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Optimization inside the General Efficiency

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Abstract. *This research work is concerned with the study of the General Efficiency and Optimization. We present the Efficiency and Optimization in their most natural context offered by the Infinite Dimensional Ordered Vector Spaces, following our recent results on these subjects. Implications and Applications in Vector Optimizaton through the agency of Isac's Cones and the new link between the General Efficiency and the Strong Optimization by the Full Nuclear Cones are presented. An important extension of our Coincidence Result between the Efficient Points Sets and the Choquet Boundaries is developed. In this way, the Efficiency is strongly connected with Potential Theory by Optimization and conversely. Several pertinent references conlude this investigation.*

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I. Introduction.

Until now, the basis on which was developed our general concept of the Efficiency in the Ordered Linear Spaces and its Applications, supplied by the Ordered Locally Convex Spaces, is represented about the vastness class of the Convex Cones introduced by Professor Isac in 1981, published in 1983, called by us and, officially recognized, as "Isac's Cone" in 2009, after the acceptance of this last agreed denomination by Professor Isac. To it is devoted this scientific research work, following the content of this International Meeting. All the elements concerning the ordered topological vector spaces used here are in acordance with *Nachbin, L., 1965* and *Peressini, A., L., 1967*.

II. General Efficiency in the Ordered Vector Spaces

Let E be any real or complex vector space ordered by a convex cone K, K_1 a non-void subset of K and A a non-empty subset of E. The following definition introduces a new concept of the efficiency which generalizes the well known Pareto type efficiency in every ordered Euclidean space and not only.

Definition 1. *(Postolică, V., 2002, 2008)***.** *We say that $a_0 \in A$ is a K_1 – minimal efficient point of A, in notation, $a_0 \in MIN(A, K, K_1)$ if it satisfies one of the following equivalent conditions:*

(i) $A \cap (a_0 - K - K_1) \subseteq a_0 + K + K_1$;

(ii) $(K + K_1) \cap (a_0 - A) \subseteq -K - K_1$;

In a similar manner one defines the K_1 – *maximal efficient points* by replacing $K + K_1$ with $-(K + K_1)$. The set of such as these points will be denoted by $MAX(A, K, K_1)$. Whenever $K_1 = \{\theta\}$, with θ the null vector of E, we consider that $MIN(A, K, K_1) = MIN(A, K)$ and $MAX(A, K, K_1) = MAX(A, K)$, respectively. *All the efficient points of the set A with respect to the convex cone K following K_1 are represented by* $eff(A, K, K_1) = MIN(A, K, K_1) \cup MAX(A, K, K_1)$. Consequently, $eff(A, K) = MIN(A, K) \cup MAX(A, K)$. Clearly,

$A \cap (a_0 \mp K) \subseteq a_0 \pm K_1 \Rightarrow A \cap (a_0 \pm K \pm K_1) \subseteq a_0 \mp K \mp K_1 \Rightarrow A \cap (a_0 \mp K_1) \subseteq a_0 \pm K$, which suggests other concepts for the *efficiency and its approximate term* in *the general Ordered Vector Spaces*.

Obviously, $a_0 \in eff\,(A, K, K_1)$ iff it is a fixed point for at least one of the multifunctions $F_{\mp} : A \to 2^A$ defined by $F_{\mp}(t) = \{a \in A : A \cap (a \mp K \mp K_1) \subseteq t \pm K \pm K_1\}$, that is, $a_0 \in F_{\mp}(a_0)$., with the corresponding signs. Consequently, for the existence of these general efficient points we can also apply appropriate fixed points theorems concerning the multi-functions (see, for instance, *Cardinali, T., Papalini, F., 1994, Park, S.,1992, Patriche, M.,2013, Zhang Cong – Jun, 2005 and any other proper scientific papers*).Whenever K is pointed, that is, $K \cap (-K) = \{\theta\}$, then $a_0 \in MIN\,(A, K, K_1)(a_0 \in MAX\,(A, K, K_1))$ means that $A \cap (a_0 \mp K \mp K_1) = \varnothing$ or, equivalently, $(\pm K + K_1) \cap (a_0 - A) = \varnothing$ for $\theta \notin K_1$ and $A \cap (a_0 \mp K \mp K_1) = \{a_0\}$, respectively, if $\theta \in K_1$. Whenever $K_1 = \{\theta\}$, from the **Definition 1** one obtains the usual concept of the efficient (*Pareto type minimal, optimal or admissible*) point. We also notice that $MIN\,(A, K)(MAX\,(A, K)) = \bigcap_{\{0\} \neq K_2 \subseteq K} MIN\,(A, K, K_2)(\bigcap_{\{0\} \neq K_2 \subseteq -K} MAX\,(A, K, K_2))$.

Moreover, $a_0 \in eff\,(A, K)$ if only if it is *a critical (ideal or balance) point* (*see, for example, Kim, W. K., Tan, K.K., 2001; Isac,G. 1985; Isac, G., Bulavsky, V. A., Kalashnikov, V., V. 2002; Isac,G., Postolică, V.,1993; Postolică, V., 2004 and the corresponding references*) for *the generalized dynamical systems* $\Gamma_{\mp} : A \to 2^A$ defined by $\Gamma_{\mp}(a) = A \cap (a \mp K),\ a \in A$. In this way , $eff\,(A, K)$ describes *the balance extremum moments* for $\Gamma_{\mp}$, which in *the market context*, expresses *the competitive* equilibrium/non-*equilibrium consisting of the general relation price/consumption*. When referring to the existence of the efficient points, the significant properties of the sets containing these points and the applications in the usual or extended contexts we mention, as a comprehensive bibliography, *(Aubin, J.P., 1993; Benson, H. P., 1977, 1979,1983;Borwein, J. M., 1977, 1983;Dauer, J. P., Gallagher, R. J., 1990; Dominiak, Cezary, 2006; Ehrgott, Matthias, 2005;Fishburn, P. C.,1984; Gass, Saul.,I., Harris, Carl., M., 2001; Geoffrion, A. M., 1968; Hartley, R., 1978; Henig, M. I., 1982; Heyne, Paul, 2000; Hyers, D., H., Isac, G., Rassias,T.M.,1997; Isac, G., 1981, 1983, 1985, 1994, Isac, G., Bahya, A. O., 2002; Isac, G., Postolică, V.,1993, 2005;; Jablonsky, J.,2006; Loridan, P., 1984; Luc, D. T., 1989; Luptáčik, M., Bőhm, B., 2006;Ng., K., F., Zheng, X., Y., 2002; Postolică, V.,1989, 1993, 1994, 1996, 1999, 2001, 2002, 2004, 2008, 2009; Songsak Sriboonchitta, Wing – Keung Wong, Sompong Dhopongsa, Hung, T., Nguyen, 2009; Sontag, Z., Zălinescu, C., 2000; Stewart, D.,J., 2004; Truong, X. D. H., 1994; Zimmermann, H., 2000)* with widely established results.

Let us consider the set $SMIN\,(A, K, K_1) = \{a_1 \in A : A \subseteq a_1 + K + K_1\}$ of all *strong minimum points* of A with respect to K by K_1 and $SMAX\,(A, K, K_1) = \{a_2 \in A : A \subseteq a_2 - K - K_1\}$ the set of all *strong maximum points* of A with respect to K by K_1, respectively, and

$S\,(A, K, K_1) = \{a_1 \in A : A \subseteq a_1 + K + K_1\} \cup \{a_2 \in A : A \subseteq a_2 - K - K_1\}$

The following theorem shows the immediate connection between the *Efficiency* and the *Strong Optimization* through the agency of *cone minimal (cone maximal) efficient points*.

Theorem 1. *(Postolică,V., 1993,1995,1996). If* $S\,(A, K, K_1) \neq \varnothing$, *then* $S\,(A, K, K_1) = eff\,(A, K, K_1)$.

It is obvious that $S\,(A, K, K_1) \subseteq eff\,(A, K, K_1)$. The previous theorem shows that every time there exists at least one strong cone minimum or cone maximum point, the set of all efficient points coincides with the set of all these points, respectively. This is important also for the *numerical methods*, because if there exist *strong efficient* points, then these are the only *general efficient* points. We also note that it is possible to have $S\,(A, K, K_1) = \varnothing$ and $eff\,(A, K, K_1) = A$. Thus, for example, if one considers $X = R^n (n \in N, n \geq 2)$, $K = R^n_+$, $K_1 = \{(0, \ldots, 0)\}$ and for each real number c we define $A_{0c} = \left\{(x_i) \in X : \sum_{i=1}^{n} x_i = c\right\}$,

$A_{1c} = \left\{ (x_i) \in X : \sum_{i=1}^{n} x_i \geq c \right\}$, $A_{2c} = \left\{ (x_i) \in X : \sum_{i=1}^{n} x_i \leq c \right\}$ then it is clear that $S\left(A_{0c}, K, K_1\right) = S\left(A_{1c}, K, K_1\right) = S\left(A_{2c}, K, K_1\right) = \varnothing$,
$eff\left(A_{0c}, K, K_1\right) = MIN(A_{0c}, K, K_1) = MAX(A_{0c}, K, K_1) = A_{0c}$,
$MIN(A_{1c}, K, K_1) = MAX(A_{2c}, K, K_1) = A_{0c}$ so, by extrapolation,
$eff\left(A_{1c}, K, K_1\right) = A_{0c} \cup \{(+\infty, +\infty, \ldots, +\infty)\}$ and $eff\left(A_{2c}, K, K_1\right) = A_{0c} \cup \{(-\infty, -\infty, \ldots, -\infty)\}$. At the same time, in the usual real linear space of all countable sequences, ordered by the convex cone $K = \left\{(x_n) : n \in N^*, n \geq 2, x_n \geq 0, \forall n \geq 2\right\}$, for $A = \left\{(x_{n\alpha}) : n \in N^*, n \geq 2, \alpha > 0\right\}$ with
$x_{n\alpha} = (n-1)^{-\alpha} - n^{-\alpha}, n \in N^*, n \geq 2, \alpha > 0$ and $K_1 = \{(0, 0, \ldots.)\}$, we have
$eff(A, K, K_1) = S(A, K, K_1) = \varnothing$. Simple examples show that, in general, the implication
$S\left(A, K, K_1\right) = \varnothing \Rightarrow eff(A, K, K_1) = A$ is not true.

III. Isac's Cones and the Efficiency

Initially, the next concept of convex cone named by its author *nuclear, supernorma*l *(Isac, G., 1980, 1981, 1983, 1985)* and, by us, *Isac's cone (Postolică, V., 2009)* was introduced in the following context, being the most appropriate for the general efficiency by optimization and conversely *(see, for instance, Isac,G., 1980,1981, 1983, 1985, 1994, 1998, 2003; Isac, G., Bahya, A.O., 2002; Isac, G., Postolică, V., 1993, 2005; Isac, G., Tammer, Chr., 2003 and so on)*. As we'll see, to ensure the existence of the efficient points for a non-empty set in an arbitrary ordered Hausdorff locally convex space, the main reason will be to replace *the compactness assumption* on the set with *the completeness* hypothesis and *the ordering cone* to be an *Isac's cone.*
Definition 2. *(Isac, G., 1981,1983). In a Hausdorff locally convex space, a pointed convex cone $K \subset X$ is nuclear (supernormal) with respect to the topolgy induced by a family $P = \{p_\alpha : \alpha \in I\}$ of seminorms if*

$\forall p_\alpha \in P, \exists f_\alpha \in X^* : p_\alpha(x) \leq f_\alpha(x), \forall x \in K$.

Initially, the regretted Professor George Isac considered that Treves'definition was more scientifically productive for his cones. Afterwards, he accepted our proposal contained in the above definition.
which is an *equivalent definition* of *Isac's cones* in separated locally convex spaces. The best special, refined and non-trivial *Isac's cones class* associated to *normal cones* in Hausdorff locally convex spaces was introduced and studied in *(Isac, G., Bahya,A., O., 2002)* as *the full nuclear cones* family defined as follows: if $K \subset X$ is a normal cone with its *dual cone* defined by $K^* = \left\{x^* \in X^* : x^*(x) \geq 0, \forall x \in K\right\}$ and its corresponding *polar* is $K^0 = -K^*$, then for any mapping $\varphi : P \to K^*$ one says that the set
$K_\varphi = \{x \in X : p(x) \leq \varphi(p)(x), \forall p \in B\}$ is *the full nuclear cone* associated to K whenever $K_\varphi \neq \{\theta\}$.
In fact, K is *supernormal* iff there exists
$\varphi : P \to K^* \setminus \{0\}$ such that $K \subseteq K_\varphi$ (**Poposition 3.1** given by *Isac, G. and Postolică, V., 2005).*Taking into account that in a real normed linear space $(E, \|.\|)$ a non-empty set $T \subset E$ is called *a Bishop-Phelps cone* if there exists y^* in the usual dual space E^* of E and $\alpha \in (0,1]$ such that $T = \left\{y \in E : \alpha\|y\| \leq y^*(y)\right\}$ and *the applications* of such as these cones in *Nonlinear Analysis*, including *the Optimization programs* with *vector-valued objective mappings*, one concludes that, for Haudorff locally convex spaces, *the full nuclear cones are natural generalizations of Bishop-Phelps cones in the normed vector spaces.*
The beginning and the considerations in **Section 4** of *(Isac, G., Bahya, A. O., 2002)* suggested us to consider for each function $\varphi : P \to K^* \setminus \{0\}$ the full nuclear cone $K_\varphi = \left\{x \in X : p_\alpha(x) \leq \varphi(p_\alpha)(x), \forall p_\alpha \in P\right\}$ in order to give the next generalization of **Theorem 7** in (*Isac,G., Bahya, A.O., 2002*), as and extension of the main result indicated by *(Isac, G., Postolică, V., 2005)*, an

important link between *the efficiency*, *the full nuclear cones*, *the strong optimization* and a significant result concerning *the multiple scalarization* for this general concept of the efficiency.

Theorem 2. *If* K *is any Isac's cone in every Hausdorff locally convex space* $(X, P = \{p_\alpha : \alpha \in A\})$*, then*

(i) $MIN(A, K, K_1) = \bigcup_{\substack{a\in A \\ \varphi\in P\to K^*\setminus\{0\}}} SMIN\left(A \cap (a - K - K_1), K_\varphi\right)$;

(ii) $MAX(A, K, K_1) = \bigcup_{\substack{a\in A \\ \varphi\in P\to K^*\setminus\{0\}}} SMAX\left(A \cap (a + K + K_1), K_{-\varphi}\right)$;

(iii)

$eff(A, K, K_1) = \bigcup_{\substack{a\in A \\ \varphi\in P\to K^*\setminus\{0\}}} [SMIN\left(A \cap (a - K - K_1), K_\varphi\right) \cup SMAX(A \cap (a + K + K_1), K_{-\varphi})]$. *for any non-empty subset* K_1 *of K.*

Corollary 2.1. *For every non-empty subset A of any Hausdorff locally convex space ordered by an arbitrary Isac's cone K with its dual cone* K^* *we have*

(i) $MIN(A, K) = \bigcup_{\substack{a\in A \\ \varphi: P\to K^*\setminus\{0\}}} SMIN\left(A \cap (a - K), K_\varphi\right)$;

(ii) $MAX(A, K) = \bigcup_{\substack{a\in A \\ \varphi: P\to K^*\setminus\{0\}}} SMAX\left(A \cap (a + K), K_{-\varphi}\right)$

(iii)

$eff(A, K) = \bigcup_{\substack{a\in A \\ \varphi\in P\to K^*\setminus\{0\}}} [SMIN\left(A \cap (a - K), K_\varphi\right) \cup SMAX(A \cap (a + K), K_{-\varphi})]$.

Important examples and significant comments together with pertinent remarks can be also found in (*Postolică, V., 2014, 2015, 2016).*

1. *Any pointed, convex cone, in every Euclidean space* R^k ($k \in N^*$) is *an Isac's cone*.
2. *In every locally convex space any well-based convex cone is an Isac's cone*.
3. *A convex cone* is *an Isac's cone* in *a normed linear space if and only if* it *is well-based*.
4. Let $n \in N^*$ be *arbitrary fixed* and let *Y* be the space of *all real symmetric (n, n) matrices ordered* by *the pointed, convex cone*

$C = \{A \in Y : x^T Ax \geq 0, \forall x \in R^n\}$. Then, *Y* is *a real Hilbert space* with respect to *the scalar product* defined by *<A, B>= =trace (A. B)* for *all A,* $B \in Y$ and *C* is *well-based by* $B=\{A \in$ **C:** $\langle A, I\rangle = 1\}$ where *I denotes the identity matrix*.

5. *Every pointed, locally* or *weakly locally compact convex cone* in *any Hausdorff locally convex space is an Isac's cone.*
6. *A convex cone is an Isac's cone in a nuclear space (Pietch, A., 1972) if and only if it is a normal cone.*
7. In *any Hausdorff locally convex space* a *convex cone* is *an weakly Isac's cone if and only if* it is *weakly normal*.
8. In $L^p([a, b])$, $(p \geq)$, *the convex cone* $K_p=\{x \in L^p([a, b]) : x(t) \geq 0$ *almost everywhere}* is *an Isac's cone if and only if p=1, being well based in this case* by *the set* $B=\{x \in K_1 : \int_a^b x(t)\,dt = 1\}$. Indeed, *if p>1*, then *the sequence* (x_n) *defined by*

$$x_n(t) = \begin{cases} n^{1/p}, & a \leq t \leq a+(b-a)/2n \\ 0, & a+(b-a)/2n < t \leq b \end{cases} \quad n \in \mathbf{N}$$

converges to 0 in the weak topology but not in the usual norm topology. Therefore, by virtue of Theorem 1, K_p *is not an Isac's cone. Generally, for every p>1,* K_p *has a base* $B=\{x \in K_p : \int_a^b x(t)\,dt = 1\}$ *which is unbounded and* *any cone generated by a closed and bounded set $B_t=\{x\in B$: $\int_a^b |x(t)|^p\, dt \le t\}$ with $t\geq 0$ is certainly an Isac's cone. A similar result holds for $L^p(R)$. Thus, if we consider a countable family (A_n) of disjoint sets which covers R such that $(A_n) = 1$ for all n in N, where μ is the Lebesgue measure, then the sequence (y_n) given by $y_n(t) = 1$ if $t\in A_n$ and $y_n(t) = 0$ for $t\in R\backslash A_n$ converges weakly to zero while it is not convergent to zero in the norm topology. Taking into account the above theorem, it follows that the usual positive cone in $L^p(R)$ is not an Isac's cone if $p>1$, that is, it is not well-based in all these cases. However, these cones are normal for every $p\geq 1$. The same conclusion concerning the non-supernormality is valid for the positive orthant convex cones in the usual Orlicz spaces.*

9. In l^p *($p\geq 1$) equipped with the usual norm* $\|\cdot\|_p$ *the positive cone*

$C_p= \{(x_n)\in l^p: x_n \in 0$ *for all* $n\in N\}$ is also *normal* with respect to *the norm topology*, but *it is not an Isac's cone excepting the case $p = 1$*. Indeed, *for every $p>1$, the sequence (e_n) having 1 at the nth coordinate* and *zeros elsewhere converges to zero in the weak topology, but not in the norm topology* and by *virtue of Theorem 3* it follows that C_p *is not an Isac's cone*. For $p = 1$, C_p is *well-based* by *the set* $B = \{x\in C_1: \|x\|_1 = 1\}$ and *Proposition 5 (Isac, G., 1983) ensures that* it is *an Isac's cone*. If we consider *in this case the locally convex topology in l^1 defined by the seminorms* $p_n((x_k)) = \sum_{k=0}^{n} |x_k|$ *for every (x_k) in l^1 and $n\in N$*, which is *weaker than its usual weak* topology, then *the usual positive cone remains an Isac's cone* with respect to *this topology* (now *it is normal in a nuclear space* and *one applies Proposition 6* of *(Isac, G., 1983)* but *it is not well based*. Taking into account *the concept* of *H-locally convex space* introduced by *Precupanu, T.* in *1969* and defined as *any Hausdorff locally convex space* with *the seminorms satisfying the parallelogram law* and *the property* that *every nuclear space is also a H-locally convex space* with respect to *an equivalent system* of *seminorms (Pietch, A., 1972), the above example shows* that in *a*
H-locally convex space a proper convex cone may be an Isac's cone without to be well-based. Moreover, if we consider in l^2 the H-locally convex topology induced by the seminorms

$$\tilde{p}_n((x_k)) = \left(\sum_{i\geq n} |x_i|^2\right)^{1/2},\ n\in N,\ (x_k)\in \ell^2,$$

then *the convex cone $C_2 = \{(x_k) \in l^2: x_k\in 0$ for all $k\in N\}$ is normal* in *the*
H-locally convex space $(l^2,\{\tilde{p}_n\}_{n\in N})$, but it is not a supernormal cone because *the same sequence (e_k) is weakly convergent to zero* while *$(\bar{p}_n(e_k))$ is convergent to 1 for each $n\in N$* and *one applies again Theorem 3. Another interesting example* of *normal cone* in *a H-locally convex space* which *is not supernormal is the usual positive cone* in *the space $L^2_{loc}(R)$ of all functions from R to C which are square integrable over any finite interval of R, endowed with the system of seminorms* $\{\bar{p}_n : n\in N\}$ defined by $\bar{p}_n(x) = \left(\int_{-n}^{n} |x(t)|^2\, dt\right)^{1/2}$ *for every x in $L^2_{loc}(R)$*. In *this case, the sequence (x_k)* given by:

$$x_k(t) = \begin{cases} 0, & t\in(-\infty, 0)\cup(1/k, +\infty) \\ \sqrt{k}, & t\in[0, 1/k] \end{cases}$$

converges weakly to zero, but *it is not convergent in the H-locally convex topology. The results follows* by *Theorem 3*. It is clear that *every weak topology is a H-locally convex topology* and, *in these cases, the supernormality of convex cones coincides* with *the normality thanks to the Corollary of Proposition 2 in (Isac, G., 1983).*

10. In *the space $C([a, b])$ of all continuous, real valued functions defined on every non-trivial, compact interval $[a, b]$ equipped with the usual supremum norm the convex cone $K = \{x \in C([a, b])$: x is concave, $x(a)=x(b) = 0$ and $x(t)\geq 0$ for all $t\in[a, b]\}$ is an Isac's cone, being well based by the set $\{x\in K: x(t_0) =1\}$ for some arbitrary $t_0\in[a,b]$. The hypothesis that all $x\in K$ are concave is essential for the supernormality.*

11. *The convex cone of all nonnegative sequences* in *the space* of *all absolutely convergent sequences* is *the dual* of *the usual positive cone* in *the space* of *all convergent sequences. Consequently, it has a weak star compact base* and hence *it is a weak star supernormal cone.*

12. In l^∞ or in c_0 *equipped with the supremum norm, the convex cone* consisting of *all sequences having all partial sums non-negative is not normal, hence it is not supernormal*.

13. *In every Hausdorff locally convex space any normal cone is an Isac's cone with respect to the weak topology.*

14. *In every locally convex lattice which is a (L)-space the ordering cone is supernormal* (*see also the Example 7 given by Isac, G*. in *1994*).

15. *If we consider the space of all locally integrable functions* on *a locally compact space Y with respect to a Radon measure μ endowed with the topology induced by the family of seminorms $\{p_A\}$ where $p_A(f)$ =* $\int_A |f(x)| d\mu$ *for every non-empty and compact subset A of Y and every locally integrable function f,* then *the convex cone $K = \{f: f(x)\geq 0, "x\in Y\}$ is supernormal.*

16. *If Z is any locally convex lattice ordered* by *an arbitrary convex cone K and Z^* is its topological dual ordered by the corresponding dual cone K^*, then the cone K is supernormal with respect* to *the locally convex topology defined on Z by the neighbourhood base at the origin $\{[-f, f]°\}_{f\in K^*}$.*

17 *In every regular vector space (E, K)* (*that is, the order dual E^* separates the points of E*) *with the property that E = K - K the convex cone K is supernormal with respect to the topology defined in the preceding example*.

18. *Any semicomplete cone in a Hausdorff locally convex space is an Isac's cone (for this concept* see *the Example 11* of *Isac, G., 1994*).

Remark 10. Clearly, *if a convex cone K is supernormal in a normed space*, then *K admits a strictly positive, linear and continuous functional*, that is, *there exists a linear, continuous functional f* such that *f(k)>0 for all $k\in K\backslash\{0\}$.Generally, the converse is not true even in a Banach space as we can see in the following examples*:

19. *If one considers in the usual space l^p* $(1 \leq p \leq \infty)$ *the convex cone*

$K = \ell^p_+ = \{x = (x_i) \in \ell^p : x_i \geq 0 \text{ for every } i \in N\}$ *of infinite vectors with non-negative components, then the functional φ defined by* $\varphi(k) = \sum_{i=1}^{\infty} k_i$ *for any $k = (k_i) \in l^p$ is linear, continuous and strictly positive. But, as we have seen in the above considerations (Example 9), this cone is supernormal if and only if p = 1.*

20. *Let K be the usual positive cone $L^p_+ = \{x\in L^p([a, b]):x(t)\geq 0$ almost everywhere$\}$ in $L^p([a, b])$ $(1\leq p\leq\infty)$. Then, the linear and continuous functional ψ on $L^p([a, b])$ given by* $\psi(x) = \int_a^b x(t)dt$ *for every $x\in L^p([a, b])$ is strictly positive on K while K is supernormal (see the above Example 8) if and only if p = 1.*

Therefore, l^1_+ and L^1_+ are *supernormal cones* with *empty topological interiors* and for *every $p\in(1,+\infty)$ it follows that l^p_+ and L^p_+ are normal cones with empty interiors which are not supernormal. Hence, these convex cones are not well based.* A *very simple example* of *supernormal cones having non-empty topological interior is R_+^n $(n\in N^*)$.*

Remark 11. In *the order complete vector lattice B([a, b])* of *all bounded, real valued* functions on *a compact non-singleton interval [a, b]* endowed with *its usual norm the standard positive cone $K = \{u\in B([a, b]): u(t)\geq$ for all $t\in([a, b])\}$ is normal but it has not a base*, that is, *it is not supernormal*. However*, this cone has non-empty interior*. *If we consider the linear space l^1 endowed with the separated locally convex topology generated by the family $\{p_n:n\in N\}$ of seminorms defined by* $p_n(x) = \sum_{k=0}^{n} |x_k|$ *for every $x = (x_k) \in l^1$*, then *the convex cone $K = \{ x = (x_k) \in l^1: x_k\geq 0$ whenever $k\in N\}$ is supernormal* but *it is not well based*.

IV. Some conclusions

The natural context of supernormality (nuclearity) *for convex cones* is *any separated locally convex space*. *Isac, G. introduced the concept of "nuclear cone" in 1981, published it* in *1983* and *he showed that in a normed space a convex cone is nuclear* if and only if it is *well based* or equivalently *iff* it is *"with plastering",* the last concept being defined by *Krasnoselski, M. A.* in *fifties (see ,for example , Krasnoselski, M. A., 1964 and so on).* Such a convex cone was initially called *"nuclear cone"* by *Isac, G. (1981)* because *in every nuclear space (Pietch, A., 1972) any normal cone is a nuclear cone in Isac's sense* (*Proposition 6 of Isac, G., 1983).* Afterwards, *since the nuclear cone introduced by Isac appears as a reinforcement of the normal cone*, it was called *supernormal. The class of supernormal cones* in *Hausdorff locally convex spaces was initially imposed by the theory and the applications* of *the efficient (Pareto minimum type) points (especially existence conditions based on completeness instead of compactness were decisive together with the main properties of the efficient points sets), the study of critical points for dynamical systems and conical support points and their importance was very well illustrated by important results, examples and comments in the specified references and in other connected papers.* It is also *very significant to mention again* that *the concept of supernormality introduced by Isac, G.* (*1981) is not a simple generalization of the corresponding notion defined in normed linear spaces by Krasnoselski, M. A. and his colleagues in the fifties*. Thus, for example, *Isac's supernormality attached to the convex cones has his sense in every Hausdorff locally convex space identically with the well known Grothendieck's nuclearity*. By *analogy* with *the fact* that *a normed space is nuclear in Grothendieck's sense if* *and only if it is isomorphe with an usual Euclidean space, a convex cone is supernormal in a normed space if and only if it is well based,* that is, *it is generated by a convex bounded set which does not contain the origin in its closure. Beside Pareto type optimization, we also mention Isac's significant contributions, through the agency of supernormal cones, to the convex cones in product linear spaces and Ekeland's variational type principles (Isac.G.,2003;Isac,G.,Tammer,Chr., 2003).* Therefore*, the more appropriate background for Isac's cones is any separated locally convex space.* On the other hand*, the extension in real or complex vector spaces of this notion can be done since any real or complex vector space can be considered as a locally convex space with respect to the locally convex topology generated by the family of the seminorms originated in the Minkowski functionals associated to the algebraic and geometric concepts of the convex hulls for the balanced (circled) and absorbing sets containing its origin (Postolică,V., 2014*)*. The behaviour of Isac's cones in some variable locally convex topologies, especially in duality, is also interesting (Postolică,V., 2015*)*.

Definition 4. *(Postolică,V., 1995)***.** *A nonempty set B of a topological vector space X ordered by a convex cone K is called K - bounded if there exists $B_0 \subseteq X$ bounded so that $B \subseteq B_0 \pm K$ and K - closed if its conical extension $B \pm \overline{K}$ is closed, where $\overline{K}$ signifies the topological closure of K.*

A synthesis concerning the existence and significant properties of the efficient points are given in the next theorem.

Theorem 4. *(Postolică, V., 1995,1997, 2008)***.**

(i) if K is an Isac's cone in a Hausdorff locally convex space X, A, $B \subseteq X$ are non-empty subsets positioned by $A \subseteq B \subseteq A + K$ ($A \subseteq B \subseteq A - K$) and $B \cap (A_0 - K)$ ($B \cap (A_0 + K)$) is a bounded and complete set for at least one non-empty set $A_0 \subseteq A$, then $MIN(A,K) \neq \emptyset$ ($MAX(A,K) \neq \emptyset$);

(ii) if X is a quasi-complete separated locally convex space, that is,any non-empty bounded and closed subset is complete and K is a closed Isac's cone in X, then for any K-bounded and K-closed non-empty set A of X we have $MIN(A,K) \neq \emptyset$, $A \subseteq MIN_K(A) + K$ ($A \subseteq MAX(A) - K$)(the domination property), $MIN_K(A) + K = A + K$ ($MAX(A,K) - K = A - K$) and $MIN(A,K)$ ($MAX(A,K)$) is K-bounded and K-closed;

(*iii) according to the same hypotheses from (ii), $MIN(A,K) \neq \emptyset$ ($MAX(A,K) \neq \emptyset$) for any non-emptyl subset A with $B \cap (A_0 - K)$ ($B \cap (A_0 + K)$) K-bounded and K-closed set whenever $A_0 \subseteq A \subseteq X$ and $A \subseteq B \subseteq A + K$ ($A \subseteq B \subseteq A - K$),with the corresponding implications for $Eff(A,K)$.*

V. Efficiency and Choquet Boundaries

Definition 5.*(Postolică,V.,2008) A real function $f : X \to R$ is called $(K + K_1)$ - increasing($K + K_1$ - decreasing) if $f(x_1) \geq f(x_2)$ whenever $x_1, x_2 \in X$ and $x_1 \in x_2 + K_1 + K$ ($x_1 \in x_2 - K_1 - K$).*

Obviously, every real increasing (decreasing) function defined on any linear space ordered by an arbitrary convex cone K is $K+K_1$-increasing ($K+K_1$-decreasing), for each non-empty subset K_1 of K .

The next coincidence of the efficient points sets and the Choquet boundaries generalizes the main result given by *Bucur, I. and Postolică, V., 1994,* and can not be obtained as a consequence of the Axiomatic Potential Theory.

Theorem 5. *(Postolică,V., 2008, 2009)***.** *If A is any non-void, compact subset of every Hausdorff linear topological vector space X and*

(i) K is an arbitrary, closed, convex, pointed cone in X;

(ii) K_1 is a non-empty subset of K such that $K + K_1$ is closed with respect to the separated topology on X.

Then, $MIN(A, K, K_1)$ ($MAX(A, K, K_1)$) coincides with the Choquet boundary $\partial_{S_1} A$ ($\partial_{S_2} A$) of A with respect to the convex cone S_1 (S_2) of all $K + K_1$ - increasing ($K + K_1$ - decreasing) real continuous functions on A. Consequently, each of these sets, endowed with the corresponding trace topology, is a Baire space and a G_δ - subset of X whenever (A, τ_A) is metrizable. Moreover, $eff(A, K, K_1) = \partial_{S_1} A \cup \partial_{S_2} A$)

Corollary 5.1. *(i) If for every $f \in C(A)$ one denotes $\overline{f}(a) = \sup\{f(a') : a' \in A \cap a + K + K_1\}$ and $\tilde{f}(a) = \sup\{f(a'') : a' \in A \cap (a - K - K_1\}$, then*

$$eff\,(A, K, K_1) = \bigcup_{a \in A} \left\{ a \in A : f(a) = \overline{f}(a)(\tilde{f}(a)), \forall f \in C(A) \right\};$$

(ii) $MIN\,(A, K, K_1)(MAX\,(A, K, K_1), MIN\,(A, K, K_1)((MAX\,(A, K, K_1)) \cap \{a \in A : s(a) \le 0\}(s \in S_1)$ *are compact sets with respect to the Choquet's topology and as usual compact subsets of A.*

Remark 12. Generally, $MIN\,(A, K, K_1)(MIN\,(A, K, K_1)$ coincides with the corresponding Choquet boundary of A only with respect to the convex cone of all real, continuous and $K + K_1$-increasing ($K + K_1$-decreasing) functions on A. Thus, for example, if A is any non-empty, compact and convex subset of every Hausdorff locally convex space and $S = \{ f : A \to R \,/\, f$ *is continuous and concave*$\}$, then its Choquet boundary with respect to S coincides with the set $ex(A)$ of all extreme points x in A, that is, if $y, z \in A$ and there exists $\alpha \in (0,1)$ with $\alpha y + (1 - \alpha) z = x$, then $y = z = x$. The hypothesis of concavity imposed on the functions is essential for the validity of this result. Generally, even in the finite dimensional cases, an extreme point for a compact convex set is not necessary an efficient point and conversely.

Remark 13. The largest class ζ of convex cones ensuring the existence for the efficient points in all non-empty compacts subsets of every separated topological vector space was defined by *Sterna – Karwat, A., 1986* as follows: if *V* is an arbitrary Hausdorff topological vector space, a convex cone *C* belongs to ζ iff for every closed vector subspace *L* of *V*, $C \cap L$ is a vector subspace whenever its closure $\overline{C \cap L}$ is a vector subspace. As we have already specified before Theorem 5, there exists more general conditions than compactness imposed upon a non-empty set A in a separated locally convex space ordered by a convex cone K ensuring that $eff\,(A, K) \ne \varnothing$. Perhaps our coincidence result suggests a natural extension of the Choquet boundary at least in these cases. Anyhow, Theorem 5 represents an important link between Vector Optimization and Potential Theory and a new way for the study of the properties of efficient points sets and the Choquet boundaries. Indeed, one of the main question in Potential Theory is to find the Choquet boundaries. This fact is relatively easy for particular cases but, in general, it is an unsolved problem. Since in a lot of cases the efficient points sets contain dense subsets which can be identified by adequate optimization methods, it is possible to determine the corresponding Choquet boundaries in all these situations. Consequently, our coincidence result has its practical consequences at first for the Axiomatic Theory of Potential and its applications. At the same time, by the above coincidence result, the Choquet boundaries offer important properties for the efficient points sets.

VI. Best Approximation in Locally Convex Spaces. Efficiency by Splines in H-locally Convex Spaces

We conclude *this research report* with some *topics on the best approximation* (*simultaneous* and *vectorial*) in locally convex spaces and *the efficiency (optimization) by splines in H-locally convex spaces*.

Let *X be a Hausdorff locally convex space with the topology induced by a family* $P = \{p_a : a \in I\}$ *of seminorms*, $x_0 \in X$ and *G a non-empty subset of X*.

Definition 6. (*Isac, G., Postolică, V.,1993)* $g_0 \in$ G *is said to be a best simultaneous approximation for* x_0 *by the elements of* G *with respect the family* P *(abbreviated* g_0 *is a* P *-b.s.a. of* x_0*) when*

$$p_a(x_0 - g_0) \le p_a(x_0 - g) \text{ for all } g \in G \text{ and } p_a \in P.$$

If, in addition, each element $x \in X$ *possesses at least one* P *-b.s.a. in* G*, then the set* G *is called* P *-simultaneous proximinal.*

Definition 7. *(Isac, G., Postolică, V., 1993)* $g_0 \in$ G *is said to be a best vectorial approximation of* x_0 *by* G *with respect to* P *(abbreviated* g_0 *is a*

P *-b.v.a. of* x_0*) if* $(p_a(x_0 - g_0)) \in MIN(\{(p_a(x_0 - g)) : g \in G\}, K)$ *where* $K = R_+^I$.

If, in addition, each element $x \in X$ *possesses at least one* P *-b.v.a. in* G*, then the set* G *is called* P *-vectorial proximinal. Several original results and properties concerning these notions were given in our above mentioned book.*

Concerning *the H-locally convex spaces, it is known that this concept of was introduced* and *studied for the first time* by *Precupanu, T. (1969)* and *defined as any Hausdorff locally convex space with the seminorms satisfying the parallelogram law.* At the same time, *we introduced the notion of spline function in H-locally convex space (Postolică, V., 1981)* and *we established the basic properties of approximation and optimal interpolation for these splines. Our splines are natural extensions in H-locally convex spaces of the usual abstract splines which appear in any Hilbert space like the minimizing elements for a seminorm subject to the restrictions given by a set of linear continuous functionals.*

So, let now *$(X, P = \{p_a : a \in I\})$ be a H-locally convex space* with *each seminorm p_a being induced by a scalar semiproduct $(.,.)_a$ $(a \in I)$* and *M a closed linear subspace* of *X for which there exist a H-locally convex space $(Y,Q = \{ q_a : a \in I\})$* with *each seminorm $q_a \in Q$ generated by a scalar semiproduct $<.,.>_a$ $(a \in I)$* and *a linear* (*continuous*) *operator $U:X \to Y$ such that $M=\{x \in X: (x,y)_a = <Ux,Uy>_a, \forall \ a \in I\}$.*

The space of spline functions with respect to U was defined by Postolică, V.(1981) as the U-orthogonal of M, that is,

$M^{\perp} = \{x \in X: <Ux,Uz>_a = 0, \forall \ z \in M, a \in I\}$. Clearly, $M^{\perp}$ *is the orthogonal of M in the H-locally convex sense.*

Let us consider *the direct sum $X' = M \oplus M^{\perp}$* and *for every $x \in X'$, we denote its projection onto $M^{\perp}$ by s_x.* Then, taking into account *the Theorem 4* obtained by *Postolică, V.* in *1981*, it follows that *this spline is a best simultaneous U-approximation of x with respect to $M^{\perp}$* since *it satisfies all the next conditions* : $p_a (x - s_x) \le p_a(x - y) \ \forall \ y \in M^{\perp}, p_a \in P$.

Moreover, *following the definition of the approximate efficiency*, *the results given in Chapter 3* of *(Isac, G., Postolică, V., 1993)* and *the conclusions obtained* by *(Postolică, V.,1981,1993, 1998)*, we have

Theorem 6.

(i) for every $x \in X'$ *the only elements of best simultaneous and vectorial approximation with respect to any family of seminorms which generates the* H*-locally convex topology on* X *by the linear subspace of splines are the spline functions* s_x. *Moreover, if* M *and* $M^{\perp}$ *supply an orthogonal decomposiion for* X, *that is* $X = M \oplus M^{\perp}$, *then* $M^{\perp}$ *is simultaneous and vectorial proximinal;*

(ii) if $K = R_+^I$, *then for each* $s \in M^{\perp}$, *every* $s \in M^{\perp}$ *is the only solution of following optimization problem* $MIN(\{(q_a(U(h-s))): h \in X' \text{ and } h - s \in M\}, K)$;

(iii) for every $x \in X'$ *its spline function* s_x *is the only solution for the next vectorial optimization problems:*

$$MIN(\{(q_a(U(h-x))): h \in M^{\perp}\}, K), MIN(\{(p_a(x-y)): y \in M^{\perp}\}, K), MIN(\{(q_a(Uy)): y-x \in M\}, K).$$

Finally, let us consider *two numerical examples* in which, following *Postolică,V.,(1981), Isac, G., Postolică, V.,(1993) and Postolică, V., (1998),* we specify *the expressions of splines* and *M and $M^{\perp}$ realize orthogonal decompositions.*

Example 1. *Let $X=H^m(R)=\{f \in C^{m-1}(R): f^{(m-1)}$ is locally absolutely continuous and $f^{(m)} \in L^2_{loc}(R)\}$, $m \ge 1$ endowed with the H-locally convex topology generated by the scalar semiproducts*

$$(x,y)_k = \sum_{h=0}^{m-1} [x^{(h)}(k)\, y^{(h)}(k) + x^{(h)}(-k)\, y^{(h)}(-k)] + \int_{-k}^{k} x^{(m)}(t)\, y^{(m)}(t)dt, \ k=0,1,2, \ldots \text{ and}$$

$Y = L^2_{loc}(R)$ with the H-locally convex topology induced by the scalar semiproducts $<x,y>_k = \int_{-k}^{k} x(t)\, y(t)dt$, $k=0,1,2, \ldots$

If $U:X \to Y$ is the derivation operator of order m, then

$M=\{x \in H^m(R): x^{(h)}(n)=0, \ \forall \ h=\overline{0, m-1}, n \in Z\}$

and $M^{\perp} = \{s \in H^m(R): \int_{-k}^{k} s^{(m)}(t)\, x^{(m)}(t)dt, \ \forall \ x \in M, k=0,1,2, \ldots\}$

We proved in (Postolică, V., 1981) that

$M^{\perp} = \{s \in H^m(R): s_{/(n,n+1)}$ is a polynomial function of degree 2m-1 at most$\}$ and if $y=(y_n)$, $y' = (y'_n)$, $y'' = (y''_n)$, $y^{(m-1)} = (y^{(m-1)}_n)$ are m sequences of real numbers, then there exists an unique spline $S \in M^{\perp}$ satisfying the following conditions of interpolation: $S^{(h)}(n) = y^{(h)}(n)$ whenever $h=\overline{0, m-1}$ and $n \in Z$. Moreover, we observed in the paragraph 3 of (Isac, G., Postolică, V., 1993) that any spline function S such as this is defined by

$$S(x) = p(x) + \sum_{h=0}^{m-1} c_1^{(h)}(x-1)_+^{2m-1} + \sum_{h=0}^{m-1} c_2^{(h)}(x-2)_+^{2m-1} + \ldots + \sum_{h=0}^{m-1} c_0^{(h)}(-x)_+^{2m-1} + \ldots$$

where $u_+=(|u|+u)/2$ for every real number u, p is a polynomial function of degree 2m-1 at most perfectly determined by the conditions $p^{(h)}(0)= y_0^{(h)}$ and $p^{(h)}(1)= y_1^{(h)}$ for all $h=\overline{0, m-1}$ and the coefficients $c_n^{(h)}$ ($h=\overline{0, m-1}$, $n \in Z$) are successively given by the general interpolation.

Therefore, for every function $f \in H^m(R)$, there exists an unique function denoted by $S_f \in M^{\perp}$ such that $S_f^{(h)}(n)= f^{(h)}(n)$, $\forall$ $h=\overline{0, m-1}$ and $n \in Z$. Hence, in this case, M and $M^{\perp}$ give an orthogonal decomposition for the space $H^m(R)$.

Example 2. *Let $X=F_m=\{f \in C^{m-1}(R): f^{(m-1)}$ is locally absolutely continuous and $f^{(m)} \in L^2(R)\}$ endowed with the H-locally convex topology induced by the scalar semiproducts*

$(x,y)_n=x(n)y(n)+ \int_R x^{(m)}(t)\, y^{(m)}(t)dt$, $n \in Z$, $Y= L^2(R)$ with the topology generated by the inner product $(x,y)_n= \int_R x(t)\, y(t)dt$, $n \in Z$ and $U:X \to Y$ be the derivation operator of order m. Then, $M= \{x \in F_m: x(n)=0$ for all $n \in Z\}$ and

$$M^{\perp} =\{ s \in F_m: \int_R x^{(m)}(t)\, y^{(m)}(t)dt=0 \text{ for every } x \in M\}.$$

In a similar manner as in Example1 it may be proved that $M^{\perp}$ coincides with the class of all piecewise polynomial functions of order 2m (degree 2m-1 at most) having their knots at the integer points. Moreover, for every function f in F_m there exists an unique spline function $S_f \in M^{\perp}$ which interpolates f on the set Z of all integer numbers, that is, S_f satisfies the equalities $S_f(n)=f(n)$ for every $n \in Z$, being defined by

$$S_f(x)= p(x)+ a_1(x-1)_+^{2m-1}+ a_2 (x-2)_+^{2m-1}+ \ldots +a_0(-x)_+^{2m-1}+ a_{-1}(-x-1)_+^{2m-1}+ \ldots$$

where u_+ has the same signification as in Example1, the coefficients a_n ($n \in Z$) are successively and completely determined by the interpolation conditions $S_f(n)=f(n)$, $n \in Z\backslash\{0, 1\}$ and p is a polynomial function satisfying the conditions $p(0)=f(0)$ and $p(1)=f(1)$. The uniqueness of S_f is ensured in Theorem 2 given by (Postolică, V., 1981).

Thus, M and $M^{\perp}$ give an orthogonal decomposition of the space F_m and, as in the preceding example, $M^{\perp}$ is simultaneous and vectorial proximinal with respect to the family of seminorms generated by the above scalar semiproducts.

Remark 14. *Our examples show that the abstract construction of splines can be used to solve also several frequent problems of optimal interpolation and approximation, having the possibility to choose the spaces and the scalar semiproducts. It is obvious that for a given (closed) linear subspace of a H-locally convex space X such a H-locally convex space Y (respectively, a linear (continuous) operator $U:X \to Y$) would not exist. Otherwise, the problem of best vectorial approximation by the corresponding orthogonal space of any (closed) linear subspace M for the elements in the direct sum $M \oplus M^{\perp}$ might be always reduced to the best simultaneous approximation. But, in general, such a possibility doesn't exist. Even in a H-locally convex space it is possible that there exist best vectorial approximations and the set of all best simultaneous approximations to be empty for some element of the space. We confine ourselves to mention the following simple example.*

Example 3. *Let $X= R^N$ endowed with the topology generated by the family $P =\{p_i : i \in N\}$ of seminorms defined by $p_i(x)= |x_i|$ ($i \in N$) for every*

$x= (x_i) \in X$ and $G=\{(x_i) \in X: x_i \geq 0$ whenever $i \in N$ and $\sum_{i \in N} x_i=1\}$

X is a P -simultaneous strictly convex (Isac, G., Postolică, V., 1993)
H-locally convex space. Nevertheless, every element of G is P -b.v.a. for the origin while its corresponding set of the best simultaneous approximations with respect to P is empty.

VII. Some Open Problems

The above context of research suggests immediately the following open problems.

Op. 5.1. If $eff\,(A,K) \neq \varnothing$ there exist a Hausdorff locally convex space Y, an Isac's cone K_0 in Y and a non-empty set $A_0 \subset Y$ with $eff\,(A,K) = eff\,(A_0,K_0)$ (or, at least, $eff\,(A,K)$ to be dense in $eff\,(A_0,K_0)$)?

Op. 5.2. If $eff\,(A,K) \neq \varnothing$, there exist a separated locally convex space X_1, a (pointed), convex cone K_1 in X_1 and a compact set $A_1 \subset X_1$ such that $eff\,(A,K) = eff\,(A_1,K_1)$ (at least $eff\,(A,K)$ to be dense in $eff\,(A_1,K_1)$ or conversely) ?

Op. 5.3. If T is a Hilbert space, K is a closed, convex, pointed cone in T and A is a non-empty, closed, convex subset of T does $eff\,(A,K)$ preserve the property of coincidence with the corresponding Choquet boundary as in the above theorem?

Op. 5.4. The same question for each of the following cases:

(i) T is a quasi-complete locally convex space, K is an Isac's cone , A is a K - bounded and K - closed set in T *(Postolică, V., 1995)*.

(ii) T is a quasi-complete locally convex space, the closure $\overline{K}$ of K has the properties given in *(Truong, X., D. H., 1994)* and A is a K - bounded and K - closed subset in T.

VIII. Future research directions

This research work can also be considered as a study in which the approximate solutions including those usual to vector optimization programs in ordered (topological) vector spaces are investigated. The listed properties of these solutions allow developments. Several new facts about the relationships between the approximate efficiency and the scalarized problems can be formulated. As we have specified , the family of Isac's cones represent the largest class of ordering cones in Hausdorff locally convex spaces ensuring the existence and the adequate properties of the efficient points sets involved in optimization, following different completeness types instead of compactness. Consequently, one of the main goal of the next research is to identify new applications of Isac's cones. The given strong connection by coincidence between the approximate solutions and Choquet's boundaries for non-empty compact sets can be extended to non-compact sets.

Target Audience

Operations Research, Multicriteria Decision Making, Mathematical Modeling.

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