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Group Divisible Design (n_1, n_2, n_3,4; λ_1, λ_2), for $n_1 = 3$,$n_2 = n$ and $n_3 = n + 1$

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Abstract
The work in this article is about Group Divisible Designs (GDDs) with three groups of sizes n_1, n_2 and n_3, where n_1=3, n_2=n and n_3=n+1 and block size four. First, we establish necessary conditions for the existence of GDD(3,n,n + 1,4;λ_1,λ_2): for n_1=3,n_2=n and n_3=n+1. Necessary conditions include an inequality $\lambda_1 \geq \frac{(2n^2+14n+6)\lambda_2}{(10n^2+30)}$.Then we prove that these conditions are sufficient for several families of GDDs. We give an example where parameters satisfy all the necessary conditions including the inequality but the GDD does not exist.

Key words*: Group divisible designs (GDDs);Balanced incomplete block designs (BIBDs); a Design.*

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I. Introduction

Group divisible designs (GDDs) are studied for their usefulness in Statistics and for their universal application in the constructions of new designs [12, 11]. Certain difficulties are present especially when the number of groups is smaller than the block size. In [1, 2, 3], the question of existence of GDDs for block size three was settled. GDDs with block size four have been studied by many authors for example see, [17]. In [14], results about GDDs with number of groups 2 or 3 and block size four were established.

A lot of work has been done for GDDs of block size 3 with different group sizes for example: Lapchinda, Punnim and Pabhapote [10] in 2014 proved that necessary conditions are sufficient for the existence of GDD (1+*n*+*n*,3;λ_1,λ_2) when $\lambda_1 \geq \lambda_2$. Chaiyasena, Hurd, Punnim and Sarvate [9] in 2012 investigated group divisible designs known as (GDDs, GADs, or PBIBDs) with block size 3 and unequal size groups. They determined the necessary and sufficient conditions for groups with size (*n*,1) for any $n \geq 3$, and (*n*,2,1) for $n \in \{2,3, \cdots ,6\}$. They also obtained some general results for (n_1, n_2, n_3). Punnim and Uiyyasathian [8] in 2012 gave necessary conditions on *m* and *n* for the existence of a GDD($v = m + n$,2,3;1,2), along with sufficient conditions for each $m<\frac{n}{2}$. Furthermore, they introduced some construction techniques to construct some GDDs($v = m + n$,2,3;1,2) when $m>\frac{n}{2}$, namely, a GDD($v = 9 + 15$,2,3;1,2) and a GDD($v = 25 + 33$,2,3;1,2).

Lapchinda and Uiyyasathian [15] in 2009, Uiyyasathian and Pabhapote [16] in 2011, Pabhapote [6] in 2012 and Pabhapote and Punnim [7] in 2011 obtained results for block size three with unequal group sizes including the cases where the number of groups is bigger than or equal to block size 3.Sakda and Uiyyasathian [13] in 2017 gave a complete solution for the existence problem of GDDs (or PBIBDs) with block size $k = 3$ for groups of sizes (*n*, *n*, *n*,1) and any two indices (λ_1, λ_2). They introduced the construction of infinitely many GDDs with *t* groups of size *n* and one group of size 1.

On the other hand, GDDs of block size is 4 with different group sizes have been studied in very few papers for example, results on GDDs with two groups and block size four with equal number of even and odd blocks were addressed in [5]. In 2019 K. Namyalo, L. Zhang and D. Sarvate [4] studied GDD (1, *n*,*n* + 1,4;λ_1,λ_2) with equal number of blocks with configuration (1,1,2) and (2,2) and proved that the necessary conditions are sufficient for the existence for GDD(1, *n*,*n*+1,4;λ_1,λ_2) whenever $\lambda_1 \geq \lambda_2$ except for two cases. Note that for a GDD(1,*n*,*n* + 1,4;λ_1,λ_2) where $\lambda_2 > \lambda_1$, if $n = 6t + 1$ and $\lambda_1 = 6s + 1$, then$\lambda_2 \leq \frac{2(6t+1)}{6t+2} \times (6s+1)$<$2(6s+1)$. Hence, if $t \geq 2s$, then a GDD(1,6*t*+1,6*t*+ 2,4;6*s* + 1,12*s* + 1) may exist. They established necessary conditions for the existence of GDD(2, *n*,*n*+1,4;λ_1,λ_2). In addition, necessary conditions for the existence of GDD(n_1,n_2,n_3,4;λ_1, λ_2) for $n_1 = 1$ and $n_1 = 2$ were proved nonexistence of these designs when equal number of blocks with configuration (1,1,2) and (2,2) are required. Finally, they obtained several examples for $n_1 = 2$. Therefore, the next step is to study a GDD(3,*n*,*n* + 1,4;λ_1,λ_2).

Definition 1.1. *A group divisible design GDD*($n_1,n_2, ...,n_m,k;\lambda_1,\lambda_2$) *is a collection of k-element subsets of a v-set V called* blocks *which satisfies the following properties:*

- *the elements of V are partitioned into m subsets (called* groups*) of sizes* $n_1,n_2,\cdots,n_m$*;*
- *points within the same group are called* first associates *of each other and appear together in* λ_1 *blocks;*
- *any two points not in the same group are called* second associates *of each other and appear together in* λ_2 *blocks.*

When m is small, we replace $n_1,n_2,\cdots,n_m$ with $n_1+n_2,\cdots + n_m$.

Example 1.2. *A GDD*(2,3,3;3,1) *is the same as a GDD*(2+3,3;3,1) *where* $G_1 = \{a,b\}$ *and* $G_2 = \{1,2,3\}$. *Collection of blocks:*

$\{1,a,b\},\{2,a,b\},\{3,a,b\},\{1,2,3\},\{1,2,3\},\{1,2,3\}$

2 GDD(3,n, n+1,4;λ_1,λ_2)

A group divisible design GDD(3,n,n+1,4;λ_1,λ_2) is a GDD with three groups of different size $n_1 = 3, n_2 = n$ and $n_3 = n_1 + 1$ where first associate pair occurs in λ_1 blocks and second associate pair occurs in λ_2 blocks. We establish necessary conditions for the existence of this GDD by obtaining the parameters for a GDD(3,n, n+1,4;λ_1,λ_2).

3 The necessary conditions for a GDD(3,n, n+1,4;λ_1,λ_2)

In this section we obtain the necessary conditions for a GDD(3,n,n+1,4;λ_1,λ_2) by counting the replication numbers r_i for elements of the i^{th}group namely; $r_1 = \frac{2\lambda_1+(2n+1)\lambda_2}{3}$, $r_2 = \frac{(n-1)\lambda_1+(n+4)\lambda_2}{3}$ and $r_3 = \frac{n\lambda_1+(n+3)\lambda_2}{3}$. Similarly in the argument for $bk= vr$,we have $4b = v_1r_1 + v_2r_2 + v_3r_3 = 3r_1 + n_1r_2 + n_3r_3$ thus, $b = \frac{(n^2+3)\lambda_1+(n^2+7n+3)\lambda_2}{6}$. Further necessary conditions for the existence of this GDD based on the values of indices λ_1,λ_2 and any value of n are shown in Table 1;

Table 1: The necessary conditions for GDD(3,n, n+1,4;λ_1,λ_2)

λ1/λ2	0	1	2	3	4	5
0	all n	None	None	None	None	None
1	None	$n \equiv 0$ (mod 6)	None	None	$n \equiv 3$ (mod 6)	None
2	None	None	$n \equiv 0$ (mod 3)	None	None	None
3	n odd	None	None	n even	None	None
4	None	None	None	None	$n \equiv 0$ (mod 3)	None
5	None	None	$n \equiv 3$ (mod 6)	None	None	$n \equiv 0$ (mod 6)

Theorem 3.1. *A necessary condition for the existence of a GDD*(3,n,n + 1,4;λ_1,λ_2) *is that number of first associate pairs must be greater than or equal to the number of blocks which is* $\left[\binom{3}{2}+\binom{n}{2}+\binom{n+1}{2}\right]\lambda_1 \geq \frac{(n^2+3)\lambda_1+(n^2+7n+3)\lambda_2}{6}$ *that is* $\lambda_1 \geq \frac{(2n^2+14n+6)\lambda_2}{(10n^2+30)}$.

Proof. As there are only three groups every block must have at least one first associate pair. $\left[\binom{3}{2}+\binom{n}{2}+\binom{n+1}{2}\right]\lambda_1 \geq \frac{(n^2+3)\lambda_1+(n^2+7n+3)\lambda_2}{6}$ $(3+\frac{n(n-1)}{2}+\frac{n(n+1)}{2})\lambda_1 \geq (\frac{(n^2+3)\lambda_1+(n^2+7n+3)\lambda_2}{6})$ which when simplified gives $(10n^2+30)\lambda_1 \geq (2n^2 + 14n + 6)\lambda_2$ which is finally $\lambda_1 \geq \frac{(2n^2+14n+6)\lambda_2}{(10n^2+30)}$. □

From Table 1 we have this corollary;

Corollary 3.2. *A GDD*(3,n,n+1,4;1,4) *may exist only for* $n \equiv 3$ (mod 6) *but using 3.1, a GDD*(3,n,n+1,4;1,4) *does not exist for* $n \leq 27$

Example 3.3. *A GDD*(3,6,7,4;1,7) *with* $r_1 = 31, r_2 = 25, r_3 = 23$ *and* $b = 101$ *blocks does not exist because the bound* $\lambda_1 \geq \frac{(2n^2+14n+6)\lambda_2}{(10n^2+30)}$ *is not satisfied, thus,* $1 \ngeq \frac{(2.6^2+14.6+6)7}{10.6^2+30}$ *and* $1 \ngeq 2.9$.

4 $\lambda_1 \equiv 1$ (mod 6) and $\lambda_2 \equiv 1$ (mod 6)

This section starts with an example of a GDD that exists as given below:

Example 4.1. *A GDD*(3,6,7,4;7,1) *exists with groups* $G_1 = \{a,b,c\}, G_2 = \{1,2,3,4,5,6\}$ *and* $G_3 = \{t,u,v,w,x,y,z\}$ *using* r_1, r_2, r_3 *and b from the above as* $r_1 = 9, r_2 = 15, r_3 = 17$ *and* $b = 59$ *thus the blocks of this GDD are;*

$$\begin{array}{cccc|cccc|cccc|c} a & a & a & a & a & a & a & a & a & b & b & c & c \\ b & b & b & b & b & b & b & 1 & 3 & 2 & 1 & 1 & 2 \\ c & c & c & c & c & c & c & 2 & 4 & 3 & 5 & 3 & 4 \\ t & u & v & w & x & y & z & 6 & 5 & 4 & 6 & 5 & 6 \end{array}$$

$$\begin{array}{cccc|cccc|cc} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 3 & 4 & 3 & 3 & 4 & 4 \\ 3 & 4 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 5 \\ 4 & 5 & 5 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \end{array}$$

add blocks of a BIBD(13,4,1) *on* $G_2 \cup G_3$ *and blocks of a BIBD*(7,4,6) *on* G_3.

To generalize, a GDD($3,6t,6t+1,4;6t+1,1$) exists with $r_1 = 12t^2 + 4t, r_2 = 12t^2 + 2t + 1$, $r_3 = 12t^2 + 4t + 1$ and $b = 36t^3 + 12t^2 + 10t + 1$.

5 $\lambda_1 \equiv 0 \pmod 6$ and $\lambda_2 \equiv 0 \pmod 6$

For $\lambda_1 \equiv 0 \pmod 6$ and $\lambda_2 \equiv 0 \pmod 6$ we have an example below,

Example 5.1. *A GDD*(3,4,5,4;12,6) *has groups* $G_1 = \{a,b,c\}, G_2 = \{1,2,3,4\}$ *and* $G_3 = \{x,y,z,p,q\}$ *with* $r_1 = 26, r_2 = 28, r_3 = 30$ *and* $b = 85$.*This design exists and it can be constructed by adding the blocks of a BIBD*(7,4,6) *on* $G_1 \cup G_2$, *blocks of a BIBD*(8,4,6) *on* $G_1 \cup G_3$, *and blocks of a BIBD* (9,4,6) *on* $G_2 \cup G_3$.

Theorem 5.2. *A GDD*$(n_1,n_2,n_3,k;2\lambda,\lambda)$ *exists if a BIBD*(n_1+n_2, k,λ), *a BIBD*(n_1+n_3,k,λ) *and a BIBD*(n_2+n_3,k,λ) *exist and for* $k = 4$ *a BIBD*$(n,4,6)$ *exists and thus a GDD*$(n_1,n_2,n_3,4;12,6)$ *will always exist.*

Hence a GDD$(n_1,n_2,n_3,4;12t,6t)$ *exist for t a positive integer.*

A GDD(3,4,5,4;6,12) may exist since the bound $\lambda_1 \geq \frac{(2n^2+14n+6)\lambda_2}{(10n^2+30)} = 6 \geq \frac{1128}{190} = 6 \geq 5.9$holds.Even though the necessary condition including $\lambda_1 \geq \frac{(2n^2+14n+6)\lambda_2}{(10n^2+30)}$may hold, the GDD(3,4,5,4;6,12) does not exist as described below: Let the groups be $G_1 = \{x,y,z\}, G_2 = \{a,b,c,d\}$ and $G_3 = \{1,2,3,4,5\}$ and $k = 4$. Using$b = \frac{(n^2+3)\lambda_1+(n^2+7n+3)\lambda_2}{6}$,the number of blocks is 113, total number of first associate pairs is114 and total number of second associate pairs is 564. If a block contains a second associate pair then it must contain at least one first associate pair. Therefore, since there are 113 blocks, 112 blocks must be of type (2,1,1) and exactly one block of type (2,2). Number of second associate pairs from (2,1,1) blocks is 5×113 and for the blocks from type (2,2), there are four 2^{nd} associate pairs. Hence total number of 2^{nd} associate pairs covered by these blocks is 569 but since we only have 564 second associate pairs, a GDD(3,4,5,6,12) does not exist.

6 A GDD($n-1, n, n+1,4;6,12$)

Using the same argument from a GDD(3,4,5,4;6,12), we obtain the number of blocks, number of first associate and second associate pairs of the GDD($n-1,n, n+1,4;6,12$) with $r_1 = \frac{(n-2)\lambda_1+(2n+1)\lambda_2}{3}, r_2 = \frac{(n-1)\lambda_1+(2n\lambda_2)}{3}$, $r_3 = \frac{n\lambda_1+(2n-1)\lambda_2}{3}$ and $b = \frac{(3n^2-3n+2)\lambda_1+(6n^2-2)\lambda_2}{12}$which means the total number of blocks in this design is$b = \frac{(3n^2-3n+2)\lambda_1+(6n^2-2)\lambda_2}{12}$. We obtain number of first associate pairsas $\left[\binom{n-1}{2}+\binom{n}{2}+\binom{n+1}{2}\right]\lambda_1 = \left[\binom{n-1}{2}+\binom{n}{2}+\binom{n+1}{2}\right]6 = 3(3n^2 - 3n + 2)$. Number of second associate pairs is $(n-1).(2n+1) + n(n+1) = 12(3n^2 - 1)$. Therefore, the number of blocks is $b = \frac{15n^2-3n+2}{2}$, number of 1^{st} associate pairs is $3(3n^2 - 3n+2)$ and number of 2^{nd} associate pairs is $12(3n^2 - 2)$.

Example 6.1. *A GDD*(3,4,5,4,18,6) *exists and its blocks can be obtained by having the blocks as shown below;*

$$\begin{array}{cccccc|ccc|ccc|ccc|ccc|ccc|ccc|} a & a & a & a & a & a & a & b & c & a & b & c & a & b & c & a & b & c & a & b & c & a & b & c \\ b & b & b & b & b & b & b & c & a & b & c & a & b & c & a & b & c & a & b & c & a & b & c & a \\ c & c & c & c & c & c & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 3 \\ 5 & 5 & 5 & 5 & 5 & 5 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \end{array}$$

$$\begin{array}{cccc|cccc|cccc|} 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 \\ 3 & 4 & 4 & 4 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \end{array}$$

$$\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc|} a & b & c & a & b & c & a & b & c & a & b & c & a & b & c & a & b & c \\ b & c & a & b & c & a & b & c & a & b & c & a & b & c & a & b & c & a \\ x & x & x & x & x & x & x & x & x & y & y & y & y & y & y & z & z & z \\ y & y & y & z & z & z & w & w & w & z & z & z & w & w & w & w & w & w \end{array}$$

added to blocks of a BIBD(9,4,6) *on* $G_2 \cup G_3$ *together with blocks of a BIBD*(5,4,6) *to get the required total number of blocks of a GDD*(3,4,5,4;18,6).

Finally, the inequality $\lambda_1 \geq \frac{(2n^2+14n+6)\lambda_2}{(10n^2+30)}$ proves that a GDD(3,4,5,4;6,18) does not exist. In fact a GDD(3,5,6,4;6,18) also does not exist, thus the inequality proves the necessity of a GDD(3,6,7,4;6,18) to exist as in this case $\lambda_1 = 6 \geq \frac{(2n^2+14n+6)\lambda_2}{(10n^2+30)}$, that is, 6 <7.4.

lemma 6.2. *The necessary conditions for the existence of a GDD*(3,1,2,4;λ_1,λ_2) *are* $\lambda_1 \equiv 0$ (mod 6) *and* $\lambda_2 \equiv 0$ (mod 6), *that is, only when* $\lambda_1 = \lambda_2$.

Proof. When $\lambda_1 = \lambda_2 = 6$, a GDD(3,1,2,4;6,6) exists as BIBD(6,4,$\lambda_1 = \lambda_2 = \lambda = 6$) and the blocks of the GDD are obtained by getting the blocks of a BIBD(6,4,6) on $G_1 \cup G_2 \cup G_3$. □

Theorem 6.3. *For all values of* $n \geq 4$, *a GDD*(3,n,n + 1,4;λ_1,λ_2) *exists if* $\lambda_1 = \lambda_2$.

Proof. The blocks of a GDD(3,n,n+1,4: λ_1,λ_2) for $\lambda_1 = \lambda_2 = 6$ are obtained by taking the sum of blocks of a BIBD(2n+1,4,6) on $G_1 \cup G_2$, blocks of a BIBD(n+4,4,6) on $G_1 \cup G_3$ and blocks of a BIBD(n,4,6) on G_2, and take away ($n^2 - 3n$) blocks giving the required numbers of blocks of a GDD(3,n, n+1,4;6,6). Thus, this GDD(3,n, n+1,4;6,6) has $r_1 = r_2 = r_3 = 4n + 6$ and $b = 2n^2 + 7n + 6$. While for a BIBD(2n + 1,4,6) has $r = 4n$ and $b = 2n^2 + n$ blocks, a BIBD(n + 4,4,6) with $r = 2n + 6$ has $b = \frac{2n^2+14n+24}{4}$ blocks and a BIBD(n,4,6) with $r = 2n - 2$ has $b = \frac{2n^2-2n}{4}$ blocks. Thetotal number of blocks from the three BIBDs is $(2n^2 + n) + \left(\frac{2n^2+14n+24}{4}\right) + \left(\frac{2n^2-2n}{4}\right) = 3n^2 + 4n + 6$ take away ($n^2 - 3n$) blocks giving $(3n^2 + 4n + 6) - (n^2 - 3n) = 2n^2 + 7n + 6$ blocks of the required GDD(3,n, n+1,4;6,6). □

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