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Unsteady Free-Convective Flow in a Vertical Annulus: A Fourier approximation approach

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Abstract
In this work, a semi-analytical approach is employed to investigate flow formation of time dependent free convection flow in a vertical annulus. The governing momentum and energy equations governing the flow are derived and closed form expressions are obtained in Laplace domain while the Fourier approximation is used in transforming the solutions from Laplace domain to time domain. During the course of investigation of flow formation with time, nature of fluid and ratio of radiuses, it is found that these governing parameters play a major role in the attainment of steady state solution.
Keywords: *Annulus; Fourier approximation; Free convection; Asymmetric heating.*

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I. Introduction

The study of heat transfer in annular geometry is of great importance because of the frequent need to either increase or decrease the rate at which heat flows between two points in the annulus. Flow in annulus is significant in drilling and exploration processes, study of blood flow, transpiration in plants and many other cylindrical designs. On the other hand, natural convection flow in annular geometry has received more attention due to it physical and industrial cooling application; these include melting of ice, heat exchanger, steam turbine, radiators, cooling circuit and air condition. Many researchers over the years have studied flow formation in annular geometry using different numerical technique. Some related works include El-Shaarawi and Sarhan [1] who reported the fully developed free convection flows in vertical annulus with isothermal and adiabatic boundaries. In other related article, Joshi [2] studied the fully developed free convection flows in vertical annulus by considering two isothermal boundaries, the inner of which is kept at a higher temperature than the outer one. Later, Lien *et al.* [3] investigated the effect of free convection and mass transfer on the flow past an impulsively moving infinite vertical circular cylinder. They found that that there is a rise in the velocity due to the presence of a foreign mass. Also, Asmaa [4] conducted an investigation to find the effect of free convection on the developing forced laminar upward and downward flow in a vertical annulus when the inner is heated uniformly and the motionless outer cylinder is kept adiabatically insulated. He concluded that the growth of the thermal boundary layer along the annulus increases as heat flux increases and Reynolds number decreases. Other related work on natural convection flow in annulus considering different physical phenomena can be found in Mahmud and Fraser [5], Leong and Lai [6] as well as Jha *et al.* [7].

Several studies have used numerical method in studying time dependent flow formation in annular geometry due to the complexity of the exact solution, but few studies talked about the influence of ratio of radiuses of the two cylinders which is of great importance in design engineering. Therefore, the objective of this article is to investigate the role of annular gap (ratio of radiuses) on fluid velocity, Nusselt number and skin-friction at the walls for air and water for both transient state and steady state.

II. Mathematical formulation

Let's consider an unsteady free-convective flow of an incompressible and viscous fluid between two vertical cylinders with asymmetric wall heating at the surfaces of the cylinders. The radiuses of the inner and outer cylinders are respectively a and b. Initially, the temperature of the fluid as well as the inner surface of outer cylinder maintained an ambient temperature T_0. After some time $(t' > 0)$, the temperature of the outer surface of the inner cylinder is raised to $T_w > T_0$. This temperature difference at the walls leads to density difference of fluid in the annulus and hence sets up natural convection. The flow is assumed to be thermally and hydrodynamically fully developed. In addition, the $r - axis$ is the axis in radial direction while the $z - axis$ is the axial coordinate which is parallel to the gravitational acceleration g but in opposite direction (see Fig. 1).
Following the above assumptions and using Boussinesq's approximation, the governing momentum and energy equations governing the flow in dimensional form are given below respectively:

$$\frac{\partial u}{\partial t'} = \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right] + g\beta(T - T_0) \quad (1)$$

$$\frac{\partial T}{\partial t'} = \alpha\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right] \quad (2)$$

Subject to the following boundary conditions:

$$t' \le 0: \quad u = 0 \quad T = T_0; \quad \text{for } a \le r \le b$$

$$t' \ge 0\begin{cases} u = 0 \quad T = T_w \quad \text{at} \quad r = a \\ u = 0 \quad T = T_0 \quad \text{at} \quad r = b \end{cases} \quad (3)$$

The above equations in dimensionless form are derived using the following dimensionless quantities:

$$R = \frac{r}{a}, \ Pr = \frac{\nu}{\alpha}, \quad t = \frac{t'\nu}{a^2}, \ \lambda = \frac{b}{a}, \quad \theta = \frac{(T - T_0)}{(T_w - T_0)}, \ U = \frac{u\nu}{g\beta a^2 (T_w - T_0)} \quad (4)$$

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R} + \theta \quad (5)$$

$$\frac{\partial \theta}{\partial t} = \frac{1}{Pr}\left[\frac{\partial^2 \theta}{\partial R^2} + \frac{1}{R}\frac{\partial \theta}{\partial R}\right] \quad (6)$$

Subject to the following boundary conditions:

$$t \le 0: \quad U = 0 \quad \theta = 0; \quad \text{for } 1 \le R \le \lambda$$

$$t \ge 0\begin{cases} U = 0 \quad \theta = 1 \quad \text{at} \quad R = 1 \\ U = 0 \quad \theta = 0 \quad \text{at} \quad R = \lambda \end{cases} \quad (7)$$

Using the Laplace transform technique, closed form expressions in Laplace domain are obtained for fluid temperature, velocity, rate of heat transfer and skin-friction respectively as:

$$\theta(R,S) = C_1 I_0\left(R\sqrt{SPr}\right) + C_2 K_0\left(R\sqrt{SPr}\right) \quad (8)$$

$$U(R,S) = C_3 I_0\left(R\sqrt{S}\right) + C_4 K_0\left(R\sqrt{S}\right) + \frac{\left[C_1 I_0\left(R\sqrt{SPr}\right) + C_2 K_0\left(R\sqrt{SPr}\right)\right]}{S(1 - Pr)} \quad (9)$$

$$Nu_1 = -\left.\frac{d\theta(R,S)}{dR}\right|_{R=1} = \sqrt{SPr}\left[C_2 K_1\left(\sqrt{SPr}\right) - C_1 I_1\left(\sqrt{SPr}\right)\right] \quad (10)$$

$$Nu_\lambda = -\left.\frac{d\theta(R,S)}{dR}\right|_{R=\lambda} = \sqrt{SPr}\left[C_2 K_1\left(\lambda\sqrt{SPr}\right) - C_1 I_1\left(\lambda\sqrt{SPr}\right)\right] \quad (11)$$

$$\tau_1 = \left.\frac{dU(R,S)}{dR}\right|_{R=1} = \sqrt{S}\left[C_3 I_1\left(\sqrt{S}\right) - C_4 K_1\left(\sqrt{S}\right)\right] + \frac{\sqrt{Pr}\left[C_1 I_0\left(\sqrt{SPr}\right) + C_2 K_0\left(\sqrt{SPr}\right)\right]}{\sqrt{S}(1 - Pr)} \quad (12)$$

$$\tau_\lambda = -\left.\frac{dU(R,S)}{dR}\right|_{R=\lambda} = \sqrt{S}\left[C_4 K_1\left(\lambda\sqrt{S}\right) - C_3 I_1\left(\lambda\sqrt{S}\right)\right] - \frac{\sqrt{Pr}\left[C_1 I_0\left(\lambda\sqrt{SPr}\right) + C_2 K_0\left(\lambda\sqrt{SPr}\right)\right]}{\sqrt{S}(1 - Pr)} \quad (13)$$

The volumetric flow rate which measures the amount of fluid passing through the annulus in dimensionless form is given by:

$$Q = \int_1^\lambda RU(R,S)dR = q_1 + q_2 + q_3 + q_4 \quad (14)$$

where I_0, K_0 are modified Bessel's function of first and second kinds respectively with order zero and C_1, C_2, C_3, C_4 and q_1, q_2, q_3, q_4 are constant defined in appendix.

Due to the complexity of the closed form expression obtained from Eqs. 8-14 in Laplace domain, a numerical procedure used in [7-9] which is based on the Fourier approximation to invert the solutions from Laplace domain to time domain is used. This method is based on the Bromwich contour inversion integral, which can be expressed as the integral of a real valued function of a real variable by choosing a specific contour. One first converts the inversion integral into the Fourier transform and then approximates the transform by a Fourier series (use trapezoidal rule) with a specific discretization error. In this method, any function in the Laplace domain can be inverted to the time domain as follows.

$$U(R,t) = \frac{e^{\varepsilon t}}{t}\left[\frac{1}{2}U(R,\varepsilon) + Re\sum_{n=1}^{M} U\left(R, \varepsilon + \frac{in\pi}{t}\right)(-1)^n\right], 1 \le R \le \lambda \quad (15)$$

Where Re refers to the real part of $i = \sqrt{-1}$ the imaginary number. M is the number of terms used in the Fourier approximation and ε is the real part of the Bromwich contour that is used in inverting Laplace transforms. The Riemann-sum approximation for the Laplace inversion involves a single summation for the numerical process its accuracy depends on the value of ε and the truncation error dictated by M. According to Tzou [10], the value of εt that best satisfied the result is 4.7. In addition, it has been shown by [11-15] that the Fourier approximation approach to Laplace inversion is a promising technique for obtaining high accuracy with exact solution for large value of n (in this work, the value of n with high accuracy is $n = 2000$)

Steady state solution

The accuracy of the Fourier approximation approach in Eqs. $(8 - 14)$ is validated by computing the steady-state solution for velocity, temperature, skin-friction and Nusselt number. This is obtained by taking $\frac{\partial()}{\partial t} = 0$ in Eqs. (5) and (6) which on solving gives the following exact solution:

$$\theta_s(R) = 1 - \frac{\ln(R)}{\ln(\lambda)} \quad (20)$$

$$U_s(R)=\frac{(R^2-1)\left[\ln\left(R/_{\lambda}\right)-1\right]}{4\ln(\lambda)}+\frac{(\lambda^2-1)\ln(R)}{4[\ln(\lambda)]^2} \quad (21)$$

$$Nu_1=\frac{d\theta(R)}{dR}\bigg|_{R=1}=\frac{1}{\ln(\lambda)} \quad (22)$$

$$Nu_\lambda=-\frac{d\theta(R)}{dR}\bigg|_{R=\lambda}=\frac{-1}{\lambda\ln(\lambda)} \quad (23)$$

$$\tau_{1,s}=\frac{(\lambda^2-1)}{4[\ln(\lambda)]^2}-\frac{(1+\ln(\lambda))}{2\ln(\lambda)} \quad (24)$$

$$-\tau_{\lambda,s}=\frac{\lambda}{2\ln(\lambda)}-\frac{[\lambda^2-1]}{4\lambda\ln(\lambda)}-\frac{(\lambda^2-1)}{4\lambda[\ln(\lambda)]^2} \quad (25)$$

It is expected that the steady state solution should correspond to the transient state solution at large dimensional time t. Also, steady state solution is independent on working fluid (Pr).

III. Results And Discussion

The solution obtained for the set objective is seen to be influenced by Prandtl number (Pr) and radius of radiuses (λ). In this present study, we consider 1.5$\leq \lambda \leq$ 2.5; to capture the cases when the ratio of the bigger cylinder radius is 1.5, 2.0 and 2.5 times that of inner cylinder, air ($Pr = 0.71$) and water ($Pr = 7.0$) as working fluid since most coolant is either water or air (e.g. air-condition, automobile radiators). The accuracy of the solutions obtained by Fourier series approach is established by comparing the steady state solution with those obtained by Fourier series approach at large time. This comparison gives an excellent agreement (see Table).

Figures 2a and b depict temperature profiles for different values of ratio of radiuses (λ) and time (t) for air and water respectively. It is observed for both fluids considered that fluid temperature increases with increase in t and λ. This can be attributed to the fact that continuous heating of surface of the cylinder transfers heat to the fluid thereby increasing the temperature of the fluid until a steady state is attained. In addition, it is found that decrease in λ speeds up the attainment of steady state temperature.

Figures 3a and 3b on the other hand exhibit velocity profiles for different values of λ and t for air and water respectively. From this figures, it is evident that fluid velocity at steady state is independent on the kind of fluid considered. Also, as time and ratio of radiuses increase, fluid velocity also increases. It is clear from this figures that attainment of fluid velocity steady state is dependent on λ and Pr (type of fluid considered). This can be explained by the fact that as Pr increases, kinematic viscosity also increases which in turn retards fluid motion until steady state is attained.

Figures 4a and 4b present the rate of heat transfer represented by Nusselt number between the fluid and the inner surface of outer cylinder for air and water respectively. It is seen that the role of ratio of radiuses as well as time is to increase the rate of heat transfer. This is because continuous heating reduces the temperature difference between the fluid and surface of the cylinder and hence reduces the Nusselt number. It is good to notice that the negativity in Eq. (11) is to present the Nusselt number as positive value, thereby giving reverse physical case. The rate of heat transfer at the outer surface of inner cylinder follows exactly the same pattern and hence not presented in this article. In addition, the negative rate of heat transfer increases with increase in ratio of radiuses. This trend can be explained due to the fact that increasing λ decreases the temperature difference between the fluid and the surface of the cylinder and hence reduces rate of heat transfer (thereby increasing the negative Nusselt number). Furthermore, the time required to attain steady state Nusselt number is found to vary with ratio of radiuses as well as nature of fluid. The negative Nusselt number is the term given to Eq. (11) due to the inclusion of negative.

The variation of time and ratio of radiuses on skin-friction at the inner surface of outer cylinder for air and water respectively is illustrated in Figures 5a and 5b. From both figures, λ and t are observed to enhance skin-friction at the surface. It is important to observe that at steady state skin-friction is independent on nature of fluid used (air or water). This is applicable in design of dams and water reservoir.

IV. Conclusion

This study is devoted to the study of free convective flow in a vertical annulus with asymmetric surface heating of the cylinder surface. The governing momentum and energy equations are derived and solved exactly using the well-known Laplace technique in Laplace domain in terms of modified Bessel's function. Due to the complexity of the Laplace inversion into time domain, the Fourier approximation technique is employed. Result shows that time as well as ratio of radiuses increase fluid temperature, velocity, Nusselt number and skin-friction. In addition, ratio of radiuses as well as nature of fluid considered play an important role in attainment of steady state temperature, velocity, Nusselt number and skin-friction. This work is applicable in design of cooing appliances that assume cylindrical shape.

Nomenclature

a radius of the inner cylinder
b radius of the outer cylinder
FA Fourier approximation
g gravitational acceleration
I_n modified Bessel function of the first kind of order n
K_n modified Bessel function of the second kind of order n
Q dimensionless volume flow rate
Pr Prandtl number
r' dimensional radial coordinate
R dimensionless radial coordinate
SS steady state
t' dimensional time
t dimensionless time
T dimensional temperature
T_c cooled temperature of inner surface of outer cylinder
T_h heated temperature of outer surface of inner cylinder
T_m fluid temperature
u dimensional axial velocity
U dimensionless axial velocity

Greek letters

α thermal diffusivity
β coefficient of thermal expansion
λ ratio of radiuses (b/a)
ν fluid kinematic viscosity
ρ density
τ skin-friction
θ dimensionless temperature

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Appendix

$C_1 = \frac{-K_0(\lambda\sqrt{SPr})}{S[K_0(\sqrt{SPr})I_0(\lambda\sqrt{SPr}) - K_0(\lambda\sqrt{SPr}) - I_0(\sqrt{SPr})]}$, $C_2 = \frac{I_0(\lambda\sqrt{SPr})}{S[K_0(\sqrt{SPr})I_0(\lambda\sqrt{SPr}) - K_0(\lambda\sqrt{SPr}) - I_0(\sqrt{SPr})]}$,

$C_3 = \frac{-K_0(\lambda\sqrt{S})}{S^2(1-Pr)[I_0(\sqrt{S})K_0(\lambda\sqrt{S}) - K_0(\sqrt{S}) - I_0(\lambda\sqrt{S})]}$, $C_4 = \frac{I_0(\lambda\sqrt{S})}{S^2(1-Pr)[I_0(\sqrt{S})K_0(\lambda\sqrt{S}) - K_0(\sqrt{S}) - I_0(\lambda\sqrt{S})]}$

$q_1 = \frac{C_3}{\sqrt{S}}\left[\lambda I_1(\lambda\sqrt{S}) - I_1(\sqrt{S})\right]$, $q_2 = \frac{C_4}{\sqrt{S}}\left[K_1(\sqrt{S}) - \lambda K_1(\lambda\sqrt{S})\right]$

$q_3 = \frac{C_1}{S(1-Pr)\sqrt{SPr}}\left[\lambda I_1(\lambda\sqrt{SPr}) - I_1(\sqrt{SPr})\right]$, $q_4 = \frac{C_2}{S(1-Pr)\sqrt{SPr}}\left[K_1(\sqrt{SPr}) - \lambda K_1(\lambda\sqrt{SPr})\right]$

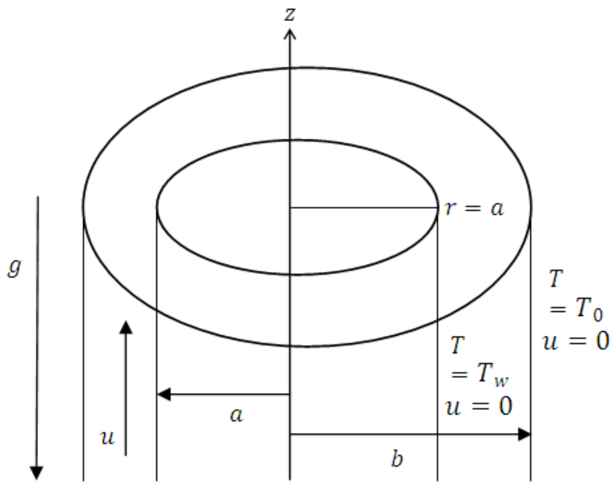
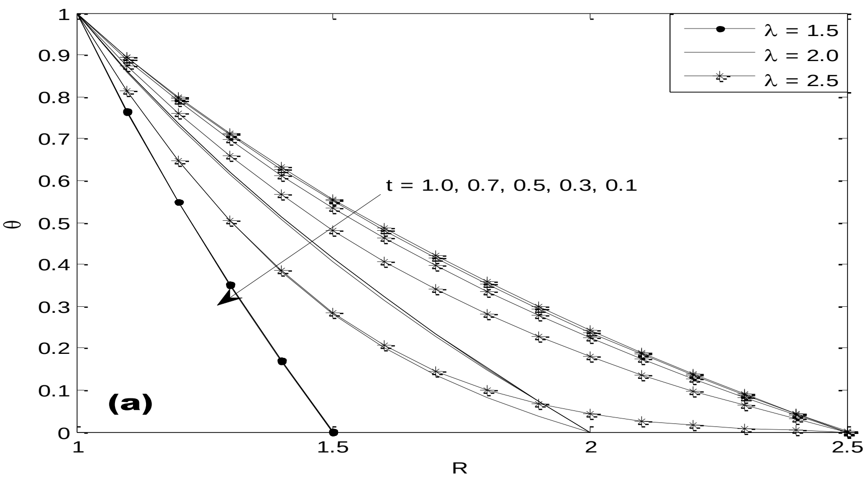


Fig. 1 Schematic diagram of the problem



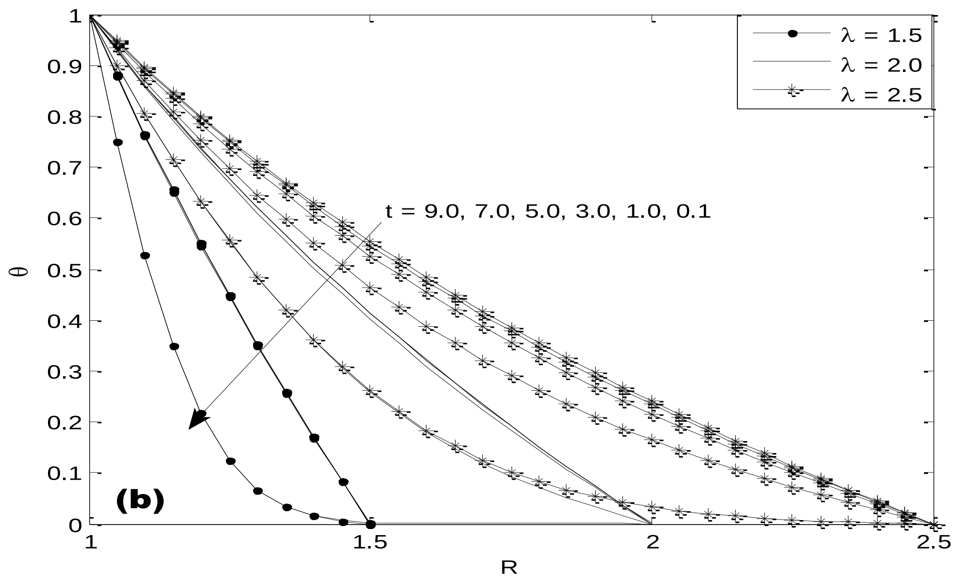
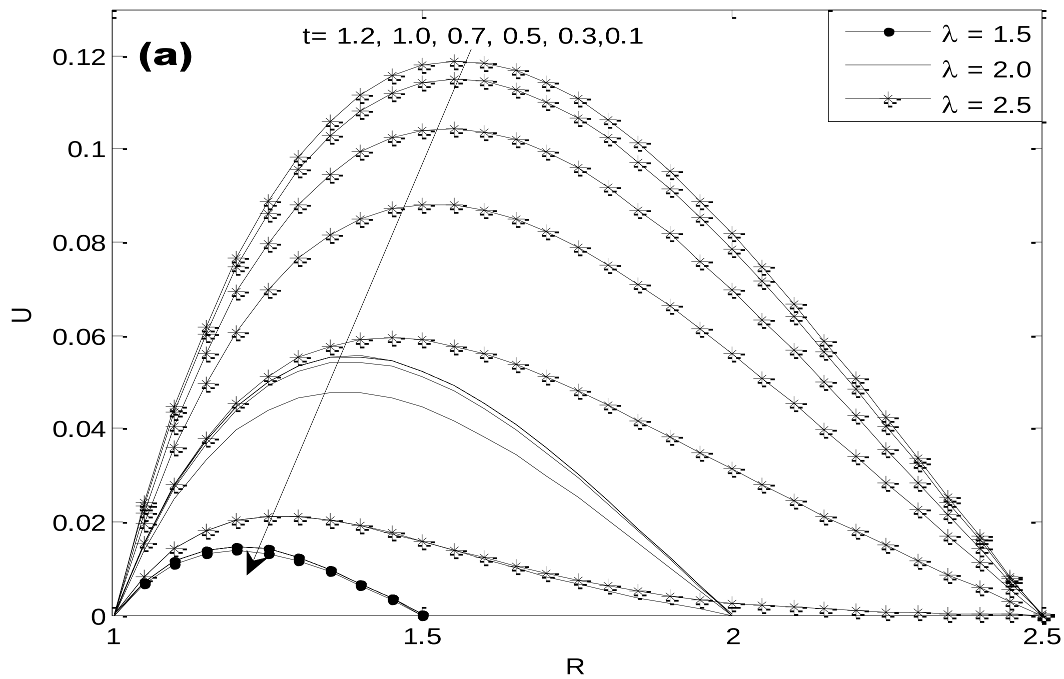


Figure 2: Temperature profiles for different values of λ and t at **(a)** $Pr = 0.71$ (air), **(b)** $Pr = 7.0$ (water)



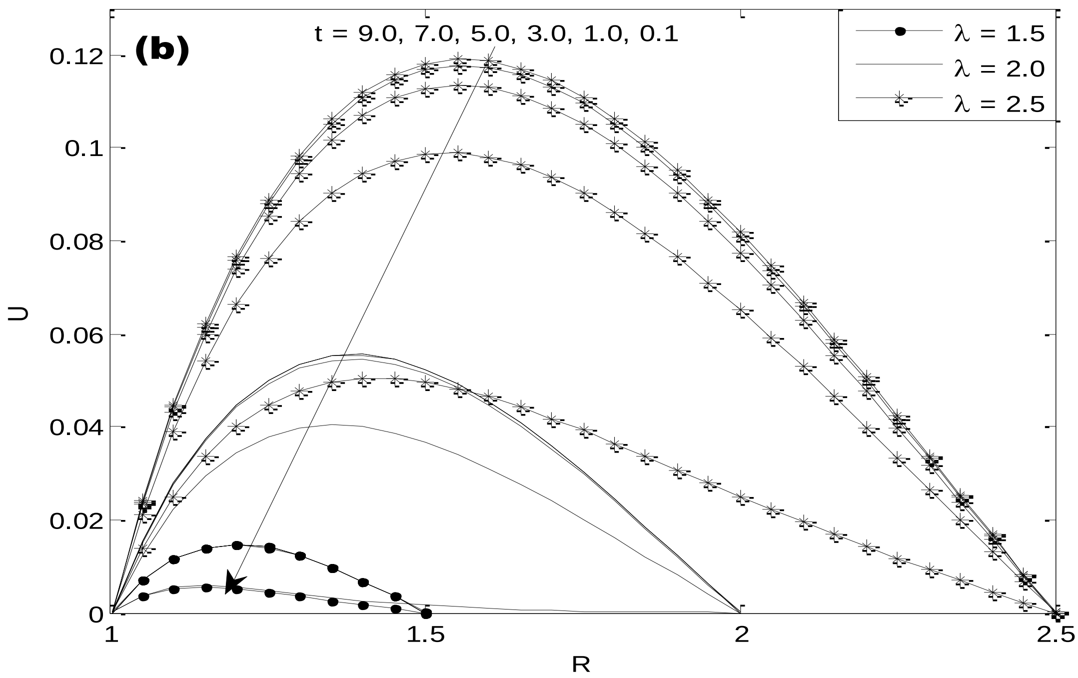
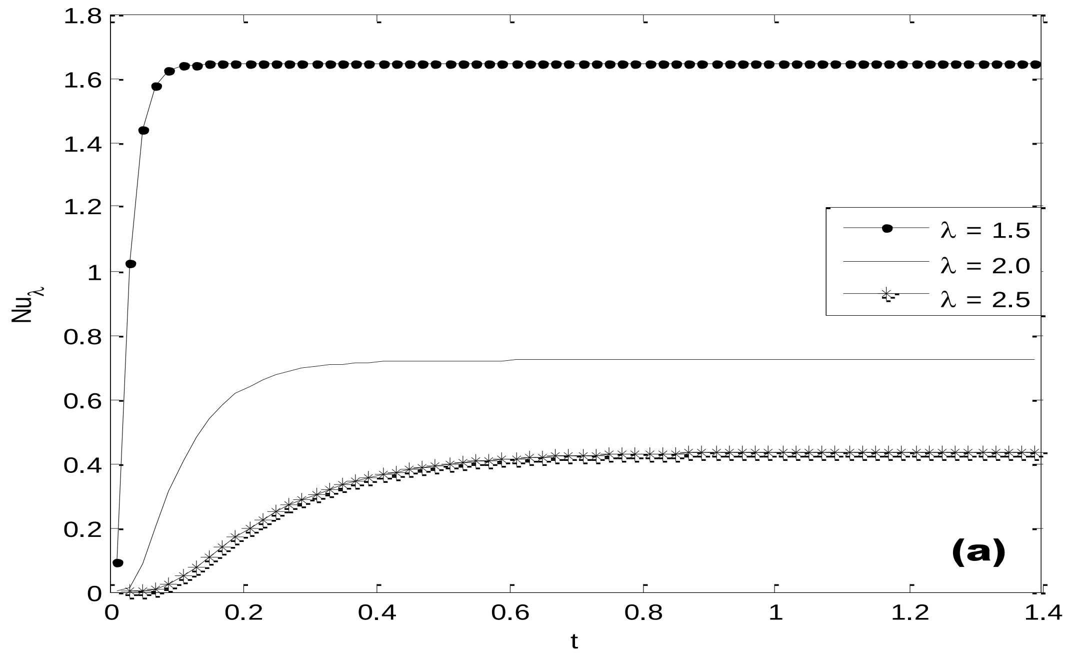


Figure 3: Velocity profiles for different values of λ and t at **(a)** $Pr = 0.71$ (air), **(b)** $Pr = 7.0$ (water)



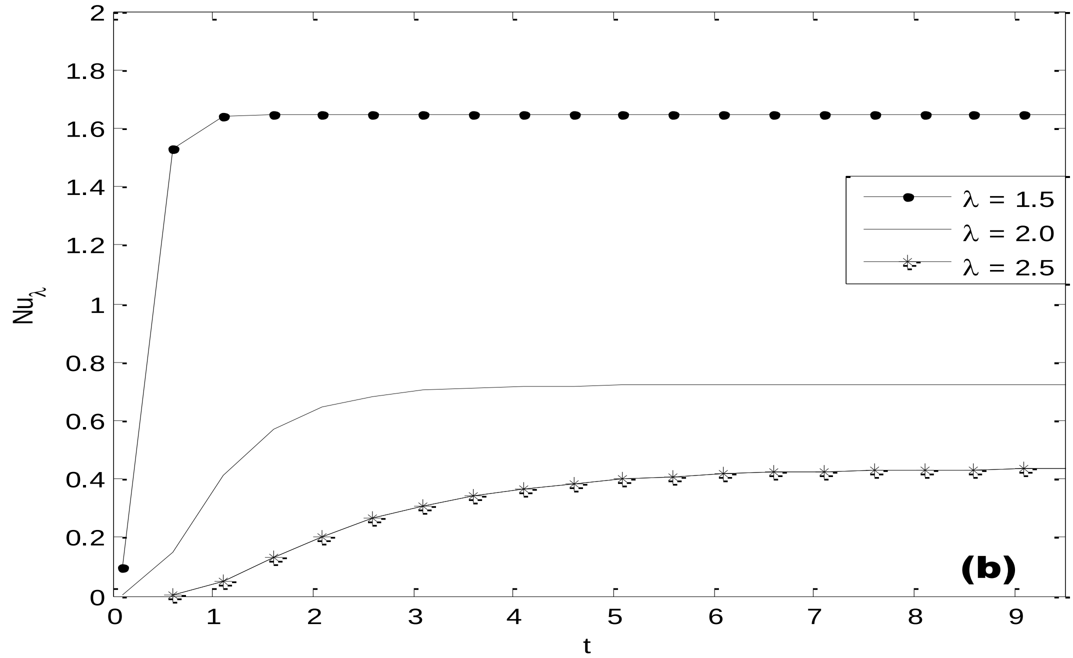
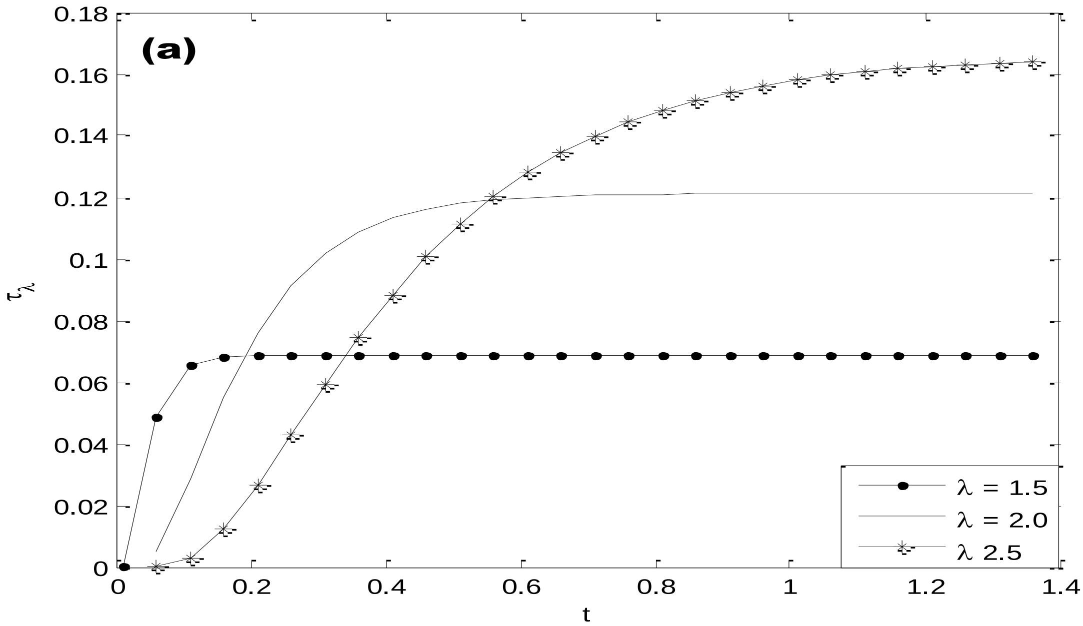


Figure 4: Nusselt number for different values of λ and t at **(a)** $Pr = 0.71$ (air), **(b)** $Pr = 7.0$ (water)



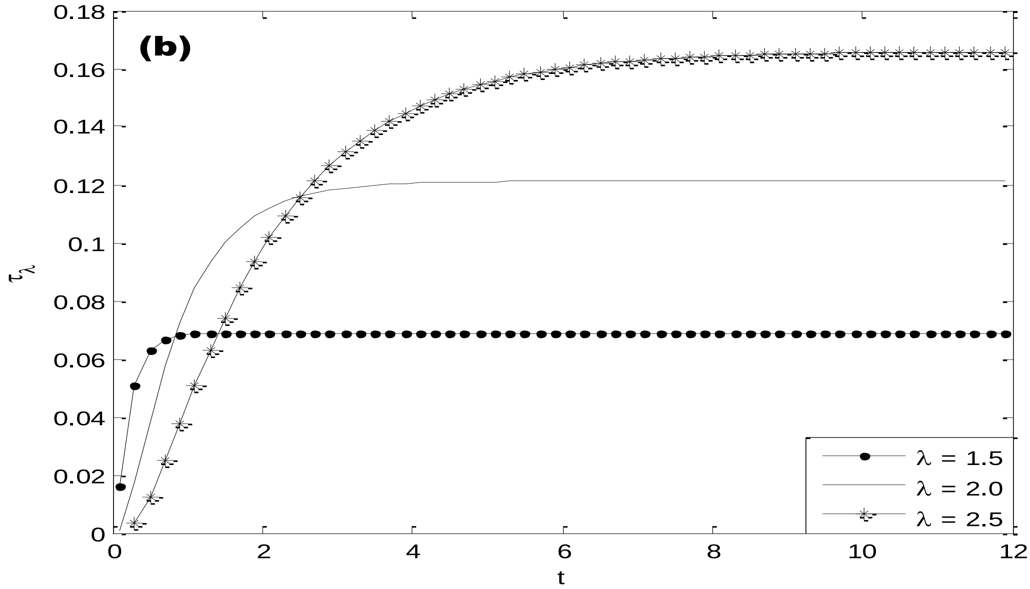


Fig. 5 Skin-friction for different values of λ and t at **(a)** $Pr = 0.71$ (air), **(b)** $Pr = 7.0$ (water)

Table 1: Numerical comparison of the values of temperature and velocity obtained by Riemann-sum approximation with those obtained at steady state (Exact solution), $\lambda = 2.0$

	$Pr = 0.71$ (air)			$Pr = 7.0$ (water)		
R	t	Velocity (FA)	Velocity (SS)	t	Velocity (FA)	Velocity (SS)
1.2	0.1	0.0202		0.1	0.0056	
	0.2	0.0330		1.0	0.0345	
	0.5	0.0440		4.0	0.0447	
	SS	0.0448	0.0448	SS	0.0448	0.0448
1.5	0.1	0.0156		0.1	0.0016	
	0.2	0.0344		1.0	0.0368	
	0.5	0.0511		4.0	0.0522	
	SS	0.0524	0.0524	SS	0.0524	0.0524
1.8	0.1	0.0051		0.1	0.0002	
	0.2	0.0149		1.0	0.0162	
	0.5	0.0238		4.0	0.0244	
	SS	0.0245	0.0245	SS	0.0245	0.0245

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