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A Note on Marriage Theorem

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Abstract:
In this paper, the marriage problem and some related problems involving matching in bipartite graph is studied. Also, it is explained how such problems can be expressed in the language of transversal theory. A modified version of marriage theorem is also discussed.

Keywords: *Bipartite Graph, Matching Theory, and Marriage Theorem*

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I. Introduction

In a graph $\mathbf{G} = (\mathrm{V}, \mathrm{E})$, a set $\mathrm{M} \subseteq \mathrm{E}$ is a **matching** if M contains no adjacent edges. A practical aspect of **matching problems** is to find a pairing of the elements so that each element is paired uniquely with an available companion. The **marriage problem**, first introduced by Gale and Shapley [**1**], is a special case that could be stated as follows: Given a set $\boldsymbol{X}$ of boys and a set $\boldsymbol{Y}$ of girls, under what condition can each boy marry a girl who cares to marry him? Marriage problem is popular because it combines the maximum of temptation with the maximum of opportunity.

A matching M in a graph $\mathbf{G} = (\mathrm{V}, \mathrm{E})$ is said to **saturate** a vertex $\mathrm{v} \in \mathrm{V}$ if it is an end of some edge in M. More generally, a matching M **saturates** $\mathrm{A} \subseteq \mathrm{V}$ if it saturates every $\mathrm{v} \in \mathrm{A}$. A matching M is a called a **perfect** (or complete) **matching** if it saturates the entire set of vertices V. Given a matching M, an **odd length path** $\boldsymbol{\pi} = \boldsymbol{e_1\, e_2 \ldots\ldots e_{2K+1}}$ is called ***M*-augmented**, if the path $\boldsymbol{\pi}$ alternates between $E \setminus M$ and M, and the ends of $\boldsymbol{\pi}$ are <u>not saturated</u>. Next, a matching M is said to be **maximum** if for no matching M', $|\mathrm{M}| < |\mathrm{M}'|$. Clearly, every perfect matching is a maximum matching. The following result is a classic *necessary and sufficient condition* for a matching to be maximum.

Theorem1.(**Berge**, 1957) A matching M in a graph $\mathbf{G}$ is **maximum** if and only if there are no M-augmented paths in $\mathbf{G}$.

In fact, one may use a part of the proof to construct a maximum matching in an iterative manner starting from any matching M and from any M-augmented path.

For a set $S \subseteq V$ of a graph $G = (V, E)$, we may write

$N_G(S) = \{v \in V \mid uv \in G, for\ some\ u \in S\}$.

If G is (X, Y)- bipartite graph, and $S \subseteq X$, then $N_G(S) \subseteq Y$.

Theorem 2.(**Hall's Marriage Theorem**, 1935) Let G be a (X, Y)- bipartite graph. Then, G contains a matching M saturating X if and only if $|S| \leq |N_G(S)|$, for all $S \subseteq X$.

Corollary 1. (**Frobenius**, 1917) If G is a k- regular bipartite graph with $k > 0$, then G has a perfect matching.

Consider a bipartite graph G with a bipartition (X, Y) of its vertex set. Each vertex $x \in G$ supplies an **order of preferences** of the vertices in $N_G(x)$. We write $u <_x v$, if x prefers v over u; here, $u, v \in Y$, if $x \in X$, and $u, v \in X$, if $x \in Y$.

The matching problem can be applied in Mathematics, Economics and Computer Science etc. to find a stable matching between two equally sized sets given an *ordering of preferences* for each element of these sets. Carvolho et al. [**2**] introduced a perfect matching problem considering all the challenging requirements of modern complexities of various equipments. Relational matching with dynamic graph structures was described by Wilson and Hancock [**3**]. Messmer [**4**] presented some efficient graph matching algorithm for preprocessed model graphs.

DEFINITION A matching M of a graph G is said to be **stable**, if for each *unmatched pair* $xy \notin M$ (with $x \in X$and $y \in Y$), it is not the case that x and y prefer each other better than their matched companions:$xv \in M \;\; and \;\; y <_x v \quad or \quad uy \in M \;\; and \;\; x <_y u.$

The stable marriage problem plays a vital role in various real-world situations. Knuth [**5**] carried out stable marriage problem and its relation to other combinational problems. Beside Mathematics and Computer science, statistical physics is also a recent application area of stable marriage problem, A. Chakraborti and Y. C. Zhang [**6, 7**] which deals with large part of particle theory.

The problem related to assignments of medical graduate students to their hospital first preference has been discussed in a project [**8**]. Sveriges Riksbank Prize in Economic Sciences was awarded jointly to Alvin E. Roth and Lloyd S. Shapley for the theory of stable allocations and the practice of market design presented **[9]**. A new approach of matching in general graph is introduced by Blum [**10**]. A case study in game theory on the evolution of the labor market for medical interns and residents has been discussed by Roth [**11**]. Recently, Burchardt [**12**] presented some sufficient conditions for the uniqueness of stable matching in the Gale-Shapley marriage classical model of even size. Maggs and Sitaraman [**13**] assigned users to servers in a large distributed internet service.

II. MARRIAGE PROBLEM – A Case Study

The marriage problem has many variations. One of them is the job assignment problem, where we are given applicants and m jobs, and we should assign each applicant to a job he is qualified. The problem is that an applicant may be qualified for several jobs, and a job may be suited for several applicants.

The marriage problem can be explained with the help of following two graphical representation of matching. In the given graphical representation the bold black vertices represents workers while empty vertices represents tasks. The bipartite graph has been applied to assign a task to the qualified worker which is shown by a bold black dark edge (line).

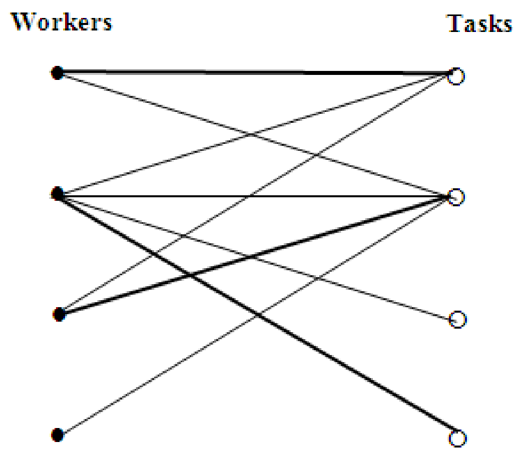


Figure -1

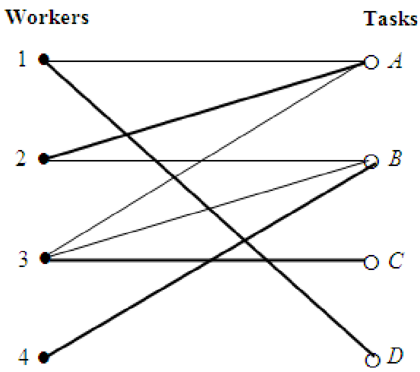


Figure - 2

The figure (1) depicts that almost three out of the four workers (worker 1 with task A, worker 2 with task D and worker 3 with task B) can be assigned to different tasks for which they are qualified, on the other hand in figure (2) all the four workers (worker 1 with task D, worker 2 with task A, worker 3 with task C and worker 4 with task B) are assigned to different tasks for which they are qualified.

In this paper marriage problem is discussed to understand how to assign tasks to all workers for which they are qualified. To state this problem, we may consider two sets, M of men and W of women, assuming that each man knows at least one woman.

We ask: **Under what condition(s) it is possible that all men marry women they know?**

We first consider the situation wherein $|M| \leq |W|$. Moreover any two, three…., men must collectively know at least two, three…., women respectively. In general, we can say that for each subset of m men, the m men collectively must know at least m women for all possible values of m. This is a necessary condition for all the men to marry women they know. What is not so obvious is that it is also a sufficient condition. This result is known as marriage theorem. Let us illustrate it with the help of the following specific cases

Case-I: Suppose A_1, A_2, A_3 and A_4 are the men and B_1, B_2, B_3 and B_4 are women.

The following table shows which women are known to which of the four men:

men	$women$ known
A_1	B_1, B_2
A_2	B_1, B_2, B_3, B_4
A_3	B_1, B_2
A_4	B_2

There are four men, so the total number of subset is 2^4. However, we need to consider only $2^4 - 1 = 15$ subsets, since we do not have a need of empty set.
The four men collectively know all the four women, so the marriage condition is satisfied for the subset $\{A_1, A_2, A_3, A_4\}$.

The following table shows **for which subset** of three men marriage condition is satisfied or not:

$Subset\ of\ men$	$Women\ known$	$No.\ of\ men\ (m)$	$No\ of\ women\ known$	$Is\ marriage\ condition\ satisfied$?
$\{A_1, A_2, A_3\}$	B_1, B_2, B_3, B_4	3	4	yes
$\{A_1, A_2, A_4\}$	B_1, B_2, B_3, B_4	3	4	yes

$\{A_1, A_3, A_4\}$	B_1, B_2	3	2	no
$\{A_2, A_3, A_4\}$	B_1, B_2, B_3, B_4	3	4	yes

For the third subsets, the three men collectively know only two women, so the marriage condition is not satisfied therefore all four men cannot marry women they know. For this example, we don't need to examine any more subset, but if the marriage condition had been satisfied for all four subsets of three men, then we should continue with the subsets containing two men and if necessary the subsets containing only one men.

Second Possibility is that consider three men A_1, A_2 and A_3 each of which know some of the four women B_1, B_2, B_3 and B_4, as shown in the following table

Men	*Women known*
A_1	B_1, B_3, B_4
A_2	B_4
A_3	B_2, B_3

The $2^3 - 1 = 7$ men which collectively know some women as shown in the following table:

Subset of men	*Women known*	*No. of men* (m)	*No of women known*	*Is marriage condition satisfied ?*
$\{A_1, A_2, A_3\}$	B_1, B_2, B_3, B_4	3	4	yes
$\{A_1, A_2\}$	B_1, B_3, B_4	2	3	yes
$\{A_1, A_3\}$	B_1, B_2, B_3, B_4	2	4	yes
$\{A_2, A_3\}$	B_2, B_3, B_4	2	3	yes
$\{A_1\}$	B_1, B_3, B_4	1	3	yes
$\{A_2\}$	B_4	1	1	yes
$\{A_3\}$	B_2, B_3	1	2	yes

From the above table we examine that the marriage condition is satisfied for every subset, so by the marriage theorem each of the three men can marry a woman which he knows.

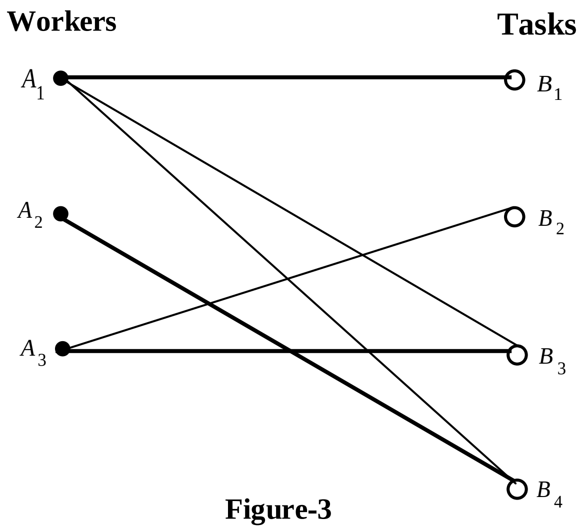


Figure-3

The bipartite graph representation of this solution is shown in the figure 3. The bold black lines indicate a maximum matching of three edges. Other matching with three edges is also possible.

In order to understand above example very clearly, we introduce the marriage theorem. The basic assumption of this theorem is that we do not require the sets of men and women to be equally distributed i.e. if the men are less, then marriage theorem condition will fail. On the other hand, if there are fewer women, the difference can be made up by desperate women willing to marry any men without disturbing the truth or falsity of the marriage condition.

Marriage Theorem A necessary and sufficient condition for there to be a solution to the marriage problem is that for every subset of m men, the m men collectively know at least m women, for every value of m, $1 \leq m \leq n$, where n is the total number of men.

Since, it is obvious that the given condition in the marriage problem is necessary, so we have only to prove that it is also sufficient. Now, we shall discuss the proof of marriage theorem by induction and constructive methods.

Induction Method: We carry out it into two steps
Step 1. In this step, we shall verify that it is true when $n = 1$ i.e there is one man who knows at least one woman, so obviously the marriage problem has a solution.
Step 2. Here, we shall prove that if the statement is true for every number of men less than n, then it must also be true for n men. Now assuming that the statement is true for every number of men less than n. If we consider n men then there are two cases, which together cover all possibilities.

Case (i). Suppose that for all $m < n$ set of m men collectively know at least $m+1$ women. In this case, the marriage condition is satisfied as at least one woman is to be spared for every set of m

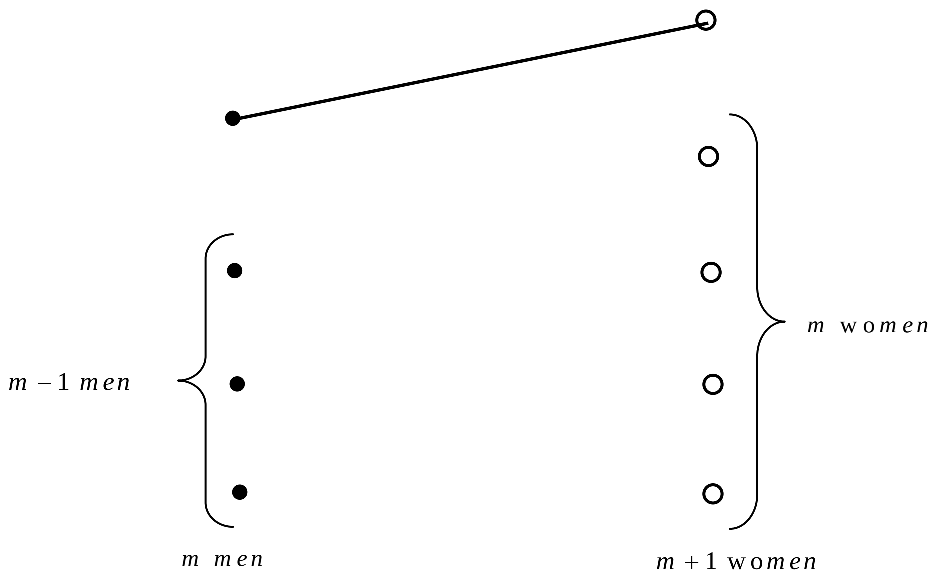


Figure -4

men. In this case we can take any man and marry to any woman he knows and the marriage condition will still be satisfied for the remaining $m-1$ men. As we are assuming that the theorem is true for every number of men less than n, so it must be true for $n-1$ men. We can therefore marry off the $n-1$ men appropriately. We have now married off all n men, so this completes the proof of step- 2 in this case (figure 4)

Case (ii). Consider that $m < n$ and there is at least one set of m men who collectively know exactly m women.
Assume that the theorem is true for every number of men says m less than n, which conclude that, we can marry these m men to the m women leaving $n-m$ men and at least $n-m$ women. Now any collection

of r men from $n-m$ men must collectively know at least r women otherwise these r men together with the set of m men would collectively know less than $r+m$ women which is contrary to the marriage condition.

Since we are assuming that the marriage theorem is true for less than n men and we can marry the $n-m$ men to women they know. This completes the proof of step 2.
We have thus proved that if the statement is true for all numbers of men less than n, then it is true for n men. Since, we know from step1 that the statement is true for $n=1$, it must be true for all positive integers n. This completes the proof of the theorem (figure 5)

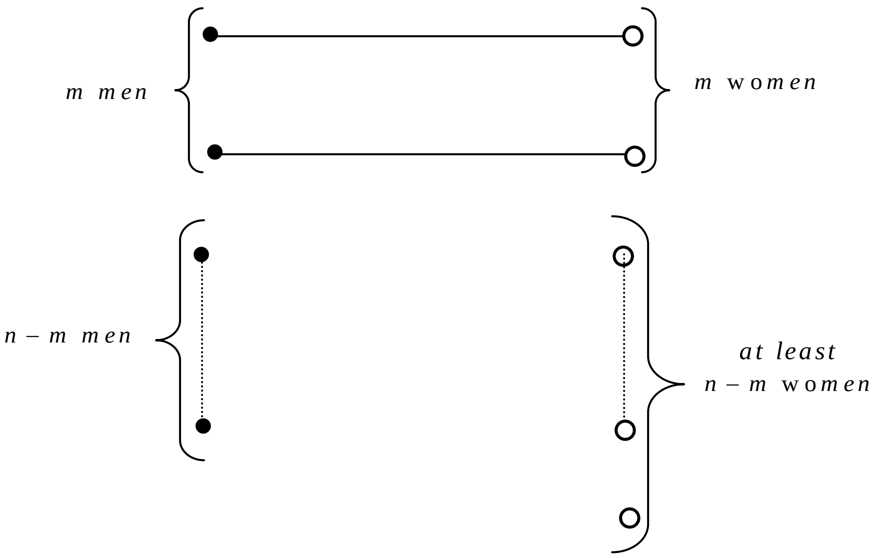


Figure -5

III. Marriage Theorem – A Constructive Proof

Consider that m men paired with m women where $m < n$ and increase it to a pairing of $m+1$ men. Initially, we can start if necessary with $m=1$ that is by pairing any man with women he knows.

Further, we can then successively increase the number of men paired until all n men are paired with women they know. We begin it with m men paired with m women whom they know. Now if there is a man left who knows women not already paired then $(m+1)^{th}$ pairing is immediately possible otherwise we proceed as follows:

Let A_0 be a man not paired with a women. Also let all the women whom he knows already paired with other men. Let B_1 be such women whom A_0 knows but already paired with say A_1. By the marriage theorem condition, A_0 and A_1 must collectively know at least two women namely B_1 and at least one other woman say B_2. If B_2 is not already paired then A_0 can be paired with B_1 and A_1 with B_2 then we can stop the procedure (figure 6)

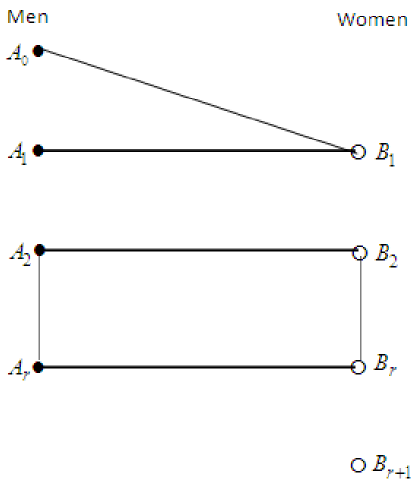


Figure-6

Further, if B_2 is paired with A_2 then the men A_0 , A_1 and A_2 must collectively know third woman say B_3 . If B_3 is paired with A_3 then we continue until unpaired women B_{r+1} is reached. All the possible results of this procedure can be represented by the following diagram (figure 7).

We have replaced the r original pairings with $r+1$ pairings. Therefore, we have a total of $m+1$ pairings including the pairings not involve in this process. If there are still any men who are not paired, we continue with this process until all n men are paired with women they can marry. The above constructive process provides a method of finding a matching of the n men to n of the women

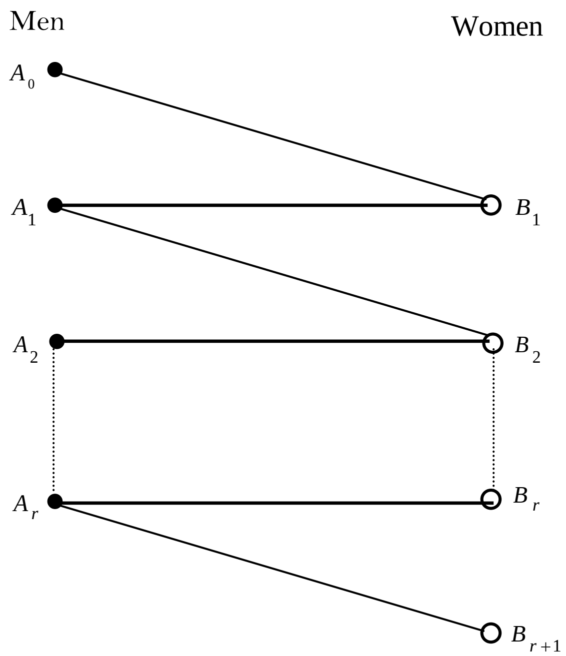


Figure-7

We have replaced the r original pairings with $r+1$ pairings. Therefore, we have a total of $m+1$ pairings including the pairings not involve in this process. If there are still any men who are not paired, we continue with this process until all n men are paired with women they can marry. The above constructive process provides a method of finding a matching of the n men to n of the women.

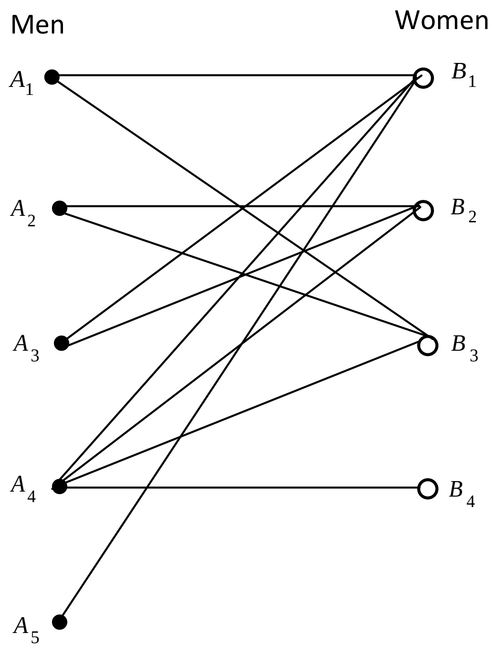
In the above discussion, we have observe that the marriage problem will not be work when the subsets of n workers is more than the subsets of given tasks. To resolve this problem we introducing "the modified marriage problem" which gives us more perfect matching as compare to marriage problem for those subsets where marriage problem will be failed.

IV. The Modified Marriage Theorem

Statement. If in a group of n men each knows some of a group of women, the maximum number of men who can marry women they know is equal to the minimum value of the expression.
(Number of women by a subset of *m* men) + (*n* - *m*), for any subset of *m* men, 1≤ *m*≤ *n*

Worked Problem. Suppose that five men A_1, A_2, A_3, A_4 and A_5 know four women $B_1, B_2, B_3,$ and B_4 as shown in the following table and bipartite graph. Discuss the maximum number of men who can marry women they know.

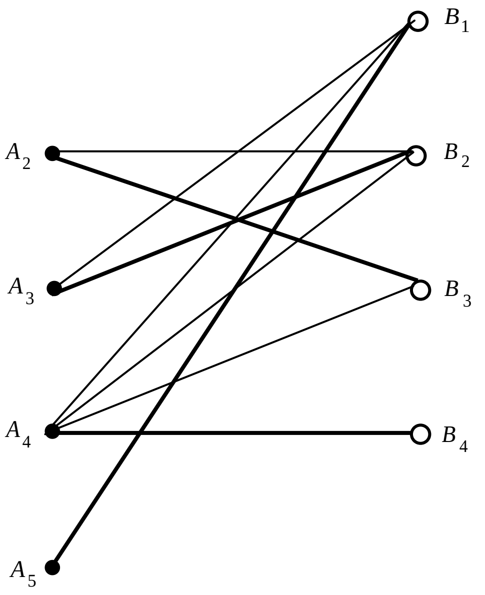
Men	Women known
A_1	B_1, B_3
A_2	B_2, B_3
A_3	B_1, B_2
A_4	B_1, B_2, B_3, B_4
A_5	B_1



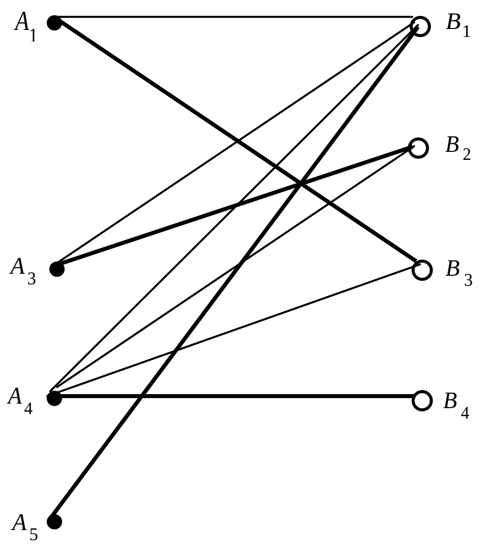
Solution. To answer this problem, we need to find the minimum value of the expression in the modified marriage theorem. For this, we need to consider only those subsets of men for which the original marriage condition is not satisfied. By inspection of the table or the graph, we can observe that there are only two such subsets, namely, $\{A_1, A_2, A_3, A_5\}$ and $\{A_1, A_2, A_3, A_4, A_5\}$. We can draw up the following table:

Subset of men	Women known	No. of men (m)	No. of women known (p)	$p+(n-m)$
$\{A_1, A_2, A_3, A_5\}$	B_1, B_2, B_3	4	3	4
$\{A_1, A_2, A_3, A_4, A_5\}$	B_1, B_2, B_3, B_4	5	4	4

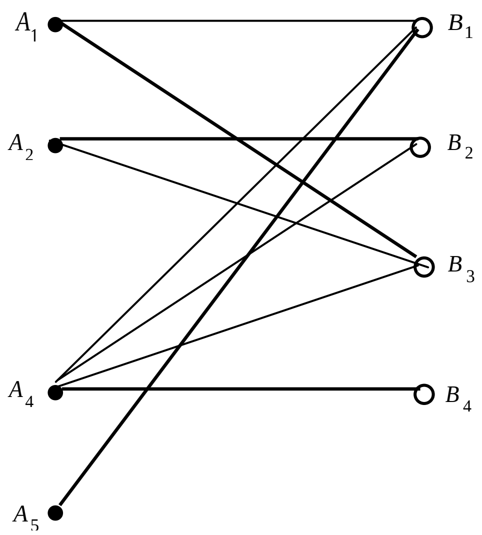
For both subsets, the value of the expression $p+(n-m)$ *is* 4. Hence by modified marriage theorem exactly four men can marry women they know



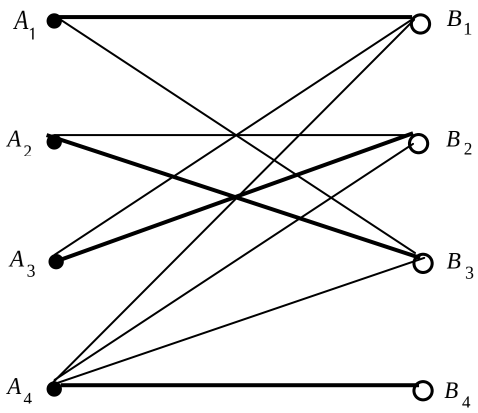
A_1 *removed*

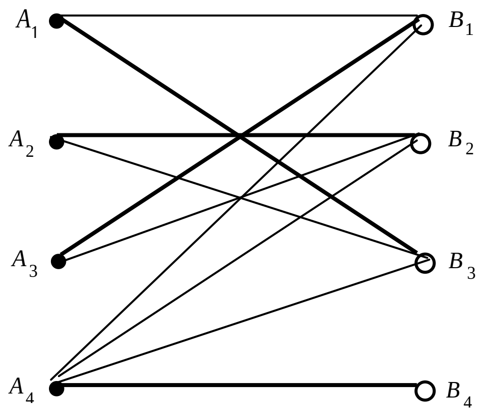


A_2 *removed*



A_3 *removed*





A_5 *removed (two possibilities)*

Figure-8

The next question which arises is that which of the four men can marry the women they know? or which one man should we remove so that the remaining four men can marry the women they know? To answer this question, we find out the two subsets for which the marriage condition is not satisfied. If we remove a man who is in both of these subsets, then clearly there are four possibilities A_1, A_2, A_3 and A_5. Now, we can remove each of A_1, A_2, A_3 and A_5 turn wise to find a marriage scheme for the remaining four men in each case. All such marriage schemes involved the maximum number of men is shown in the (figure 8).

If A_4 is removed, then it is obvious that all the remaining four men cannot marry women they know, since none of them knows B_4.

DEFINITION Let $\boldsymbol{S} = \{\boldsymbol{S_1}, \boldsymbol{S_2}, \cdots, \boldsymbol{S_m}\}$ be a family of finite nonempty subsets of a set S, where S_i need not be distinct. A **transversal** (or a system of distinct representatives) of S is a subset $T \subseteq S$ of m distinct elements one from each S_i.

The connection of **transversals** to the Marriage Theorem is as follows: Let $\boldsymbol{S} = Y$ and $X = [1, m]$. Form an (X, Y)-bipartite graph G such that there is an edge (i, s) if and only if $s \in S_i$. The possible transversals T of $\boldsymbol{S}$ are then obtained from the matchings M saturating X in G by taking the ends in Y of the edges of M.

V. Conclusion

In this paper, we presented marriage and modified marriage problems with theoretical solution and numerical simulations. These observations help us to further understand the structure and properties of marriage problem solution. It also sheds a light on the matching process of Mathematics and resource allocations who deal with many of the real bipartite matching problems.

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