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Mathematical modeling of oscillatory flow in micro-tube and micro-annulus in the presence of second order velocity slip

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Abstract
This paper presents analytical solutions to oscillatory flow in micro-tube and micro-annulus in the presence of second order velocity slip. The flow is set up as a result of fluctuations of pressure gradient in a sinusoidal form. The condition on which the second order slip is significant in micro-tube and micro-annulus is also established. Result shows that the critical frequency is periodic and also higher for micro-tube than micro-annulus. It is established that the more complicated second order velocity slip should be used for non-critical frequency.
Keywords: *Oscillatoryflow; Micro-tube; Micro-annulus; Second order velocity slip.*

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I. Introduction

The study of fluid flow in microchannel has attracted researcher's attention in the last decades due to its applications in Biomedicine, industries and engineeringwhich include DNA sequencing, drug delivery, separation, pathological diagnostics, fuel cells, electronic chip cooling, micropumps, micronozzles and microsensors.A proper understanding of fluid flow in this microscale system is therefore vital in designs and implementations.

In microfluidic system, the small dimensions encountered in microfluidic devices can result in gas rarefaction. The dimensionless Knudsen number (Kn) determines the division of rarefied gas flow. The Knudsen number is defined as the ratio of the molecular mean free path (λ) to characteristics length scale. A classification of different flow regimes based on Knudsen number (Kn) can be found in Schaaf and Chambre [1]. Based on the Knudsen number, gas flows can be *qualitatively* classified as continuum($Kn \leq 0.01$), slip flow($0.01 \leq Kn \leq 0.1$), transition ($0.1 \leq Kn \leq 10$), and free-molecular ($Kn \geq 10$) flow.In this present study, we concern ourselves with a rarefied gas in the range $0.001 < Kn < 0.1$.

Several studies have been conducted on forced convective gas flow in micro tubes [2-13].The analyses of laminar heat transfer in slip flow regime were first undertaken by Sparrow et al. [2] and Inman [3] for tubes with uniform heat flux and a parallel plate channel or a circular tube with uniform wall temperature using continuum theory subject to slip-velocity and temperature-jump boundary conditions. Their works show the Nusselt numbers decrease in the presence of slip.

On forced convection in micro-tubes/annulus [4-6], Haddad et al. [5]presented the effect of forced convection gaseous slip flow in a circular porous channel and concluded that skin-friction increases with decrease in Knudsen number.Recently, the second order slip was introduced in addition to first order slip in order to capture some more physical situation [7-9]. Nian et al. [7] studied microtube gas flow with second-order slip flow and temperature jump boundary conditions and concluded that second-order boundary conditions assuming an effective mean free path model predicts a lower slip velocity than that of first order slip. Also, [8,9] analyses gaseous flow in micropipe and microchnnel respectively with second order velocity slip and temperature jump boundary conditions.

Fluctuating flow has also been of great applications in day to day activity and cooling systems. Jian et al. [10] studied time periodic electro-osmotic flow through microannulus. Motivation of this current paper was the work of Hamdan et al. [11] where they discussed the effect of second order velocity-slip/ temperature-jump on basic gaseous fluctuating micro-flows. Critical values of frequency for Couette flow, pressure driven flow, stokes problem and transient natural convection problem were presented. For frequency at these critical values, they concluded that the first order slip should be used instead of the second order slip velocity. Other related works onoscillating frequency can be seen in Haddad et al. [12-22].

The solution presented by [11] fails when a cylindrical geometry is considered. Therefore, the aim of this paper is to examine the effect of second order velocity-slip on fluctuating fluid flow in micro-tube and micro-annulus while the objective is to establish criteria for using the more complicated second-order slip instead of the first order slip and compare the critical frequency of micro-tube and micro-annulus.

II. Mathematical Analysis

Case I

Consider an unsteady fully developed forced convection flow of viscous, incompressible, fluid in a micro-tube with oscillating pressure gradient$\left(\frac{dp}{dz}=\sin(\varpi t)\right)$. The $z-$ axis is taken along the axis of the cylinders in parallel direction and $r-$ axis is the radial direction fig. 1. The axial heat conduction in the fluid and in the wall is assumed to be negligible. Further, the viscous dissipation and compressibility effects in the fluid are neglected. The governing equation with second order velocity slip of this case in dimensional form is given as:

$$\frac{\partial u}{\partial t}=\nu\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right]-\frac{1}{\rho}\frac{dp}{dz} \qquad (1)$$

In dimensionless form, we have;

$$\frac{\partial U}{\partial \tau}=\frac{\partial^2 U}{\partial R^2}+\frac{1}{R}\frac{\partial U}{\partial R}+sin(\omega\tau)$$

(2)

Subject to

$$\frac{\partial U}{\partial R}=0 \qquad \text{at} \qquad R=0$$

$$U=\frac{-(2-\sigma_v)}{\sigma_v}\left[Kn\frac{\partial U}{\partial R}+D\frac{\partial^2 U}{\partial R^2}\frac{Kn^2}{2}\right] \qquad \text{at} \qquad R=1 \qquad (3)$$

Case II

Consider the same problem as stated in case I, but this time to be in an annulus,The radius of the inner cylinder and outer cylinder is$a$ and b respectively, as shown in Fig.1. In dimensionless form, the governing equation is given exactly the same as case I but with the following boundary conditions:

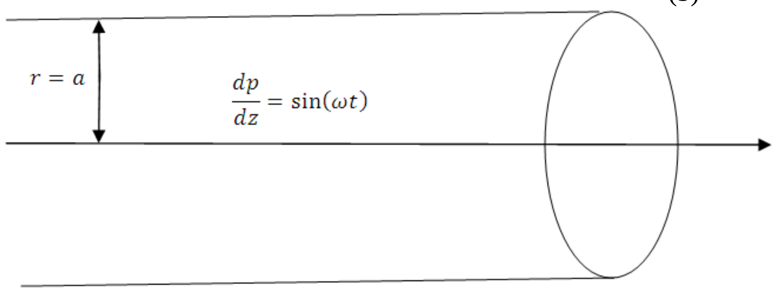
$$U=\frac{(2-\sigma_v)}{\sigma_v}\left[Kn\frac{\partial U}{\partial R}+C\frac{\partial^2 U}{\partial R^2}\frac{Kn^2}{2}\right] \qquad \text{at} \qquad R=1$$

$$U=-\frac{(2-\sigma_v)}{\sigma_v}\left[Kn\frac{\partial U}{\partial R}+D\frac{\partial^2 U}{\partial R^2}\frac{Kn^2}{2}\right] \qquad \text{at} \qquad R=\delta \qquad (4)$$

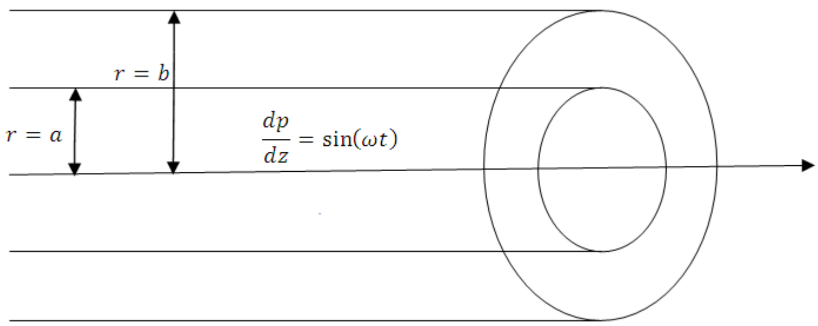
The following dimensionless quantities are used in the above equations

$$R=\frac{r}{a}, \qquad Z=\frac{z}{a}, \qquad \tau=\frac{\nu t}{a^2}, \ \omega=\frac{\varpi a^2}{\nu}, \qquad U=\frac{u}{u_0}, Kn=\frac{\lambda}{a}, u_0=a^2\frac{dp}{dz}, \qquad \delta=\frac{b}{a}$$

(5)



CASE I



CASE II

An exact solution of Eqs. (2-4) can be obtained by assuming the solution is of this form [11]:

$$U(\tau,R)=Im[e^{i\omega\tau}V(R)] \qquad (6)$$

Where"Im" stands for imaginary part of the assumed complex solution, $i=\sqrt{-1}$, ω is the dimensionless frequency of oscillation and τ is the dimensionless time.Substituting Eq. (6) into Eqs. (2-4), the solutions are obtained as follow:

Case I

$$V(R) = \frac{1}{i\omega}\left[1 + \frac{I_0(R\sqrt{i\omega})}{X_8}\right] \quad (7)$$

Skin-friction on the surface of the cylinder is given by:

$$\tau_0 = \left.\frac{\partial U}{\partial R}\right|_{R=1} = Im\left[e^{i\omega\tau}\frac{I_1(R\sqrt{i\omega})}{X_8\sqrt{i\omega}}\right] \quad (8)$$

Case II

$$V(R) = \frac{X_9}{i\omega}\left[(X_5 - X_7)I_0(R\sqrt{i\omega}) + (X_6 - X_4)K_0(R\sqrt{i\omega})\right] + \frac{1}{i\omega} \quad (9)$$

Similarly, the skin-friction on the outer surface of the inner cylinder and inner surface of outer cylinder is respectively given as:

$$\chi_0 = \left.\frac{\partial U}{\partial R}\right|_{R=1} = Im\left\{e^{i\omega\tau}\left(\frac{1}{\sqrt{i\omega}[X_5X_6 - X_4X_7]}\left[(X_5 - X_7)I_1(\sqrt{i\omega}) - (X_6 - X_4)K_1(\sqrt{i\omega})\right]\right)\right\} \quad (10)$$

$$\chi_1 = \left.\frac{\partial U}{\partial R}\right|_{R=\delta} = Im\left\{e^{i\omega\tau}\left(\frac{1}{\sqrt{i\omega}[X_5X_6 - X_4X_7]}\left[(X_5 - X_7)I_1(\delta\sqrt{i\omega}) - (X_6 - X_4)K_1(\delta\sqrt{i\omega})\right]\right)\right\} \quad (11)$$

Where $X_i$for $i = 1,2,3, ...$ are constants defined in the appendix?

III. Results And Discussion

To see the effect of using first order or second order velocity slip, graphs and tables are shown using MATLAB. All governing parameters entering the solution are carefully selected to give the physical meaning of stated problem. Knudsen number($Kn = 0.1$), since it serves as the region where the Navier-Stoke equation with velocity slip is applicable. Frequency of oscillation (ω) is taken so as to account for when ω is small and large. The general trend in these figures can be summarized as: at small slip factor (less than 0.01) the obtained velocity profiles are similar to the corresponding classical macro-flow profiles.

Table 1 computes the critical values of frequency (ω). It is observed from the table that the critical values have regular pattern. At the critical points, the use of second order slip is negligible as it corresponds exactly with that of first order slip. It can also be observed that it is higher in the tube compare to the surfaces of the annulus.

Fig. 2 presents the effect of Knudsen number (Kn)on flow velocity in the tubefor first and second order slip.It is observed that for small (($Kn) \leq 0.01,$)the first and second order sliphas insignificant effect on flow velocity. Also as (Kn)increases, velocity also increases; this is because the fluid molecules have higher mean free path length and as a result, the molecules travel longer distance without colliding with other molecules. In addition, velocity of the fluid is seen to be higher for second order slip than first order slip and a point of inflexion is observed around the surface of the tube (at this point, the use of second order slip is insignificant).

Fig. 3 on the other hand depicts the effect of using first or second order slip on flow velocity varying frequency of oscillation at different point in the cylinder. The flow is seen to be sinusoidal and has a point of inflexion.The use of first or second order slip from the figure is seen to be almost insignificant.

Effect of Knudsen number (Kn) on flow velocity in annulus for first and second order slip is considered in Fig. 4. As expected, it is observed that for small value of Knudsen number(($Kn) \leq 0.01,$) the first and second order slips have the same velocity. Also as (Kn)increases, velocity slip at the wall also increases and a point of inflexion is observed around the center of the annulus (for$(Kn) = 0.1$) .

Table 1. Critical values of frequency (ω)for dimensionless velocity at $Kn = 0.1$, $\tau = \pi/4$ for $0 \leq \omega \leq 50$

CASE I		CASE II	
$R = 0$	$R = 1$	$R = 1$	$R = \delta$
0	0	0	0
6.46	4.69	5.09	4.70
11.46	8.86	9.08	8.96
16.16	13.00	13.09	13.10
20.79	17.12	17.10	17.21
25.30	21.21	21.11	21.29
29.75	25.29	25.12	25.35
34.16	29.35	29.12	29.41
38.54	33.34	33.12	33.46
42.87	37.46	37.12	37.50

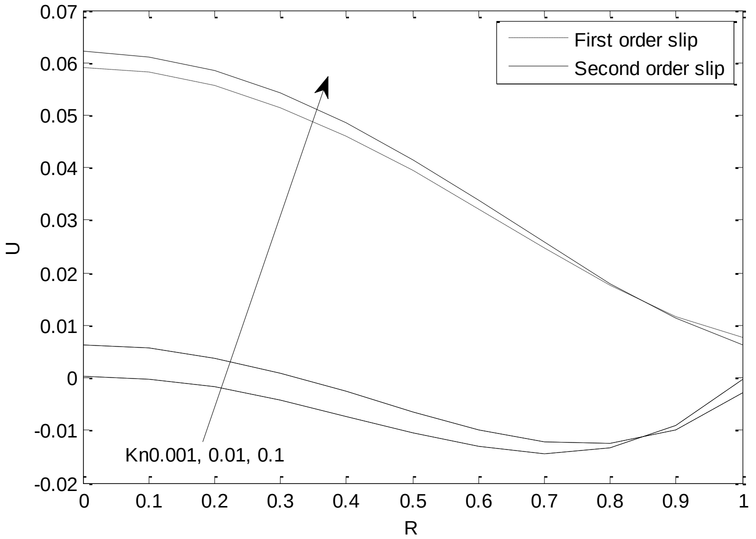


Fig. 2 Effect of velocity slip on velocity distribution for CASE I varying R and Kn at $\tau=\pi/4$, $\omega=5.0$

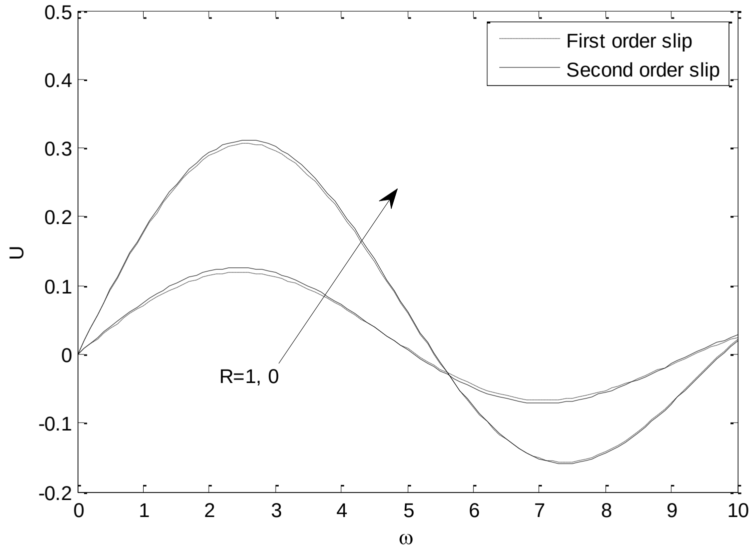


Fig. 3 Normalized velocity distribution for CASE I varying frequency and R at $\tau=\pi/4$, Kn=0.1

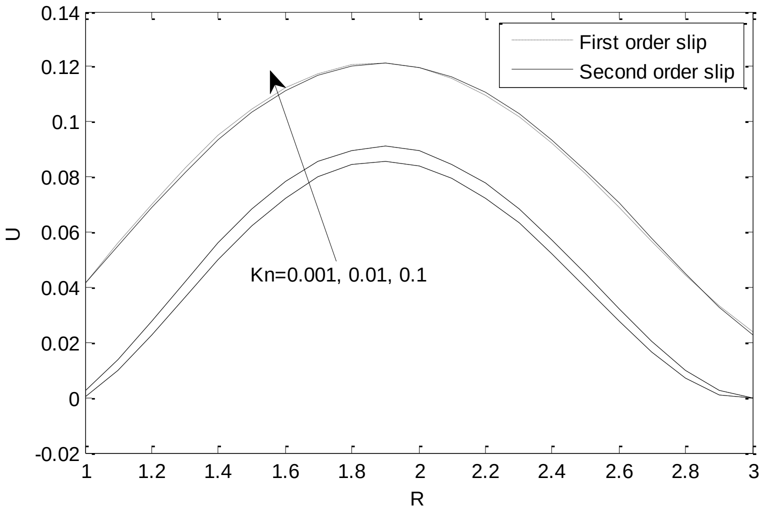


Fig. 4 Normalize velocity distribution for CASE II varying R and Kn at τ=pi/4, δ=3.0 and ω=5.0

As in Fig. 2, Fig. 5 gives the significant of using first or second order slip on flow velocity varying frequency of oscillation at different points in the annulus. It is observed that the flow is characterize as a sinusoidal function of ω. Periodic points of inflexion are also observed in the annulus as ω becomes lager; at these points, using ether the first or second order sliphas no effect on fluid velocity.

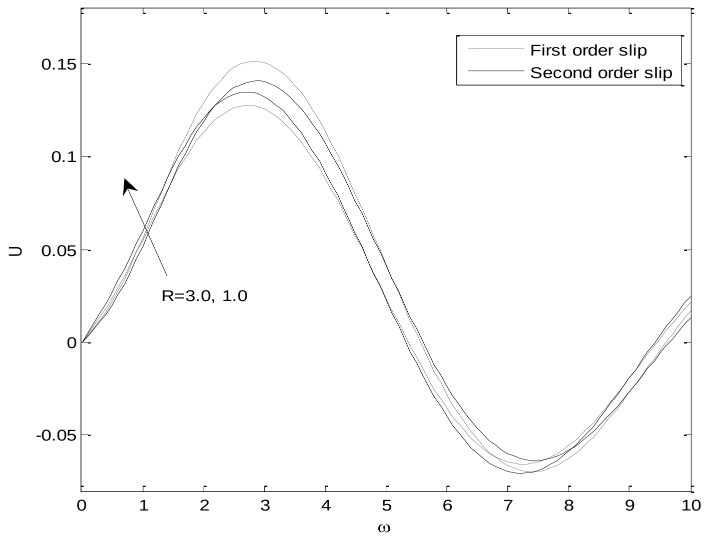


Fig. 5 Velocity distribution for CASE II varying frequency and R at τ=π/4, Kn=0.1, δ=3.0

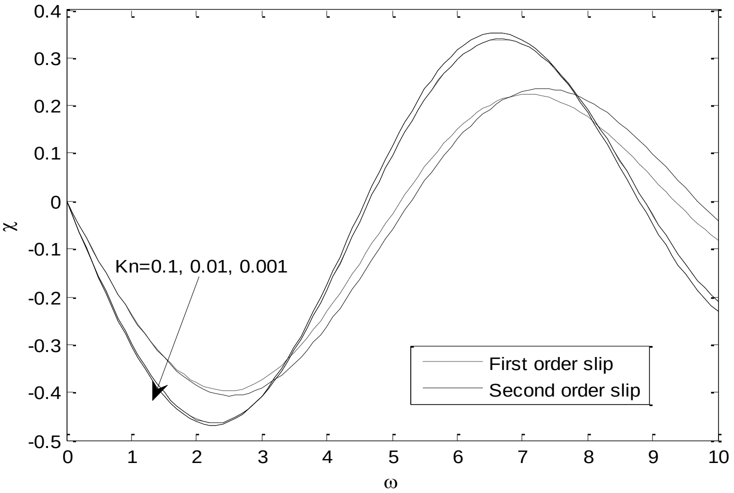


Fig. 6 Effect of velocity slip on Skin friction at the surface of the tube for CASE I varying ω and Kn at τ=π/4

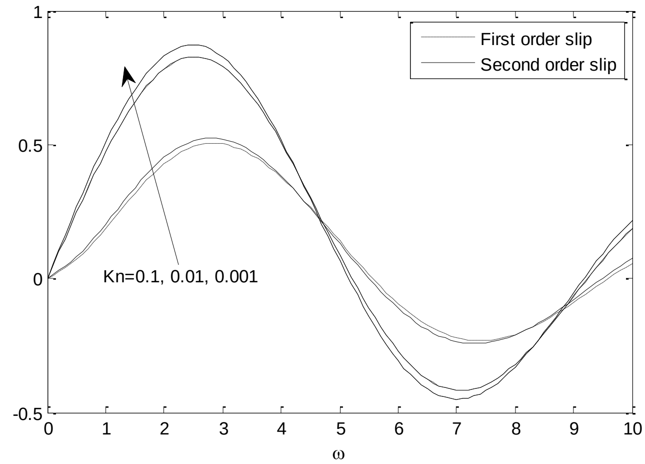


Fig. 7 Effect of velocity slip on skin-friction at the outer surface of inner cylinder CASE II varying Kn and ω for δ =3.0, τ=π/2

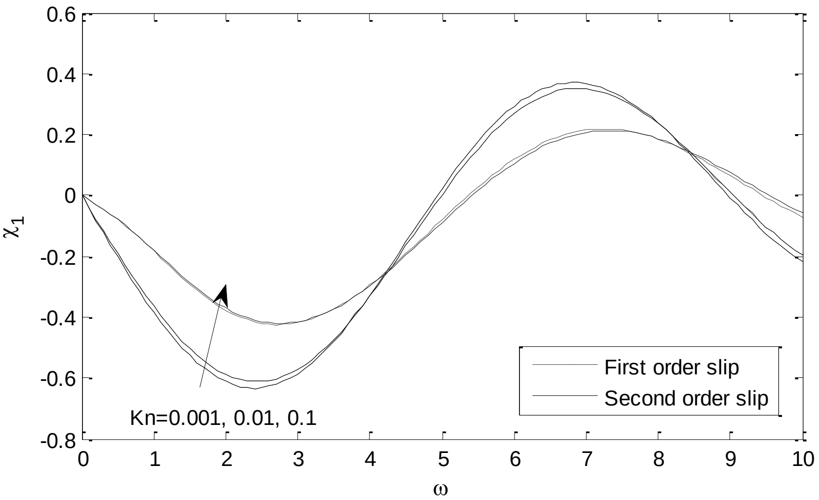


Fig. 8 Effect of velocity slip on skin friction at the inner surface of outer cylinder CASE II varying ω and Kn at δ=3.0, τ=π/4

Figs. 6-8 show the effect of using first or second order slip on skin-friction at the surfaces of the walls for tube and annulus. In fig. 6, the skin-friction at the surface of the tube is seen to increase with increase in (Kn) and also in a periodic manner. The effect of using first or second order slip is seen to be significant when

ω is not a point of inflexion. In similar manner, the skin-friction at the outer surface of inner cylinder also decreases with increase in $(Kn)$from fig. 7. Fig. 8 analyses the combined effect of oscillation frequency (ω), Knudsen number (Kn) and slip order on skin-friction at the inner surface of the outer cylinder. At $\omega \approx 4.3,8.8$ and $\omega \approx 5.0,9.0$in figs 7 and 8 respectively, the governing parameters are seen to be insignificant on the skin-friction on the surface.

IV. Conclusions

This paper presents an analytical solution for oscillatory flow in micro-tube and micro-annulus inspired by fluctuating pressure gradient with first and second order velocityslip at the surfaces of the micro-cylinders.The critical values of frequency $(\omega)$are seen to be periodic in nature. For tube, the critical values at $R = 0$, is given as $\omega \approx 5n$ for $n = 0,1,2,3,\ldots$ while at $R = 1, \omega \approx 4n$ for $n = 0,1,2,3,\ldots$. On the other hand, in annulus, the critical values of frequency at $R = 1$ is given as $\omega \approx 0, 5.09 + 4n$ for $n = 0,1,2,3,\ldots$ while at $R = \delta,\omega \approx 0, 4.7 + (4.1)n$ for $n = 0,1,2,3,\ldots$. Any value of frequency at these critical points is independent of second order slip while any frequency value outside these critical frequencies should use the more complicated second order slip.

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Appendix

$X_1 = \frac{(2-\sigma_v)}{\sigma_v}$ $\qquad X_2 = Kn\sqrt{i\omega}$ $\qquad X_3 = \frac{Kn^2\sqrt{i\omega}}{2}$

$$X_4 = X_1X_2I_1\left(\sqrt{i\omega}\right) + CX_1X_3\left[\sqrt{i\omega}\, I_0\left(\sqrt{i\omega}\right) - I_1\left(\sqrt{i\omega}\right)\right] - I_0\left(\sqrt{i\omega}\right)$$

$$X_5 = CX_1X_3\left[\sqrt{i\omega}\, K_0\left(\sqrt{i\omega}\right) + K_1\left(\sqrt{i\omega}\right)\right] - \left[K_0\left(\sqrt{i\omega}\right) + X_1X_2K_1\left(\sqrt{i\omega}\right)\right]$$

$$X_6 = DX_1X_3\left[\frac{I_1(\delta\sqrt{i\omega})}{\delta} - \sqrt{i\omega}\, I_0(\delta\sqrt{i\omega})\right] - \left[\, I_0(\delta\sqrt{i\omega}) + X_1X_2I_1(\delta\sqrt{i\omega})\right]$$

$$X_7 = X_1X_2K_1(\delta\sqrt{i\omega}) - DX_1X_3\left[\sqrt{i\omega}\, K_0(\delta\sqrt{i\omega}) + \frac{K_1(\delta\sqrt{i\omega})}{\delta}\right] - K_0(\delta\sqrt{i\omega})$$

$$X_8 = X_1\left(DX_3[I_1(\sqrt{i\omega}) - i\omega I_0(\sqrt{i\omega})] - X_2I_1(\sqrt{i\omega})\right) - I_0(\sqrt{i\omega})$$

$$X_9 = \frac{1}{[X_5X_6 - X_4X_7]}$$

Nomenclature

a	radius of inner cylinder
b	radius of outer cylinder
I_0	modified Bessel's function of first kind of order zero
K_0	modified Bessel's function of second kind of order zero
I_1	modified Bessel's function of first kind of order one
K_1	modified Bessel's function of first kind of order one
Kn	Knudsen number
p	pressure
r	dimensional radius
R	dimensionless radius
t	time
TP	periodic time
u	dimensional velocity
u_0	reference velocity
U	dimensionless velocity
V	complex solution function for velocity
z	axial coordinate

Greek symbols

α	thermal diffusivity
β	coefficient of thermal expansion
δ	aspect ratio
γ	specific heat ratio
λ	mean free path length
ν	kinematic viscosity
ϖ	frequency
ρ	fluid density (kgm^{-3})
σ	electrical conductivity of the fluid
σ_v	tangential momentum accommodation coefficient
τ	dimensionless time
ω	dimensionless frequency
χ	skin-friction on the surfaces of the cylinders

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