IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 17, Issue 3 Ser. IV (May – June 2021), PP 47-54
www.iosrjournals.org

The Comparison of The Symmedian Triangle Area and The Original Triangle

Hutri Defniati[1], Mashadi[2], Sri Gemawati[3]
[1,2,3]*(Department of Mathematics, University of Riau, Indonesia)*
Corresponding Author: Mashadi and Hutri Defniati

Abstract:
Background*: This article discusses on determining the area of a bisector triangle, determining the area of the symmedian triangle, and the comparison of both. The proof is done by using trigonometry rule to find the area of the triangle and Steiner's Theorem to determine the sides that formed the triangle.*
Keywords*: area of bisector triangle, area of symmedian triangle, bisector line, symmedian line*

Date of Submission: 20-06-2021 Date of Acceptance: 05-07-2021

I. Introduction

Several discussions about symmedian have been made. In the early history of the symmedian point, [10] discussed this topic and continued discuss the symmedian of the triangle and its accompanying circle [11]. The development of the symmedian and the circle formed was developed [6] which discusses tucker circles. Symmedian points are also called lemoine points. It gives an idea for further discussion about third lemoine circles [4]. For instance, a study of the First Lemoine Circle [6, 8, 16]. This theorem proves that if P is the symmedian point of triangle and three parallel lines are drawn to the sides of the triangle where all three lines through point symmedian P then the six points all lie on one circle. Similarly, for Second Lemoine Circle that constructed by drawing three anti parallel lines to the sides of the triangle are through the symmedian point so it will intersect in six points with a side of the triangle. Then, the sixth points will be on one circle [6, 8, 16]. Furthermore, there is Third Lemoine Circle. Let O be a symmedian point on triangle ABC. Then the triangle is partitioned into three triangles, namely ΔBCO, ΔACO and ΔABO. From each triangle, a circle can be made with the center points P, Q, and R. If the points P, Q and R are connected by a line segment, it will form ΔPQR. Additionally, from constructing the three circles of the ΔBCO, ΔACO and ΔABO, will form the intersection of the circle with side and side triangle extension ABC at six points [4, 6, 16]. Futhermore, if M is the centroid point of triangles ABC, and L are the center points of the third Lemoine's circle while K is the simedian point of triangle ABC, then the three points are collinear [16].

In this paper, the researcher is interested in determining an area of the symmedian triangle $A_sB_sC_s$ with the original triangle ABC. Triangle $A_sB_sC_s$ is a triangle formed from the intersection of the symmedian lines and the connected sides of the triangle. In other words, let AA_s, BB_s and CC_s is a symmedian line, then the points A_s, B_s and C_s are connected will form the $A_sB_sC_s$ symmedian triangle. Area of the original triangle and triangle symmedian will be calculated using trigonometric formulas. Furthermore, the study will discuss about the area of the triangle formed by the bisector and the median line.

II. Material And Methods

Several papers have discussed the previous definition of symmedian [3, 5, 7 10, 11].

Definition. 1 (Symmedian Line). Given triangle ABC where *a*, *b* and *c* respectively are the side lengths of *BC*, *AC*, and *AB*. If A_m and A_b are respectively the median line and the bisector line drawn from the angle *A*, then the reflection of the line A_m against A_b produces the line A_s, which is called a symmedian line.

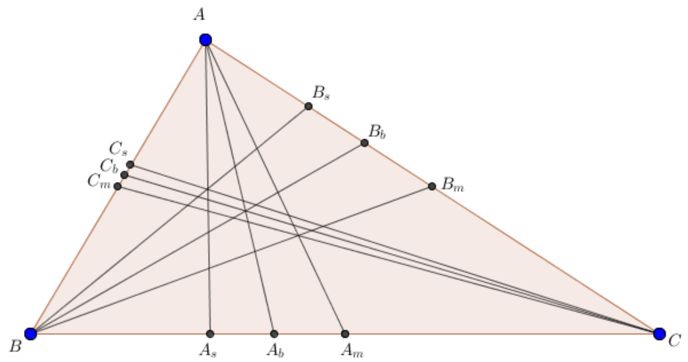

Figure 1: Symmedian lines on Triangle ABC

To determine the area of a symmedian triangle, the lengths of BA_s and CA_s from side BC, CB_s and AB_s from side AC, BC_s and AC_s from side AB,are needed. The Steiners theorem is used to calculate it. [17].

Theorem1 (Steiners Theorem**):** At any triangle ABC, let *D* and *E* be the points on the line *BC* with *AE* being the bisector line. If *AD* is reflected on *AE* and produces *AF*, it applies

$$\frac{BD}{CD} \cdot \frac{BF}{CF} = \frac{(AB)^2}{(AC)^2}$$

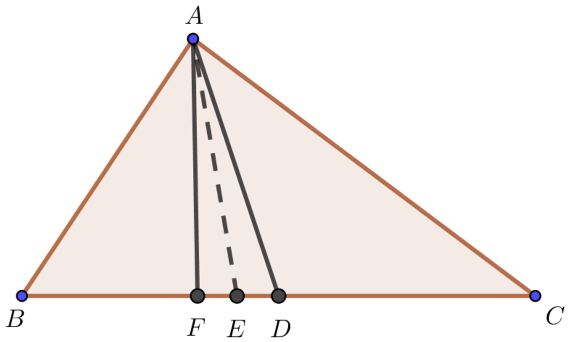

Figure 2: Steiner's theorem

Proof. The proof is discussed in [8]. ■

To calculate the area of a triangle, a trigonometric formula will be used.

III. Result

Theorem 3: Given any *ABC*, if *BC* = *a*, *AC* = *b*, and *AB* = *c* then A_b, B_b, and C_b are the points of intersection of the bisector lines on sides *a*, *b* and *c*, respectively. Thus, the area of $A_bB_bC_b$ is

$$L\Delta A_bB_bC_b = \frac{2(abc)}{(a+b)(a+c)(b+c)} \cdot L\Delta ABC$$

Proof: Before showing the area of $\Delta A_sB_sC_s$, we will determine the area of ΔAC_sB_s, ΔC_sBA_s, and ΔB_sA_sC. Therefore, it is necessary to determine the length of the side BA_s, CA_s, CB_s, BC_s, AB_s and AC_s. using the concept of congruent triangles

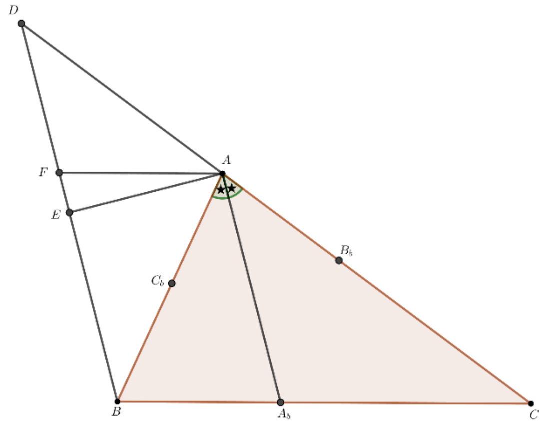

Figure 3: The Illustration Determines the Length

In figure 3, extend the line *CA*, through point *B* make a parallel line AA_b which intersects the extension of *CA* at point *D*. Next draw a line through point *A* parallel to *CB* which intersects *BD* at point *F*. Then, suppose that *E* is the midpoint of *BD* then by considering ΔCAA_b *and* ΔCDB, $\angle ACA_b = \angle BCD$ and $\angle CAA_b = \angle CDB$ we get $\Delta\, CAA_b \sim \Delta CDB$ so that it applies

$$\frac{CA}{CD} = \frac{CA_b}{CB}$$

Because $CD = CA + AD$ and $CA_b = CB - BA_b$, so that

$$\frac{CA}{CA + AD} = \frac{BC - BA_b}{BC}$$

$$BC \cdot CA = (BC - BA_b)(CA + AD)$$

$$BC - BA_b = \frac{BC \cdot CA}{CA + AD}$$

$$BC - BA_b = \frac{BC \cdot CA}{CA + AB}$$

$$BA_b = BC - \frac{BC \cdot CA}{CA + AB}$$

$$BA_b = \frac{BC \cdot AB}{AC + AB}$$

$$BA_b = \frac{a \cdot c}{b + c} \quad (1)$$

then, we will determine the length of the CA_b. because $CA_b = BC - BA_b$, then

$$CA_b = BC - BA_b$$

$$= a - \frac{ac}{b + c}$$

$$= \frac{a(b + c) - ac}{b + c}$$

$$= \frac{ab + ac - ac}{b + c}$$

$$CA_b = \frac{ab}{b + c} \quad (2)$$

In the same way it is obtained

$$B_bC = \frac{ab}{a+c} \quad (3)$$

$$AB_b = \frac{bc}{a+c} \quad (4)$$

$$AC_b = \frac{bC}{a+b} \quad (5)$$

$$BC_b = \frac{ac}{a+b} \quad (6)$$

After obtaining the Length of BA_b, CA_b, CB_b, BC_b, AB_b and AC_b, then determine the area of ΔAC_bB_b, ΔC_bBA_b, and ΔB_bA_bC using trigonometry

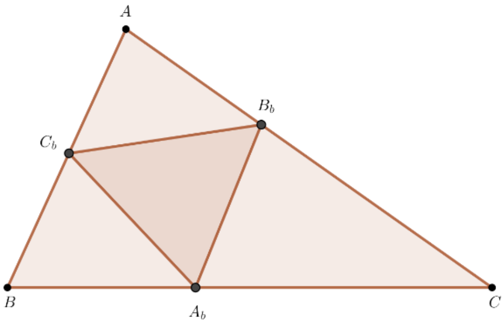

Figure 4: Bisector Triangle

To show the area of ΔA_bBC_b can be done using the trigonometric formula from the angle *B* obtained

$$L\Delta A_bBC_b = \frac{1}{2} \cdot BC_b \cdot BA_b \cdot \sin\angle\mathrm{B} \quad (7)$$

Substitute equation (1) and (6) into equation (7)

$$L\Delta A_bBC_b = \frac{1}{2} \cdot \frac{ac}{b+c} \cdot \frac{ac}{a+b} \cdot \sin\angle\mathrm{B}$$

$$L\Delta A_bBC_b = \frac{1}{2} \cdot \frac{a^2c^2}{(b+c)(a+b)} \cdot \sin\angle\mathrm{B}$$

$$\sin\angle\mathrm{B} = \frac{2(b+c)(a+b)}{a^2c^2} \cdot L\Delta A_bBC_b \quad (8)$$

Because the area of LΔABC from angle *B* is

$$L\Delta ABC = \frac{1}{2} \cdot AB \cdot BC \cdot \sin\angle\mathrm{B}$$

$$L\Delta ABC = \frac{1}{2} \cdot c \cdot a \cdot \sin\angle\mathrm{B} \quad (9)$$

Substitute equation (8) into equation (9) to get

$$L\Delta ABC = \frac{1}{2} ca \frac{2(b+c)(a+b)}{a^2c^2} \cdot L\Delta A_bBC_b$$

$$L\Delta ABC = \frac{(b+c)(a+b)}{ac} \cdot L\Delta A_bBC_b$$

$$L\Delta A_bBC_b = \frac{ac}{(b+c)(a+b)} L\Delta ABC$$

$$L\Delta A_bBC_b = \frac{ac}{(b+c)(a+b)} L\Delta ABC \quad (10)$$

In the same way it is obtained

$$L\Delta A_bB_bC = \frac{ab}{(b+c)(a+c)} L\Delta ABC \quad (11)$$

$$L\Delta AB_bC_b = \frac{bc}{(a+b)(a+c)} L\Delta ABC \quad (12)$$

After obtaining $L\Delta A_bBC_b$, $L\Delta A_bB_bC$, and $L\Delta AB_bC_b$, then determining the area of $\Delta A_bB_bC_b$

$$L\Delta A_bB_bC_b = L\Delta ABC - L\Delta A_bBC_b - L\Delta A_bB_bC - L\Delta AB_bC_b \quad (13)$$

Substitute equations (10), (11) and (12) into equation (13) so that

$$L\Delta A_bB_bC_b = L\Delta ABC - \frac{ac}{(b+c)(a+b)} L\Delta ABC - \frac{ab}{(b+c)(a+c)} L\Delta ABC - \frac{bc}{(a+b)(a+c)} L\Delta ABC$$

$$= \left[1 - \frac{ac}{(b+c)(a+b)} - \frac{ab}{(b+c)(a+c)} - \frac{bc}{(a+b)(a+c)}\right] L\Delta ABC$$

$$= \frac{2(abc)}{(a+b)(a+c)(b+c)} L\Delta ABC$$

Theorem 4: Given any ABC, if $BC = a$, $AC = b$, and $AB = c$ then A_s, B_s, and C_s are the points of intersection of the simedian lines on sides a, b and c, respectively. Thus, the area of $A_sB_sC_s$ is

$$L\Delta A_sB_sC_s = \frac{2(abc)^2}{(a^2+b^2)(a^2+c^2)(b^2+c^2)} \cdot L\Delta ABC$$

Proof: Before showing the area of $\Delta A_sB_sC_s$, we will determine the area of ΔAC_sB_s, ΔC_sBA_s, and ΔB_sA_sC. Therefore, it is necessary to determine the length of the side BA_s, CA_s, CB_s, BC_s, AB_s and AC_s. Using the steiners theorem is obtained

$$\frac{BA_s}{CA_s} \cdot \frac{BA_m}{CA_m} = \frac{(AB)^2}{(CA)^2} = \frac{c^2}{b^2} \quad (14)$$

Since $A_m$is the midpoint, then BA_m / $CA_m = 1$. So that equation (14) becomes

$$\frac{BA_s}{CA_s} = \frac{c^2}{b^2}$$

$$\frac{BA_s}{c^2} = \frac{CA_s}{b^2} \quad (15)$$

The line length $BA_s = a - CA_s$, substitute it for equation (15) is obtained

$$\frac{a - CA_s}{c^2} = \frac{CA_s}{b^2}$$

$$b^2(a - CA_s) = c^2CA_s$$

$$ab^2 - b^2CA_s = c^2CA_s$$

$$ab^2 = c^2CA_s + b^2CA_s$$

$$ab^2 = CA_s(c^2 + b^2)$$

$$CA_s = \frac{ab^2}{b^2+c^2} \quad (16)$$

Furthermore, the BA_s length is determined by substituting equation (16) into equation (15), so that it is obtained

$$\frac{BA_s}{c^2} = \frac{CA_s}{b^2}$$

$$\frac{BA_s}{c^2} = \frac{\frac{ab^2}{b^2+c^2}}{b^2}$$

$$\frac{BA_s}{c^2} = \frac{a}{b^2+c^2}$$

$$BA_s = \frac{ac^2}{b^2+c^2} \quad (17)$$

In the same way it is obtained

$$BC_s = \frac{ca^2}{a^2+b^2} \quad (18)$$

$$CB_s = \frac{ba^2}{a^2+c^2} \quad (19)$$

$$AB_s = \frac{bc^2}{a^2+c^2} \quad (20)$$

$$AC_s = \frac{cb^2}{a^2+b^2} \quad (21)$$

After obtaining the Length of BA_s, CA_s, CB_s, BC_s, AB_s and AC_s, then determine the area of ΔAC_sB_s, ΔC_sBA_s, and ΔB_sA_sC using trigonometry

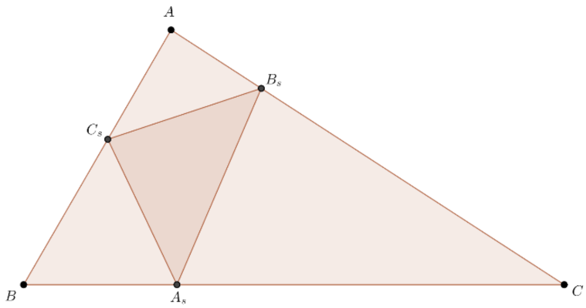

Figure 5: Symmedian Triangle

To show the area of ΔA_sBC_s can be done using the trigonometric formula from the angle B obtained

$$L\Delta A_sBC_s = \frac{1}{2} \cdot BC_s \cdot BA_s \cdot \sin\angle B \quad (22)$$

Substitute equation (17) and (18) into equation (22)

$$L\Delta A_sBC_s = \frac{1}{2} \cdot \frac{ca^2}{a^2+b^2} \cdot \frac{ac^2}{b^2+c^2} \cdot \sin\angle B$$

$$L\Delta A_sBC_s = \frac{1}{2} \cdot \frac{a^3c^3}{(a^2+b^2)(b^2+c^2)} \cdot \sin\angle B$$

$$\sin\angle B = \frac{2(a^2+b^2)(b^2+c^2)}{a^3c^3} \cdot L\Delta A_sBC_s \quad (23)$$

Because the area of L∆ABC from angle B is

$$L\Delta ABC = \frac{1}{2} \cdot AB \cdot BC \cdot \sin\angle B$$

$$L\Delta ABC = \frac{1}{2} \cdot c \cdot a \cdot \sin \angle B \quad (24)$$

Substitute equation (23) into equation (24) to get

$$L\Delta ABC = \frac{1}{2} ca \frac{2(a^2 + b^2)(b^2 + c^2)}{a^3 c^3} \cdot L\Delta A_s BC_s$$

$$L\Delta ABC = \frac{(a^2 + b^2)(b^2 + c^2)}{a^2 c^2} \cdot L\Delta A_s BC_s$$

$$L\Delta A_s BC_s = \frac{a^2 c^2}{(a^2+b^2)(b^2+c^2)} L\Delta ABC$$

$$L\Delta A_s BC_s = \frac{(ac)^2}{(a^2+b^2)(b^2+c^2)} L\Delta ABC \quad (25)$$

In the same way it is obtained

$$L\Delta A_s B_s C = \frac{(ab)^2}{(b^2+c^2)(a^2+c^2)} L\Delta ABC \quad (26)$$

$$L\Delta AB_s C_s = \frac{(bc)^2}{(a^2+b^2)(a^2+c^2)} L\Delta ABC \quad (27)$$

After obtaining $L\Delta A_s BC_s$, $L\Delta A_s B_s C$, and $L\Delta AB_s C_s$, then determining the area of $\Delta A_s B_s C_s$

$$L\Delta A_s B_s C_s = L\Delta ABC - L\Delta A_s BC_s - L\Delta A_s B_s C - L\Delta AB_s C_s \quad (28)$$

Substitute equations (24), (25) and (26) into equation (27) so that

$$L\Delta A_s B_s C_s = L\Delta ABC - \frac{(ac)^2}{(a^2+b^2)(b^2+c^2)} L\Delta ABC - \frac{(ab)^2}{(b^2+c^2)(a^2+c^2)} L\Delta ABC - \frac{(bc)^2}{(a^2+b^2)(a^2+c^2)} L\Delta ABC$$

$$= \left[1 - \frac{(ac)^2}{(a^2+b^2)(b^2+c^2)} - \frac{(ab)^2}{(b^2+c^2)(a^2+c^2)} - \frac{(bc)^2}{(a^2+b^2)(a^2+c^2)}\right] L\Delta ABC$$

$$= \frac{2(abc)^2}{(a^2+b^2)(a^2+c^2)(b^2+c^2)} L\Delta ABC$$

Thus the ratio of the area of the symmedian triangle to the original triangle is

$$\frac{L\Delta A_s B_s C_s}{L\Delta ABC} = \frac{2(abc)^2}{(a^2 + b^2)(a^2 + c^2)(b^2 + c^2)}$$

From theorem 3 and theorem 4, the comparison of the bisector triangle with the symmedian triangle is obtained,

$$\frac{L\Delta A_b B_b C_b}{L\Delta A_s B_s C_s} = \frac{2(abc) \cdot L\Delta ABC}{(a + b)(a + c)(b + c)} : \frac{2(abc)^2 \cdot L\Delta ABC}{(a^2 + b^2)(a^2 + c^2)(b^2 + c^2)}$$

$$\frac{L\Delta A_b B_b C_b}{L\Delta A_s B_s C_s} = \frac{2(abc) \cdot L\Delta ABC}{(a + b)(a + c)(b + c)} \cdot \frac{(a^2 + b^2)(a^2 + c^2)(b^2 + c^2)}{2(abc)^2 \cdot L\Delta ABC}$$

$$\frac{L\Delta A_b B_b C_b}{L\Delta A_s B_s C_s} = \frac{(a^2 + b^2)(a^2 + c^2)(b^2 + c^2)}{(abc)(a + b)(a + c)(b + c)}$$

IV. Conclusion

Symmedian lines are formed from the reflection of the median line against the bisector. To determine the area of a symmedian triangle, the Steiner's theorem and the trigonometric area formula can be used To determine the ratio of the area of the bisector triangle and the area of a symmedian triangle, simply using the area formula.

References

[1]. Amelia, Mashadi and S. Gemawati, Alternative proofs for the lenght of angle bisectors theorem on triangle, International Journal of Mathematics Trens and Technology, 66(2020), 163-166.
[2]. O. Bottterna and Erne, Elementary geometry, Springer, 2008
[3]. D. Dekov, Computer-generated mathematics: The symmedian point, Mathematika Pannonica, 10 (2008), 1-36.
[4]. D. Grinberg, Ehrmanns Third Lemoine Circle, Journal of Classical Geometry, 1(2012). 40-52
[5]. C. Kimberling, Trilinear distances inequalities for the symmedian point, centroid, and other triangle center, Forum Geometricorum, 10 (2010), 135-139.
[6]. S. N Kiss and P. Yiu, On the tucker Circles, Forum Geometricorum, 17 (2017), 157-175
[7]. B. Kolar, Symmedian and symmedian center of the triangle in an isotropic plane, Mathematika Pannonica, 17/2 (2006), 287-301
[8]. S. Luo and C. Pohoata, Let's talk about symmedians!, NC School of Science and Mathematics, Princeton University, USA, 1993.
[9]. Mashadi, Advanced Geometry (in Indonesian: *Geometri Lanjut*), UR Press, Pekanbaru, 2015.
[10]. J. S. Mackay, Early history of the symmedian point, Proceedings of the Edinburgh Mathematical Society 11 (1892-93), 92-103.
[11]. J. S. Mackay, Symmedians of a triangle and their concomitant circles, Proceedings of the Edinburgh Mathematical Society 14 (1895), 37-103
[12]. J. Sadek, M.B Yaghoub and N.H Rhee, Isogonal conjugates in a tetrahedron, Forum Geometricorum, 16 (2016), 43-50.
[13]. N. Singhal, On the ortogonality a median and a symmedian, Mathematika Pannonica, 17 (2017), 203-206
[14]. R. K. Smither, The symmedian point: Constructed and applied. The College Mathematical Journal, 2 (2011), 115-117
[15]. D. Trisna, Mashadi and S.Gemawati, Angle trisector in the three angles, IOSR Journal of Mathematical 16 (2020), 11-18.
[16]. A. Wardiyah, Mashadi and S. Gemawati, Relationship of lemoine circle with a symmedian point, Journal of Mathematical Sciences, 17(2016), 23-33.
[17]. Y. Silfiani, Mashadi, S. Gemawati, Comparison of the area of the triangle formed from the symmedian line and the median line, international journal of mathematics trands and Technology, 62 (2021), 68-73.
[18]. Y. Zhao. Three lemmas in geometry, Canada IMO Training Handout, Winter Camp, 2010.