2.2. (Fundamental Theorem of Plane Curves). Given a function $\kappa(t)$, there exists a regular curve of class C^2 parametrized by arc length $\vec{r}(t) : I \to R^2$ with the curvature function $\kappa(t)$. Furthermore, the curve is uniquely determined up to a rigid motion in the plane, [1, 4, 8].

III. Plane Curves: Global Properties

Now we turn to the study of global properties of curves. It's important to note that, global properties concern attributes about the curve taken as a whole. These global properties will deal with integration along the curve and topological properties and geometric properties of how the curve lies in R^2, [1, 4, 5, 7].

3.1 Basic Properties

Definition 3.1. A parametrized curve *C* is called closed if there exists a parametrization $\vec{r}:[a,b]\rightarrow R^2$ of *C* such that

$$\vec{r}(a)=\vec{r}(b). \quad 21$$

A regular curve is called closed if, in addition, all the derivatives of $\vec{r}$ at a and at b are equal, i.e.,

$$\vec{r}(a)=\vec{r}(b), \vec{r}\,'(a)=\vec{r}\,'(b)=\vec{r}\,''(a)=\vec{r}\,''(b),\ldots, \quad 21$$

A curve *C* is said to be simple if it has no self-intersections; a closed curve is called simple if it has no self-intersections except at the endpoints, [1, 4, 5, 7].

The conditions on the derivatives of $\vec{r}$ in definition (2.1) seem insufficient but attempt to establish the fact that the vector function $\vec{r}$ behaves identically at *a* and at *b*. A more topological approach involves using a circle S^1 rather than an interval as the domain for the map $\vec{r}$. A topological circle S^1 is defined as the set [0, 1] with the points 0 and 1 identified. The topology of S^1 is the identification topology, which in this case means that an open neighborhood U of *p* contains, for some ε, the subsets, [1, 4, 5, 7];

$$\left.\begin{array}{ll} (p-\varepsilon,\ p+\varepsilon)\ \text{with}\ \varepsilon < min\{p, 1-p\}, & \text{if}\ \ p\neq 0, \\ [0,\ \varepsilon)\cup(1-\varepsilon, 1], & \text{if}\ \ p=0\ \ (=1) \end{array}\right\} \quad 22$$

The curve *C* is said to be closed if there exists a continuous surjective function $\beta: S^1\rightarrow C$. Furthermore, a closed curve *C* is simple if there exists a bijection $\beta: S^1\rightarrow C$ that is continuous and such that β^{-1} is also continuous.

Proposition 3.1. Let *C* be a closed curve with parametrization $\vec{r}:[a,b]\rightarrow R^2$. C is bounded as a subset of R^2.

Theorem 3.1. (Green's Theorem). Let C be a simple, closed, regular, plane curve with interior region *R*, and let $\vec{F}=\left(P(x,y), Q(x,y)\right)$ be a differentiable vector field, [1, 4, 5, 7]. Then,

$$\int_C P\,dx+Q\,dy=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy \quad 23$$

Thus,

$$\int_C P\,dx+Q\,dy=\int_C \vec{F}\,d\vec{x}$$

$$=\int_a^b\left(\begin{array}{l} P\left(x(t),(y(t))\right)x'(t)+ \\ Q\left(x(t),(y(t))\right)y'(t)\end{array}\right)dt \quad 24$$

Corollary 3.1. Let *C* be a simple closed regular plane curve with interior region *R*. Then the area A of R is, [1, 4, 5, 7],

$$A=\int_C x\,dy=\int_C -y\,dx=\frac{1}{2}\int_C x\,dy-y\,dx \quad 25$$

3.2 Rotation Index

Proposition 3.2. Let *C* be a closed, regular, plane curve parametrized by $\vec{r}: I\rightarrow R^2$. Then the quantity $\frac{1}{2\pi}\int_C \kappa\,ds$ is an integer. This integer is called the rotation index of the curve, [1, 4, 5, 7].

Definition 3.2. Let $f: S^1\rightarrow S^1$ be a continuous map between circles. Let $\overline{f}:[0, 2\pi]\rightarrow R$ be the lifting of f. Then the degree of f is the integer, [1, 4, 5, 7],

$$\deg f = \frac{1}{2\pi}\left(\overline{f}(2\pi) - \overline{f}(0)\right) \tag{26}$$

Proposition 3.3. Let *C* be a regular closed curve parametrized by $\vec{r} : I \to R^2$. The rotation index of *C* is equal to the degree of $\vec{T}$, [1, 4, 5, 7].

Definition 3.3. Let *C* be a closed, regular, plane curve parametrized by $\vec{r} : I \to R^2$, and let $\vec{p}$ be a point in the plane not on *C*. Since *C* is a closed curve, we may view $\vec{r}$ as a function on S^1. We define the winding number $w(p)$ of *C* around $\vec{p}$ as the degree of the function, [1, 4, 5, 7],

$w : S^1 \to S^1$ defined by

$$w(t) = \frac{\vec{r}(t) - \vec{p}}{\left\|\vec{r}(t) - \vec{p}\right\|} \tag{27}$$

Proposition 3.4. Let *C* be a closed, regular, plane curve parametrized by $\vec{r} : I \to R^2$, and let p_0 and p_1 be two points in the same connected component of $R^2 - C$. The winding number of *C* around p_0 is equal to the winding number of *C* around p_1, [1, 4, 5, 7].

Theorem 3.2. (Regular Jordan Curve Theorem). Consider a simple, regular, closed, plane curve *C* parametrized by arc length $\vec{r} : (0, l) \to R^2$. Then the open set $R^2 - C$ has exactly two connected components with common boundary $C = \vec{r}(0, l)$. Only one of the connected components is bounded, and we call this component of $R^2 - C$ the interior, [1, 4, 5, 7].

3.3 Isoperimetric Inequality

The Jordan Curve Theorem and the Green's Theorem can be used to prove an inequality between the length of a simple closed curve and the area contained in the interior. This inequality is called isoperimetric inequality, is a global property of a curve as it relates quantities that take into account the entire curve at once, [1, 4, 5, 7].

Theorem 3.3. Let *C* be a simple, closed, plane curve with length l, and let A be the area bounded by *C*. Then $l^2 - 4A\pi \geq 0$ and equality holds if and only if *C* is a circle.

3.4 Curvature, Convexity, and the Four-Vertex Theorem

Definition 3.4. A subset of S of R^n is a convex if for all p and q in S, the line segment $[p, q]$ is a subset of S. If *C* is not convex, then it is concave, shown in Figure 1, [1, 4, 5, 7].

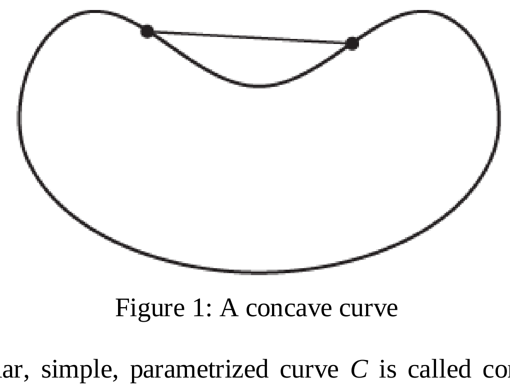

Figure 1: A concave curve

Definition 3.5. A closed, regular, simple, parametrized curve *C* is called convex if the union of *C* and the interior of *C* form a convex subset of R^2, [1, 4, 5, 7].

Proposition 3.5. A regular, closed, plane curve *C* is convex if and only if it is simple and its curvature κ does not change sign, [1, 4, 5, 7].

Proposition 3.6. A regular, simple, closed curve *C* is convex if and only if it lies on one side of every tangent line to *C*, [1, 4, 5, 7].

Theorem 3.4. (Four-Vertex Theorem). Every simple, closed, regular, convex, plane curve C has at least four vertices, [1, 4, 5, 7].

IV. Curves in Space: Local Properties

The theory for plane curves and the local theory of space curves are similar, but differences arise in the richer variety of configurations available for loci of curves in R^3. In the study of plane curves, the curvature function is introduced, which is a measure of how much a curve deviates from being a straight line. In the study of space curves, a curvature function and a torsion function will be introduced as measure of how much a space curve deviates from being linear and a measure of how much the curve twists away from being planar, respectively, [1, 4, 5, 7].

4.1 Definitions and Differentiation

If I is an interval of R, to call a space curve any function $\vec{r} : I \to R^3$ would allow for separate pieces or even a set of scattered points. By a curve, one typically thinks of a connected set of points, and, just as with plane curves, the desired property is continuity, [1, 4, 5, 7].

Definition 4.1. Let I be an interval of R. A parametrized curve in R^n is a continuous function $\vec{r} : I \to R^n$. If $n = 3$, we call $\vec{r}$ a space curve. If for all $t \in I$ we have

$$\vec{r}(t) = \left(x_1(t), x_2(t), \ldots\ldots, x_n(t)\right), \tag{28}$$

then the functions $x_i : I \to R$ for $i = 1, 2, \ldots\ldots, n$ are called the coordinate functions or parametric equations of the parametrized curve, [1, 4, 5, 7].

Definition 4.2. Let I be an interval in R, and let $t_0 \in I$. If $\vec{f} : I \to R^n$ is a continuous vector function, we say that $\vec{f}$ is differentiable at (t_0) with derivative $\vec{f}\,'(t_0)$ if there exists a vector function $\vec{\varepsilon}$ such that, [1, 4, 5, 7],

$$\vec{f}(t_0 + h) = \vec{f}(t_0) + h\vec{f}\,'(t_0) + h\vec{\varepsilon}(h) \tag{29}$$

and

$$\lim_{h \to 0} \left\| \vec{\varepsilon}(h) \right\| = 0 \tag{30}$$

It follows as a consequence of Proposition (1.1) that a vector function $\vec{r} : I \to R^n$ is differentiable at a given point if and only if all its coordinate functions are differentiable at that point. Further, if $\vec{r} : I \to R^n$ is a continuous vector function that is differentiable at $t = t_0$ and if the coordinate functions of $\vec{r}$ are

$$\vec{r}(t) = \left(x_1(t), x_2(t), \ldots\ldots, x_n(t)\right) \tag{31}$$

then the derivative $\vec{r}\,'(t_0)$ is

$$\vec{r}\,'(t) = \left(x_1'(t), x_2'(t), \ldots\ldots, x_n'(t)\right) \tag{32}$$

As in R^2, for any vector function $\vec{r} : I \to R^n$, borrowing from concept of trajectories in mechanics, we call $\vec{r}$ the position function, $\vec{r}\,'$ the velocity function, and $\vec{r}\,''$ the acceleration function, [1, 4, 5, 7]. Similarly, we define the speed function associated to $\vec{r}$ as the function $s : I \to R$ defined by

$$\vec{s}(t) = \left\| \vec{r}\,'(t) \right\| \tag{33}$$

Also the tangent line to a parametrized curve $\vec{r}(t)$ at $t = t_0$ as long as $\vec{r}(t_0) \neq \vec{0}$ can be defined as;

$$\vec{l}(u) = \vec{r}(t_0) + u\vec{r}\,'(t_0) \tag{34}$$

Proposition 4.1. Proposition 1.2.3 holds for differentiable vector functions $\vec{u}$ and $\vec{v}$ from $I \subset R$ into R^n. Furthermore, suppose that $\vec{u}(t)$ and $\vec{v}(t)$ are vector functions into R^3 that are defined on and differentiable over an interval $I \subset R$. If $\vec{r}(t) = \vec{u}(t) \times \vec{v}(t)$, then $\vec{r}(t)$ is differentiable over I and, [1, 4, 5, 7],

$$\vec{r}\,'(t) = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t) \qquad 35$$

4.2 Curvature, Torsion, and the Frenet Frame

Let I be an interval of R, and let $\vec{r} : I \to R^3$ be a differentiable space curve. For plane curves, the tangent vector $\vec{T}(t)$ defined by, [1, 4, 5, 7];

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\left\|\vec{r}\,'(t)\right\|} \qquad 36$$

It's is important to note that the unit tangent vector is not defined at a value $t = t_0$ if $\vec{r}$ is not differentiable at t_0 or if $\vec{r}\,'(t_0) = 0$.

Definition 4.2. Let $\vec{r} : I \to R^n$ be a parametrized curve. We call any point $t = t_0$ a critical point if $\vec{r}\,'(t_0)$ is not defined or is equal to $\vec{0}$; otherwise the point is a regular point. A parametrized curve $\vec{r} : I \to R^n$ is called regular if it is of class C^1 and if $\vec{r}(t) \neq \vec{0}$ for all $t \in I$, [1, 4, 5, 7].

In practice, if $\vec{r}\,'(t) = \vec{0}$ but $\vec{r}(t) \neq \vec{0}$ for all t in some interval *J* around t_0, it is possible for $\lim_{t \to t_0} \frac{\vec{r}\,'(t)}{\left\|\vec{r}\,'(t)\right\|}$ to exist. Since $\vec{T}(t)$ is a unit vector for all $t \in I$, we have:

$\vec{T}(t) \cdot \vec{T}(t) = 1$ and $\vec{T}(t) \cdot \vec{T}\,'(t) = 0$ for all $t \in I$.

We define the principal normal vector $\vec{N}(t)$ to the curve at t as,

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\left\|\vec{T}\,'(t)\right\|} \qquad 37$$

Lastly, for a space curve defined over any interval *J*, where $\vec{T}(t)$ and $\vec{N}(t)$ exist, we complete the set $\{\vec{T}, \vec{N}\}$ to an orthonormal frame by adjoining the so-called binormal vector $\vec{B}(t)$ given as

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) \qquad 38$$

Definition 4.3. Let $\vec{r} : I \to R^3$ be a continuous space curve of class C^2. To each point $\vec{r}(t)$ on the curve, we associate the Frenet frame as the triple of vectors $\left(\vec{T}(t), \vec{N}(t), \vec{B}(t)\right)$, [1, 4, 5, 7].

Similar to the basis $\{\vec{T}(t), \vec{N}(t)\}$ for plane curves, the Frenet frame provides a geometrically natural basis in which to study local properties of space curves.

We now analyze the derivatives of a space curve $\vec{r}\,'(t)$ in reference to the Frenet frame. By definition of the unit tangent vector, we have,

$$\vec{r}\,'(t) = s\,'(t)\vec{T}\,'(t). \qquad 39$$

If $\vec{r}(t)$ is twice differentiable at t, then $\vec{T}\,'(t)$ exists. If, in addition, $\vec{N}(t)$ is defined at t, then by definition of the principal normal vector, $\vec{T}\,'(t)$ is parallel to $\vec{N}(t)$.

Definition 4.4. Let $\vec{r}(t)$ be a parametrized curve of class C^2. The curvature $\kappa : I \to R^+$ of a space curve is

$$\kappa(t) = \frac{\left\|\vec{T}\,'(t)\right\|}{s\,'(t)} \qquad 40$$

Note that at any point where $\vec{N}(t)$ is defined, $\kappa(t)$ is the nonnegative number defined by,

$$\vec{T}'(t) = s'(t)\kappa(t)\vec{N}(t) \tag{41}$$

Definition 4.5. Let $\vec{r} : I \to R^3$ be a regular space curve of class C^2 for which the Frenet frame is defined everywhere. We define the torsion function $\tau : I \to R$ as the unique function such that,

$$\vec{B}'(t) = s'(t)\tau(t)\vec{N}(t) \tag{42}$$

We now need to determine $\vec{N}'(t)$, and we already know that it has the form $f(t)\vec{T}(t) + g(t)\vec{B}(t)$, [1, 4, 5, 7].

Since $\left(\vec{T}(t), \vec{N}(t), \vec{B}(t)\right)$ form an orthonormal frame for all t, we have the following equations:

$$\vec{T}(t)\cdot\vec{N}(t) = 0 \text{ and } \vec{N}(t)\cdot\vec{B}(t) = 0 \tag{43}$$

Taking derivatives with respect to t, we have

$$\vec{T}'(t)\cdot\vec{N}(t) + \vec{T}(t)\cdot\vec{N}'(t) = 0 \tag{44}$$

and

$$\vec{N}'(t)\cdot\vec{B}(t) + \vec{N}(t)\cdot\vec{B}'(t) = 0 \tag{45}$$

that is,

$$\vec{T}'(t)\cdot\vec{N}(t) = -\vec{T}(t)\cdot\vec{N}'(t) \tag{46}$$

and

$$\vec{N}'(t)\cdot\vec{B}(t) = -\vec{N}(t)\cdot\vec{B}'(t) \tag{47}$$

Thus, we deduce that

$$\vec{N}'(t) = -s'(t)\kappa(t)\vec{T}(t) + s'(t)\tau(t)\vec{B}(t) \tag{48}$$

As provided, the definitions for curvature and torsion of a space curve do not particularly lend themselves to direct computations when given a particular curve. We now obtain formulas for $\kappa(t)$ and $\tau(t)$ in terms of $\vec{r}(t)$. In order for our formulas to make sense, we assume for the remainder of this section that the parametrized curve $\vec{r} : I \to R^3$ is regular and of class C^3, [1, 4, 5, 7].

First, to find the curvature of a curve, we take the following derivatives:

$$\left.\begin{aligned} \vec{r}'(t) &= s(t)\vec{T}(t) \\ \vec{r}''(t) &= s''(t)\vec{T}(t) + \left(s'(t)\right)^2\kappa(t)\vec{N}(t) \end{aligned}\right\} \tag{49}$$

Taking the cross product of these two vectors, we now obtain

$$\vec{r}'(t)\times\vec{r}''(t) = \left(s'(t)\right)^3\kappa(t)\vec{N}(t) \tag{50}$$

However, by definition of curvature for a space curve, $\kappa(t)$ is a nonnegative function. Furthermore,

$$s'(t) = \left\|\vec{r}(t)\right\| \tag{51}$$

so we get

$$\kappa(t) = \frac{\vec{r}'(t)\times\vec{r}''(t)}{\left\|\vec{r}'(t)\right\|^3} \tag{52}$$

Secondly, to obtain the torsion function from $\vec{r}(t)$ directly, we will need to take the third derivative and making $\tau(t)$ the subject we get,

$$\tau(t) = \frac{\left(\vec{r}'(t)\times\vec{r}''(t)\right)\cdot\vec{r}'''(t)}{\left\|\vec{r}'(t)\times\vec{r}''(t)\right\|^2} \tag{53}$$

4.3 Osculating Plane

Definition 4.6. Let $\vec{r}: I \to R^3$ be a parametrized curve, and let $t_0 \in I$. Suppose that $\vec{r}$ is of class C^3 over an open interval containing t_0 and that $\kappa(t_0) \neq 0$. The osculating plane to $\vec{r}$ at $t = t_0$ is the plane through $\vec{r}(t_0)$ spanned by $\vec{T}(t_0)$ and $\vec{N}(t_0)$. In other words, the osculating plane is the set of points $\vec{u} \in R^3$ such that, [1, 4, 5, 7],

$$\vec{B}(t_0) \cdot \left(\vec{u} - \vec{r}(t_0)\right) = 0 \qquad 54$$

Proposition 4.2. Let $\vec{r}: I \to R^3$ be a regular parametrized curve of class C^2. The osculating plane to $\vec{r}$ at $t = t_0$ is the unique plane in R^3 of contact order 2 or greater. Furthermore, assuming $\vec{r}$ is of class C^3, the osculating plane has contact order 3 or greater if and only if $\tau(t_0) = 0$ or $\kappa(t_0) = 0$, [1, 4, 5, 7].

One can now interpret the sign of the torsion function $\tau(t)$ of a parametrized curve in terms of the curve's position with respect to its osculating plane at a point. In fact, $\tau(t_0) < 0$ at $\vec{r}(t_0)$ when the curve comes up through the osculating plane and $\tau(t_0) > 0$ when the curve goes down through the osculating plane, [1, 4, 5, 7].

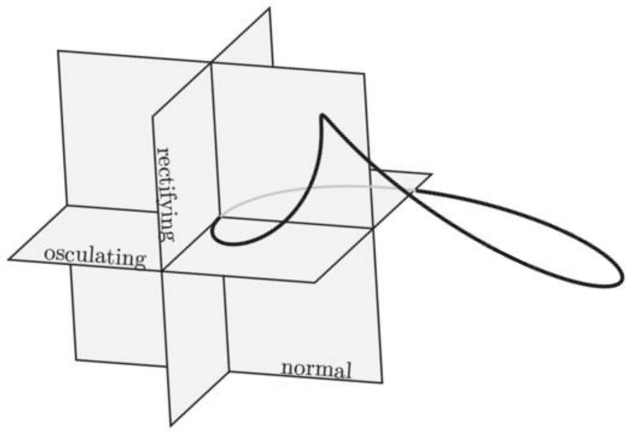

Figure 2: Moving trihedron

The osculating plane along with two other planes form what is called the moving trihedron, shown in Figure 2, which consists of the coordinate planes in the Frenet frame shown in Figure 3. The plane through $\vec{r}(t_0)$ and spanned by the principal normal and binormal is called the normal plane and is the set of points $\vec{u} \in R^3$ that satisfy

$$\vec{T}(t_0) \cdot \left(\vec{u} - \vec{r}(t_0)\right) = 0 \qquad 55$$

The plane through the tangent and binormal is called the rectifying plane and is the set of points $\vec{u} \in R^3$ that satisfy

$$\vec{N}(t_0) \cdot \left(\vec{u} - \vec{r}(t_0)\right) = 0 \qquad 56$$

Proposition 4.3. Let $\vec{r}: I \to R^3$ be a regular parametrized curve of class C^3. Let $t_0 \in I$ be a point on the curve where $\kappa(t_0) \neq 0$. The curve $\vec{r}$ admits an osculating sphere at $t = t_0$ if either (1) $\tau(t_0) \neq 0$ or (2) $\tau(t_0) = 0$ and $\kappa'(t_0) = 0$. Define $R(t) = \dfrac{1}{\kappa(t)}$, [1, 4, 5, 7]. If $\tau(t_0) \neq 0$, then at $t = t_0$ the curve admits a unique osculating sphere with center $\vec{c}$ and radius r, where

$$\vec{c} = \vec{r}(t_0) + R(t_0)\overrightarrow{N}(t_0) + \frac{R'(t_0)}{s'(t_0)\tau(t_0)}\vec{B}(t_0) \quad 57$$

and

$$r = \sqrt{R(t_0)^2 + \left(\frac{R'(t_0)}{s'(t_0)\tau(t_0)}\right)^2} \quad 58$$

If $\tau(t_0) = 0$ and $\kappa'(t_0) = 0$, then at t0 the curve admits as an osculating sphere any sphere with center $\vec{c}$ and radius *r* where

$$\vec{c} = \vec{r}(t_0) + R(t_0)\overrightarrow{N}(t_0) + c_B\vec{B}(t_0) \quad 59$$

and

$$r = \sqrt{R(t_0)^2 + c_B^2} \quad 60$$

where c_B is any real number.

4.4 Natural Equations

For space curves, one must introduce the torsion function for a measurement of how much the curve twists away from being planar, and Equation (3.8) shows how the torsion function appears as a component of $\vec{r}\,'''(t)$ in the Frenet frame, Figure 3. Since we know how $\vec{T}$, $\vec{N}$, and $\vec{B}$ change with respect to t, we can express all higher derivatives $\vec{r}^{(n)}(t)$ of $\vec{r}(t)$ in the Frenet frame in terms of $s'(t)$, $\kappa(t)$, and $\tau(t)$ and their derivatives. This leads one to posit that a curve is to some degree determined uniquely by its curvature and torsion functions, [1, 2, 4, 5, 7].

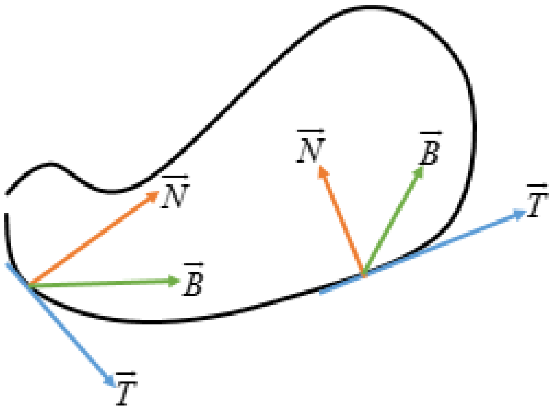

Figure 3: Moving Frenet frame

Theorem 4.1. (Fundamental Theorem of Space Curves). Given functions $\kappa(s) \geq 0$ and $\tau(s)$ continuously differentiable over some interval $J \subseteq R$ containing 0, there exists an open interval *I* containing 0 and a regular vector function $\vec{r}: I \to R^3$ that parametrizes its locus by arc length, with $\kappa(s)$ and $\tau(s)$ as its curvature and torsion functions, respectively. Furthermore, any two curves C^1 and C^2 with curvature function $\kappa(s)$ and torsion function $\tau(s)$ can be mapped onto one another by a rigid motion of R^3, [1, 4, 5, 7].

V. Curves in Space: Global Properties

As seen in section 2, global properties of curves are properties that involve the curve as a whole as opposed to properties that are defined in the neighborhood of a point on the curve. The Jordan Curve and Green's Theorems do not apply to curves in R^3, while the isoperimetric inequality and theorems connecting curvature and convexity do not have an equivalent for space curves. On the other hand, curves in space exhibit new types of global properties of knottedness and linking, [1, 3, 4, 8].

5.1 Basic Properties

Definition 5.1. A parametrized space curve C is called closed if there exists a parametrization $\vec{r}:[a,b]\to R^3$ of C such that $\vec{r}(a)=\vec{r}(b)$. A regular curve is called closed if, in addition, all the derivatives of $\vec{r}$ at a and at b are equal, i.e.,

$$\vec{r}(a)=\vec{r}(b),\vec{r}\,'(a)=\vec{r}\,'(b),\vec{r}\,''(a)=\vec{r}\,''(b),\ldots\ldots \quad 61$$

A curve C is called simple if it has no self-intersections, and a closed curve is called simple if it has no self-intersections except at the endpoints.

A closed space curve can be understood as a function $f:S^1\to R^3$ that is continuous as a function between topological spaces. In this context, to say that the curve is simple is tantamount to saying that f is injective. Our first result has an equivalent in the theory of plane curves, [1, 3, 4, 8].

Proposition 5.1. If a regular curve is closed, then it is bounded.

Stokes' Theorem presents a global property of curves in space in that it relates a quantity calculated along the curve with a quantity that depends on any surface that has that curve as a boundary. However, the theorem involves a vector field in R^3, and it does not address geometric properties of the curve, which means properties are independent of the curve's location and orientation in space. Therefore, we simply state Stokes' Theorem as an example of a global theorem and trust the reader has seen it in a previous calculus course, [1, 4, 8].

Theorem 5.1. (Stokes' Theorem). Let S be an oriented, piecewise smooth surface bounded by a closed, piecewise regular curve C. Let $\overline{F}:R^3\to R^3$ be a vector field over R^3 that is of class C^1. Then

$$\int_C \vec{F}\cdot d\vec{s}=\iint_S\left(\nabla\times\vec{F}\right)\cdot d\vec{S} \quad 62$$

If S has no boundary curve, then the line integral on the left is defined to be zero, [1, 3, 4, 8].

Corollary 5.1. Let C be a simple, regular, closed, space curve parametrized by $\vec{r}:I\to R^3$. There exists no function $f:R^3\to R$ of class C^2 such that the gradient satisfies $\vec{\nabla} f=\vec{T}(t)$ at all points of the curve, [1, 3, 4].

5.2 Indicatrices and Total Curvature

Definition 5.2. Given a space curve $\vec{r}:I\to R^3$, define the tangent, principal and binormal indicatrices respectively, as the loci of the space curves given by $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$, as defined in Section 3.2.

Since the vectors of the Frenet frame have a length of 1, the indicatrices are curves on the unit sphere S^2 in R^3. In contrast to the tangent indicatrix for plane curves, there are more possibilities for curves on the sphere than for curves on a circle. Therefore, results such as the theorem on the rotation index or results about the winding number do not have an immediate equivalent for curves in space, [1, 3, 4, 8].

Definition 5.3. The total curvature of a closed curve C parametrized by $\vec{r}:[a,b]\to R^3$ is

$$\int_C \kappa\, ds=\int_a^b \kappa(t)\,s'(t)dt \quad 63$$

This is a nonnegative real number since $\kappa(t)\ge 0$ for space curves. Though we cannot define the concept of a rotation index, Fenchel's Theorem gives a lower bound for the total curvature of a space curve. Since $\vec{T}\,'(t)=s'(t)\kappa(t)\vec{N}(t)$, the speed of the tangent indicatrix is $s'(t)\kappa(t)$. Therefore, the total curvature of $\vec{r}(t)$ is the length of the tangent indicatrix. One can also note that the tangent indicatrix has a critical point at the point where $\kappa(t)=0$, [1, 3, 4, 8].

Theorem 5.2. (Fenchel's Theorem). The total curvature of a regular closed space curve C is greater than or equal to 2π. It is equal to 2π if and only if C is a convex plane curve, [1, 3, 4, 8].

Lemma 5.1. (Horn's Lemma). Let Γ be a regular curve on the unit sphere S^2. If Γ has length less than 2π, then there exists a great circle C on the sphere such that Γ does not intersect C, [1, 3, 4, 8].

Corollary 5.2. If a regular, closed, space curve has a curvature function κ that satisfies $\kappa \le \frac{1}{R}$, then C has length greater than or equal to $2\pi R$, [1, 3, 4, 8].

Theorem 5.3. (Jacobi's Theorem). Let $\vec{\alpha} : I \to R^3$ be a closed, regular, parametrized curve whose curvature is never 0. Suppose that the principal normal indicatrix $\vec{N} : I \to S^2$ is simple. Then the locus $\vec{N}(I)$ of the principal normal indicatrix separates the sphere into two regions of equal area, [1, 3, 4, 8].

Theorem 5.4. (Crofton's Theorem). Let Γ be a curve of class C^1 on S^2. The measure of the great circles of S^2 that meet Γ is equal to four times the length of Γ, [1, 3, 4, 8].

5.3 Knots and Links

The reader should be forewarned that the study of knots and links is a vast and fruitful area that one usually considers as a branch of topology. In this section, we only have space to give a cursory introduction to the concept of a knotted curve in R^3 or two linked curves in R^3. However, the property of being knotted or linked is a global property of a curve or curves, which motivates us to briefly discuss these topics in this chapter. This section presents the Fary-Milnor Theorem, which shows how the property of knottedness imposes a condition on the total curvature of a curve, and Gauss's formula which is used to link a number of two curves together, [1, 4].

5.3.1 Knots

A knot is a simple closed curve in R^3 that cannot be deformed into a circle without breaking the curve and reconnecting it. This means that a knot can't be deformed into a circle by a continuous process without passing through a stage where it is not a simple curve, Figure 4, [1, 4].

Definition 5.4. A simple closed curve Γ in R^3 is called unknotted if there exists a continuous function $H : S^1 \times [0,1] \to R^3$ such that $H\left(S^1 \times \{0\}\right) = \Gamma$ and $H\left(S^1 \times \{1\}\right) = S^1$ and such that $H\left(S^1 \times \{t\}\right) = \Gamma_t$ is a curve that is homeomorphic to a circle. If there does not exist such a function H, the curve Γ is called knotted, [1, 4].

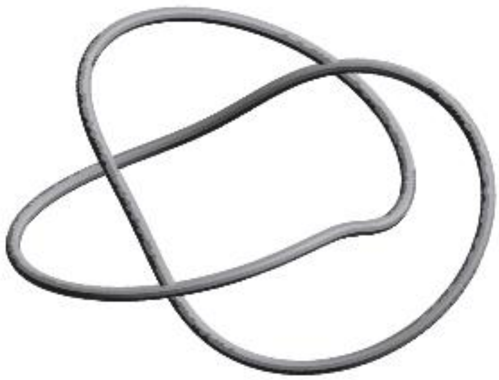

Figure 4: A knot

5.3.2 Links

A simple, closed, space curve in itself resembles a circle. Knottedness is a property that concerns how a simple closed curve is oriented in the ambient space. This implies that any simple closed curve is homeomorphic to a circle, but knottedness concerns how the curve is embedded in R^3. In a similar way, the notion of linking between two simple closed curves is a notion that considers how the curves are embedded in space in relation to each other, [1, 4].

Figure 4.6 illustrates a basic scenario of linked curves. Since it is difficult to accurately sketch curves in R^3 and to see which curve is in front of which, one commonly uses a diagram to depict the curves. In Figure 4.6, the diagram on the right corresponds to the curves on the left. The important part of the diagram is to clearly indicate the crossings, i.e., given the perspective of the diagram, which curve is on top, [1, 4].

Definition 5.5. The linking number link $\left(C^1, C^2\right)$ of two simple closed curves C^1 and C^2 is half the sum of the signs of all the crossings, as in Figure 5, in the diagram of the pair $\left(C^1, C^2\right)$, [1, 4].

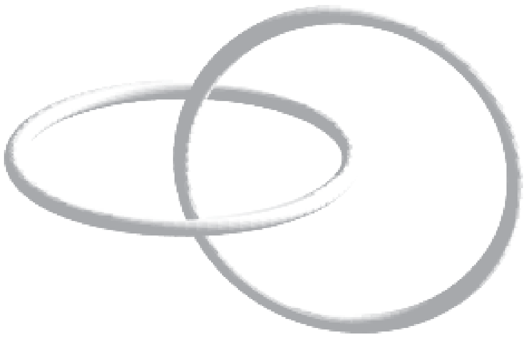

Figure 5: Linked curves

Theorem 5.6. (Gauss's Linking Formula). Let C^1 and C^2 be two closed, regular, space curves parametrized by $\vec{\alpha}: I \to R^3$ and $\vec{\beta}: I \to R^3$, respectively. The linking number between C^1 and C^2 is, [1, 4],

$$link\left(C^1, C^2\right) = \frac{1}{4\pi} \int_I \int_J \frac{\det\begin{pmatrix} \vec{\alpha}(u) - \\ \vec{\beta}(v), \vec{\alpha}'(u), \vec{\beta}'(v) \end{pmatrix}}{\left\| \vec{\alpha}(u) - \vec{\beta}(v) \right\|^3} du dv \quad 64$$

VI. Conclusion

The paper has given an elaborate introduction to advanced theory of curves. It is the recommendation of the author to have a paper with proofs and examples of the definitions, prepositions and theorems that have been mentioned in this paper.

References

[1]. Banchoff, T.F. and S.T. Lovett, Differential geometry of curves and surfaces. 2016: Chapman and Hall/CRC.

[2]. Kinyua, K.D. and K.J. Gikonyo, Differential Geometry: An Introduction to the Theory of Curves. International Journal of Theoretical and Applied Mathematics, 2017. **3**(6): p. 225.

[3]. Abbena, E., S. Salamon, and A. Gray, Modern differential geometry of curves and surfaces with Mathematica. 2017: Chapman and Hall/CRC.

[4]. Do Carmo, M.P., Differential Geometry of Curves and Surfaces: Revised and Updated Second Edition. 2016: Courier Dover Publications.

[5]. Ivancevic, V.G. and T.T. Ivancevic, Applied differential geometry: a modern introduction. 2007: World Scientific.

[6]. Raussen, M., Elementary Differential Geometry: Curves and Surfaces. Department of Mathematical Sciences, Aalborg University: Aalborg, Denmark, 2008.

[7]. Tapp, K., Differential geometry of curves and surfaces. 2016: Springer.

[8]. Toponogov, V.A., Differential geometry of curves and surfaces. 2006: Springer.

Kande Dickson Kinyua. " The Differential Geometry of Curves." *IOSR Journal of Mathematics (IOSR-JM)*, 17(4), (2021): pp. 14-28.