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Local Stability Analysis of Epidemiological Stage Structured Predator-Prey Model with Michaelis Menten Predation and Prey Refuge

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Abstract: *This paper aims to analyse the dynamical behaviour of eco-epidemiological prey predator population model (SI) giving stage structure for the predator and categorizing the prey as susceptible and infectious entity. The adult predator follows Michaelis Menten Holling type II response function while the prey takes refuge. The disease is transmitted following linear incidence rate. Existence, uniqueness and boundedness of the solution for the model are investigated. The equilibrium points are found out and the local stability analysis is carried out using Routh-Hurwitz criterion. The analytical results are verified through numerical simulation with help of MATLAB.*

Key words*: Michaelis Menten Holling Type, stage structure, prey refuge, local stability*.

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I. Introduction

In all ecological systems numerous living genera rely on one another for their continual subsistence. The reliance is for food, protection, locomotion, reproduction, etc. The interdependency among them has been explored in the past from the period of Volterra (1920) to interpret their dynamical behaviors. The researchers and mathematicians considered differential and difference equations conducive in modeling the interrelationship and interactions amidst various species.

The interrelationship among lives has a significant role in the food web. The prey-predator model associated with the food web expresses one such interdependency of the predator for food. In the recent past, this type of interrelationship is being studied by considering many key factors like stage structure for prey and for predator, different types of predation adopted by the predator and time delay due to gestation. The factors also included camouflage provided by nature to the prey, group defense adopted by the prey and anti-predator attributes of the prey to predation. The species might move from one place to another in search of forage, water, shelter leading to diffusion which also has been incorporated in studying these models. Incorporating all the dependency components simultaneously in the model systems developed have increased the complexity in acquiring solution to a particular problem. Hence the components are introduced and studied in a sequential manner by the researchers to arrive at the most appropriate solution.

The extinction of certain species has been caused by factors like environmental pollution, spread of deadly disease, over predation, over population and harvesting. Therefore, it is equally important to study the eco-epidemiological model to analyse and observe the effect of various diseases on the densities of these population model. The different disease models include SI, SIS, SEIR… and the disease transmission rate could be linear, bilinear, non-linear, Beddington DeAngelis model.

Chauhan and Misra in 2012, considered single species population model and showed that the pollution could cause adverse effect on the species leading to extinction whereas under the pollution free environment the susceptible population can never vanish [3]. Lotka-Volterra model with infected prey and stage structured predator was studied by Wuhaib and Abu-Hassan. Routh Hurwitz criterion was employed to study the local stability behavior. It was concluded through numerical simulation that stage structure affected the spread of disease and the disease did not become epidemic [16]. Dai et el. in 2015 observed Hopf bifurcation and global periodic solution including two unequal delays in the prey predator population model. Normal form method and Center Manifold theory had been used to show the occurrence of Hopf bifurcation [4].

An eco-epidemiological population model with infected prey and Lotka Volterra functional response was researched for its global stability through Lyapunov function. Here it was also assumed that a constant proportion of virus was also released into the environment [13]. The predator was stage structured and followed Holling Type II functional response. The predator fed on healthy prey only. It was shown that there was a possibility of occurrence of Hopf bifurcation when the intraspecific competition among the species crossed a threshold value [11]. The stability analysis was investigated and local bifurcations were studied at each equilibrium point for a refuge stage structure prey density [14].

SI_cIR model with Beddington DeAngelis type of incidence rate and saturated treatment rate were considered simultaneously. The Beddington DeAngelis type incidence rate took into consideration both the saturation factor and the influence of infectious individuals crowding. Global stability analysis was studied with the help of Lyapunov method. Bifurcation analysis have been carried out using Sotomayor's theorem [12]. Banerjee et al. had made a detailed and elaborated analysis of the effect of predator feeding on healthy and infected prey simultaneously [2].

Persistence and dynamical stabilities were examined by Abdulghafour et al. for the population model where the interaction was between the refuging prey that were susceptible & infected and the predator [1]. Local bifurcations were established for the epidemic population model with prey refuging. Conditions for the existence of Hopf bifurcation at a parameter value a_{13} was studied [5]. Majeed considered the model where both healthy and infected prey were consumed by the predator following Holling Type II predation. Existence, uniqueness, local and global stability qualities were discussed at all the equilibrium points. The transmission of disease followed linear incidence rate. The effect of various parameters was thoroughly studied by varying the parameters and thus conclusions were drawn [8]. It has been shown that refuging factor had a major impact on each population and there arose a periodic oscillation when the refuge factor lies within a range [9]. Xiao et al. proposed and studied a model considering gestation delay in the predator. Persistence of the model had been investigated along with local and global stability analysis [17]. Conditions were derived for the system to be stable both locally and globally with prey refuging and infected which were consumed by the mature predators [6]. Predator prey model with stage structure on both prey and predator with two different types of functional responses Holling Type II and IV was analysed for its local stability and Hopf bifurcation [10].

Sambath and Balachandran introduced and explored the impact of cross diffusion with prey refuge and Michaelis Menten type predation [7]. Stage structure has been introduced for the predator there in and the stability analysis has been done [15]. The goal of this paper is to study the local stability analysis of the predator prey model where the prey has been infected following [15].

This article is structured as follows. In section 2 the model system is presented and rescaled. In section 2 boundedness has been verified. In section 3 various equilibrium points are determined and the conditions for their existence are arrived. In section 4 local dynamical behaviour at all the equilibrium points are analysed. In section 5 numerical simulations using matlab are performed to demonstrate the behaviour of the system at the equilibrium points.

II. Model Formulation

The epidemiological prey-predator model system considered for study is as follows:

$$
\begin{aligned}
\dot{U}_s &= RU_s\left[1-\frac{U_s+U_I}{K}\right]-\frac{A_1(1-\lambda)U_sV_2}{U_s(1-\lambda)+K_1V_2}-\beta U_sU_I\\
\dot{U}_I &= \beta U_sU_I-d_1U_I\\
\dot{V}_1 &= \frac{e_1(1-n_1)A_1(1-\lambda)U_sV_2}{U_s(1-\lambda)+K_1V_2}-DV_1-d_2V_2\\
\dot{V}_2 &= DV_1-d_3V_2+\frac{e_1n_1A_1(1-\lambda)U_sV_2}{U_s(1-\lambda)+K_1V_2}
\end{aligned}
\qquad (2.1)
$$

The system includes prey species U, predator species V, involving a prey refuge $\lambda\in[0,1)$. The epidemic divides the prey population as susceptible prey U_s and infected prey U_I with a disease transmission rate β within the prey species. Only the susceptible prey population is capable of producing offspring and contributes to the growth of the prey population with a growth rate R with environmental carrying capacity K, but still the infected prey competes for food and habitat with the vulnerable prey. The predator population is stage structured as juvenile predator V_1 and adult predator V_2. Here d_1, d_2, d_3 represents the mortality rate of infected prey, juvenile predator and mature predator respectively. Also, it is assumed that only adult predator attacks the healthy prey with Michaelis Menten Holling type II predation and the juvenile predator depends on adults for their nourishment. A_1 is the attack rate of the predator, e_1 is the conversion coefficiency of the prey biomass and $n_1\in[0,1)$, the portion of food uptaken and K_1 is predators benefit from cofeeding.

After rescaling the system gets the new form as

$$\dot{u}_s = u_s\left[1 - u_s - (1+b_1)u_I - \frac{b_2 v_2}{u_s(1-\lambda) + K_1 v_2}\right] = f_1(u_s, u_I, v_1, v_2)$$
$$\dot{u}_I = u_I\left[b_1 u_s - b_3\right] = f_2(u_s, u_I, v_1, v_2)$$
$$\dot{v}_1 = \frac{b_4 u_s v_2}{u_s(1-\lambda) + K_1 v_2} - (b_5 + b_6)v_1 = f_3(u_s, u_I, v_1, v_2)$$
$$\dot{v}_2 = \frac{b_7 u_s v_2}{u_s(1-\lambda) + K_1 v_2} + b_5 v_1 - b_8 v_2 = f_4(u_s, u_I, v_1, v_2) \quad (2.2)$$

where

$\frac{\beta K}{R} = b_1;\quad \frac{A_1(1-\lambda)}{R} = b_2;\quad \frac{d_1}{R} = b_3;\quad \frac{e_1(1-n_1)A_1(1-\lambda)}{R} = b_4$

$\frac{D}{R} = b_5;\quad \frac{d_2}{R} = b_6;\quad \frac{e_1 n_1 A_1(1-\lambda)}{R} = b_7;\quad \frac{d_3}{R} = b_8$

with $u_s(0) \ge 0,\ u_i(0) \ge 0,\ v_1(0) \ge 0,\ v_2(0) \ge 0$. The functions defined in (2.2) are continuous and have continuous partial derivatives on the following four dimensional space

$\mathbb{R}^4_+ = \{(u_s, u_1, v_1, v_2) \in R^4 : u_s(0) \ge 0,\ u_i(0) \ge 0,\ v_1(0) \ge 0,\ v_2(0) \ge 0\}.$

Therefore, these functions are Lipschitzian on R^4 and hence the solutions exist for the system (2.2) and is unique.

III. Boundedness

Theorem 3.1: The solutions (u_s, u_I, v_1, v_2) of the system (2.2) are uniformly bounded.

Proof: Assume $(u_s(t), u_I(t), v_1(t), v_2(t))$ be any solution of the system (2.2) with non-negative initial values $(u_s(0), u_I(0), v_1(0), v_2(0)) \in \mathbb{R}^4_+$. Then from the first equation of the system (2.2) we have, $\dot{u}_s \le u_s(1-u_s)$. Now utilizing comparison theorem from differential inequality, we get

$\lim_{t\to\infty} \sup u_s(t) \le 1$.

Define the function

$G(t) = u_s + u_I + v_1 + v_2$

$$\dot{G}(t) = \dot{u}_s + \dot{u}_I + \dot{v}_1 + \dot{v}_2$$
$$= 2u_s - u_s^2 - u_s u_I - \frac{\left[b_2 - (b_4 + b_7)\right]u_s v_2}{u_s(1-\lambda) + K_1 v_2} - u_s - b_3 u_I - b_6 v_1 - b_8 v_2$$

From the biological perspective, the conversion rate constant of prey biomass into predator population cannot exceed the maximum predation rate constant we have $b_4 + b_7 < b_2$. So

$\dot{G}(t) < 2 - MG$ where $M = \min\{1, b_3, b_6, b_8\}$.

Solving the above differential inequality with the initial value $G(0) = G_0$, we arrive

$G = \frac{2}{M} + \left[G_0 - \frac{2}{M}\right]e^{-Mt}$. As $t \to \infty,\ 0 \le G(t) < \frac{2}{M}$. As a consequence, all the solutions of the system (2.2) are uniformly bounded.

IV. Existence Of Equilibrium Points

It is found that the system has at most five equilibrium points.

1. The trivial equilibrium point $E_0(0,0,0,0)$ always exist.
2. The axial equilibrium point $E_1(1,0,0,0)$ also exists.
3. The predator free equilibrium point $E_2(\hat{u}_s, \hat{u}_I, 0, 0)$ exists under the condition

$$b_1 > b_3 \quad (4.1)$$

where

$\hat{u}_s = \frac{b_3}{b_1}$ and $\hat{u}_I = \frac{b_1 - b_3}{b_1(1+b_1)}$

4. The infection free equilibrium point $E_3(\hat{\hat{u}}_s, 0, \hat{\hat{v}}_1, \hat{\hat{v}}_2)$ exists under the condition

$$b_7 + \frac{b_4 b_5}{b_5 + b_6} > b_8(1-\hat{\lambda}) \text{ and} \tag{4.2}$$

$$\frac{b_2}{K_1\left[(1-\hat{\lambda}) + S\right]} < 1 \tag{4.3}$$

where

$\hat{\hat{u}}_s = 1 - \frac{b_2 S}{K_1\left[(1-\hat{\lambda})b_8 + S\right]}$; $\hat{\hat{v}}_1 = \frac{b_4 \hat{\hat{u}}_s \hat{\hat{v}}_2}{(b_5 + b_6)\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1 \hat{\hat{v}}_2\right]}$; $\hat{\hat{v}}_2 = \frac{\hat{\hat{u}}_s S}{K_1 b_8}$

and

$$S = \left[b_7 + \frac{b_4 b_5}{b_5 + b_6} - b_8(1-\hat{\lambda})\right] \tag{4.4}$$

5. The positive equilibrium $E_4(\tilde{u}_s, \tilde{u}_I, \tilde{v}_1, \tilde{v}_2)$ where

$\tilde{u}_s = \frac{b_3}{b_1}$; $\tilde{u}_I = \frac{1}{1+b_1}\left\{1 - \left[\frac{b_3}{b_1} + \frac{b_2 S}{\left[(1-\hat{\lambda})b_8 + S\right]K_1}\right]\right\}$;

$\tilde{v}_1 = \frac{b_4 b_3 \tilde{v}_2}{(b_5 + b_6)\left[b_3(1-\hat{\lambda}) + K_1 b_1 \tilde{v}_2\right]}$; $\tilde{v}_2 = \frac{b_3 S}{b_1 K_1 b_8}$

exists under the condition (4.2) and

$$1 > \left[\frac{b_3}{b_1} + \frac{b_2 S}{\left[(1-\hat{\lambda})b_8 + S\right]K_1}\right] \text{ and} \tag{4.5}$$

S as defined in (4.4).

V. Local Stability Analysis

In this section, the Variational matrix is computed and local stability analysis is accomplished at each of the equilibrium points by determining the eigen values at these points.
The Variational matrix at any point (u_s, u_I, v_1, v_2) is given by

$$J = \begin{bmatrix} 1 - 2u_s - (1+b_1)u_I - \frac{b_1 K_1 v_2^2}{\left[u_s(1-\hat{\lambda}) + K_1 v_2\right]^2} & -(1+b_1)u_S & 0 & -\frac{b_2 u_s^2(1-\hat{\lambda})}{\left[u_s(1-\hat{\lambda}) + K_1 v_2\right]^2} \\ b_1 u_I & b_1 u_s - b_3 & 0 & 0 \\ \frac{b_4 K_1 v_2^2}{\left[u_s(1-\hat{\lambda}) + K_1 v_2\right]^2} & 0 & -(b_5 + b_6) & \frac{b_4 u_s^2(1-\hat{\lambda})}{\left[u_s(1-\hat{\lambda}) + K_1 v_2\right]^2} \\ \frac{b_7 K_1 v_2^2}{\left[u_s(1-\hat{\lambda}) + K_1 v_2\right]^2} & 0 & b_5 & \frac{b_7 u_s^2(1-\hat{\lambda})}{\left[u_s(1-\hat{\lambda}) + K_1 v_2\right]^2} - b_8 \end{bmatrix} \tag{5.1}$$

5.1 Local stability analysis at the zero-equilibrium point.

Theorem 5.1: The equilibrium point $E_0(0,0,0,0)$ is unstable.

Proof: Evaluating the Variational matrix at the point $E_0(0,0,0,0)$ we have

$$J(E_0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -b_3 & 0 & 0 \\ 0 & 0 & -(b_5 + b_6) & 0 \\ 0 & 0 & b_5 & -b_8 \end{bmatrix} \tag{5.2}$$

The eigen values are $\lambda_{u_s} = 1 > 0$; $\lambda_{u_I} = -b_3 < 0$; $\lambda_{v_1} = -(b_5 + b_6) < 0$; $\lambda_{v_2} = -b_8 < 0$.

Since one of the eigen values is positive the equilibrium point $E_0(0,0,0,0)$ is unstable. Hence the theorem.

5.2 Local stability analysis at $E_1(1,0,0,0)$.

Theorem 5.2: The equilibrium point $E_1(1,0,0,0)$ is locally asymptotically stable provided the following conditions are satisfied.

$$b_1 < b_3 \tag{5.3}$$

$$\frac{b_7}{(1-\hat{\lambda})} < b_5 + b_6 + b_8 \text{ and} \tag{5.4}$$

$$(b_5 + b_6)b_8 > \frac{b_7(b_5 + b_6) + b_5 b_4}{(1-\hat{\lambda})} \tag{5.5}$$

Proof: The Variational matrix at the point $E_1(1,0,0,0)$ is given by

$$J(E_1) = \begin{bmatrix} -1 & -(1+b_1)u_S & 0 & -\dfrac{b_2}{(1-\hat{\lambda})} \\ 0 & b_1 - b_3 & 0 & 0 \\ 0 & 0 & -(b_5+b_6) & \dfrac{b_4}{(1-\hat{\lambda})} \\ 0 & 0 & b_5 & \dfrac{b_7}{(1-\hat{\lambda})} - b_8 \end{bmatrix} \tag{5.6}$$

The characteristic equation at E_1 is given by

$$(-1-\lambda)(b_1 - b_3 - \lambda)(\lambda^2 + tr(A) + \det(A)) = 0 \tag{5.7}$$

where $A = \begin{bmatrix} -(b_5+b_6) & \dfrac{b_4}{(1-\hat{\lambda})} \\ b_5 & \dfrac{b_7}{(1-\hat{\lambda})} - b_8 \end{bmatrix}$

(5.7) gives $(-1-\lambda) = 0,\ (b_1 - b_3 - \lambda) = 0$ and $\lambda^2 + tr(A) + \det(A) = 0$

Hence first two terms gives $\lambda_{u_s} = -1 < 0;\ \lambda_{u_I} = b_1 - b_3$.

Now consider $\lambda^2 + tr(A) + \det(A) = 0$.

$tr(A) = \dfrac{b_7}{(1-\hat{\lambda})} - (b_5 + b_6 + b_8)$ and $\det(A) = (b_5 + b_6)b_8 - \dfrac{(b_5 + b_6)b_7 + b_4 b_5}{(1-\hat{\lambda})}$

It is obvious, by Routh Hurwitz criterion all the eigen values have negative real parts if it satisfies (5.3), (5.4) and (5.5). Hence the theorem.

5.3 Local stability analysis at $E_2(u_s, u_I, 0, 0)$.

Theorem 5.3: The equilibrium point $E_2(\hat{u}_s, \hat{u}_I, 0, 0)$ is locally asymptotically stable provided it satisfies condition (5.4) and (5.5) in addition to the condition (4.1).

Proof: The Variational matrix at the point $E_2(\hat{u}_s, \hat{u}_I, 0, 0)$ is given by

$$J = \begin{bmatrix} 1 - 2\hat{u}_S - (1+b_1)\hat{u}_I & -(1+b_1)\hat{u}_S & 0 & -\dfrac{b_2}{(1-\hat{\lambda})} \\ b_1\hat{u}_I & 0 & 0 & 0 \\ 0 & 0 & -(b_5+b_6) & \dfrac{b_4}{(1-\hat{\lambda})} \\ 0 & 0 & b_5 & \dfrac{b_7}{(1-\hat{\lambda})} - b_8 \end{bmatrix} \tag{5.8}$$

The characteristic equation at E_2 is given by

$$(\lambda^2 + tr(A) + \det(A))(\lambda^2 + tr(B) + \det(B)) = 0 \tag{5.9}$$

where

$$A = \begin{bmatrix} 1 - 2u_s - (1+b_1)\hat{u}_I & -(1+b_1)\hat{u}_S \\ b_1\hat{u}_I & 0 \end{bmatrix}; \quad B = \begin{bmatrix} -(b_5+b_6) & \dfrac{b_4}{(1-\hat{\lambda})} \\ b_5 & \dfrac{b_7}{(1-\hat{\lambda})} - b_8 \end{bmatrix}$$

$$tr(A) = 1 - 2\hat{u}_s - (1+b_1)\hat{u}_I + b_1\hat{u}_s - b_3 = -\frac{b_6}{b_1}; \quad \det(A) = b_1(1+b_1)\hat{u}_S\hat{u}_I = \frac{b_3(b_1-b_3)}{b_1}$$

$$tr(B) = -(b_5+b_6) + \frac{b_7}{(1-\hat{\lambda})} - b_8 = \frac{b_7}{(1-\hat{\lambda})} - \left(b_5+b_6+b_8\right)$$

$$\det(B) = -(b_5+b_6)\left(b_8 - \frac{b_7}{(1-\hat{\lambda})}\right) - \frac{b_5b_4}{(1-\hat{\lambda})} = (b_5+b_6)\left(b_8 - \frac{b_7}{(1-\hat{\lambda})}\right) - \frac{b_5b_4}{(1-\hat{\lambda})}$$

By Routh Hurwitz criterion all the eigen values have negative real parts when (5.4), (5.5) and (4.1) is true. Hence the theorem

5.4 Local stability analysis at $E_3(\hat{\hat{u}}_s, 0, \hat{\hat{v}}_1, \hat{\hat{v}}_2)$.

Theorem 5.4: The equilibrium point $E_3(\hat{\hat{u}}_s, 0, \hat{\hat{v}}_1, \hat{\hat{v}}_2)$ is locally asymptotically stable provided the following conditions are satisfied.

$$\hat{\hat{u}}_s < \frac{b_3}{b_1}; \qquad 2\hat{\hat{u}}_s + \frac{b_1K_1\hat{\hat{v}}_2^2}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1\hat{\hat{v}}_2\right]^2} > 1; \qquad b_8 > \frac{b_7\hat{\hat{u}}_s^2(1-\hat{\lambda})}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1\hat{\hat{v}}_2\right]^2}; \qquad a_{33}a_{44} > a_{43}a_{34}$$

$$a_{14}\left[a_{41}(a_{11}+a_{44}) + a_{31}a_{43}\right] > 0$$

Proof: The Variational matrix at the point $E_3(\hat{\hat{u}}_s, 0, \hat{\hat{v}}_1, \hat{\hat{v}}_2)$ is given by

$$J = \begin{bmatrix} 1 - 2\hat{\hat{u}}_s - \dfrac{b_1K_1\hat{\hat{v}}_2^2}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1\hat{\hat{v}}_2\right]^2} & -(1+b_1)\hat{\hat{u}}_S & 0 & -\dfrac{b_2\hat{\hat{u}}_s^2(1-\hat{\lambda})}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1\hat{\hat{v}}_2\right]^2} \\ 0 & b_1\hat{\hat{u}}_s - b_3 & 0 & 0 \\ \dfrac{b_4K_1\hat{\hat{v}}_2^2}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1\hat{\hat{v}}_2\right]^2} & 0 & -(b_5+b_6) & \dfrac{b_4\hat{\hat{u}}_s^2(1-\hat{\lambda})}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1\hat{\hat{v}}_2\right]^2} \\ \dfrac{b_7K_1\hat{\hat{v}}_2^2}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1\hat{\hat{v}}_2\right]^2} & 0 & b_5 & \dfrac{b_7\hat{\hat{u}}_s^2(1-\hat{\lambda})}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1\hat{\hat{v}}_2\right]^2} - b_8 \end{bmatrix} \tag{5.10}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \tag{5.11}$$

The characteristic equation at $E_3(\hat{\hat{u}}_s, 0, \hat{\hat{v}}_1, \hat{\hat{v}}_2)$ is given by

$$(a_{22} - \lambda)\left(\lambda^3 + R_1\lambda^2 + R_2\lambda + R_3\right) = 0 \tag{5.12}$$

where

$$R_1 = -(a_{11} + a_{22} + a_{33} + a_{44})$$

$$R_2 = a_{11}(a_{33} + a_{44}) + a_{33}a_{44} - a_{34}a_{43} - a_{41}a_{14}$$

$$R_3 = -a_{11}(a_{33}a_{44} - a_{34}a_{43}) - a_{14}(a_{31}a_{43} - a_{33}a_{41})$$

From (5.12) $(a_{22} - \lambda) = 0$ and

$$\left(\lambda^3 + R_1\lambda^2 + R_2\lambda + R_3\right) = 0 \tag{5.13}$$

$(a_{22} - \lambda) = 0$ gives $\lambda_{u_I} = a_{22} < 0$ when $\hat{\hat{u}}_s < \dfrac{b_3}{b_1}$.

However, for the equation $\left(\lambda^3 + R_1\lambda^2 + R_2\lambda + R_3\right) = 0$ by Routh Hurwitz Criterion all the eigen values have negative real parts if and only if $R_1 > 0,\ R_3 > 0$ and $R_1R_2 - R_3 > 0$

$R_1 > 0$ provided that

$$2\hat{\hat{u}}_s + \frac{b_1 K_1 \hat{\hat{v}}_2^2}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1 \hat{\hat{v}}_2\right]^2} > 1 \tag{5.14}$$

and

$$b_8 > \frac{b_7 \hat{\hat{u}}_s^2 (1-\hat{\lambda})}{\left[\hat{\hat{u}}_s(1-\hat{\lambda}) + K_1 \hat{\hat{v}}_2\right]^2} \tag{5.15}$$

$R_3 > 0$ provided that $a_{33}a_{44} > a_{43}a_{34}$.

In addition to the above condition if $a_{14}\left[a_{41}(a_{11}+a_{44}) + a_{31}a_{43}\right] > 0$ holds, then all the eigen values of (5.12) have negative real parts. Hence $E_3(\hat{\hat{u}}_s, 0, \hat{\hat{v}}_1, \hat{\hat{v}}_2)$ is locally asymptotically stable.

5.5 Local stability analysis at $E_4(\tilde{u}_s, \tilde{u}_I, \tilde{v}_1, \tilde{v}_2)$.

Theorem 5.5: The equilibrium point is $E_4(\tilde{u}_s, \tilde{u}_I, \tilde{v}_1, \tilde{v}_2)$ is locally asymptotically stable provided the following conditions are satisfied.

$$\frac{\left[1-(1+b_1)\tilde{u}_I\right]\left[\tilde{u}_s(1-\hat{\lambda}) + K_1\tilde{v}_2\right]^2 - b_2 K_1 \tilde{v}_2^2}{2\left[\tilde{u}_s(1-\hat{\lambda}) + K_1\tilde{v}_2\right]^2} < \tilde{u}_s \tag{5.16}$$

$$b_8 > \frac{b_7 \tilde{u}_s^2(1-\hat{\lambda})}{\left[\tilde{u}_s(1-\hat{\lambda}) + K_1\tilde{v}_2\right]^2} \tag{5.17}$$

$$(b_5 + b_6)\left(b_8 - \frac{b_7 \tilde{u}_s^2 (1-\hat{\lambda})}{\left[\tilde{u}_s(1-\hat{\lambda}) + K_1\tilde{v}_2\right]^2}\right) > \frac{b_4 b_5 \tilde{u}_s^2 (1-\hat{\lambda})}{\left[\tilde{u}_s(1-\hat{\lambda}) + K_1\tilde{v}_2\right]^2} \tag{5.18}$$

and

$$q_1 > q_2 \tag{5.19}$$

where $q_1,\ q_2$ are defined as in (5.21) and (5.22).

Proof: The Variational matrix at the point $E_4(\tilde{u}_s, \tilde{u}_I, \tilde{v}_1, \tilde{v}_2)$ is given by

$$J(E_4) = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$$

where

$$c_{11} = 1 - 2\tilde{u}_s - (1+b_1)\tilde{u}_I - \frac{b_1 K_1 \tilde{v}_2^2}{\left[\tilde{u}_s(1-\hat{\lambda}) + K_1\tilde{v}_2\right]^2}; \quad c_{12} = -(1+b_1)\tilde{u}_S < 0; \; c_{13} = 0; \quad c_{14} = -\frac{b_2 \tilde{u}_s^2 (1-\hat{\lambda})}{\left[\tilde{u}_s(1-\hat{\lambda}) + K_1\tilde{v}_2\right]^2} < 0$$

$$c_{21} = b_1 \tilde{u}_I; \quad c_{22} = 0; \quad c_{23} = 0; \quad c_{24} = 0$$

$$c_{31} = \frac{b_4 K_1 \tilde{v}_2^2}{\left[\tilde{u}_s(1-\hat{\lambda}) + K_1\tilde{v}_2\right]^2}; \quad c_{32} = 0; \quad c_{33} = -(b_5 + b_6) < 0; \quad c_{34} = \frac{b_4 \tilde{u}_s^2 (1-\hat{\lambda})}{\left[\tilde{u}_s(1-\hat{\lambda}) + K_1\tilde{v}_2\right]^2}$$

$$c_{41} = \frac{b_7 K_1 \tilde{v}_2^2}{\left[\tilde{u}_s(1-\hat{\lambda}) + K_1\tilde{v}_2\right]^2}; \quad c_{42} = 0; \quad c_{43} = b_5; \quad c_{44} = \frac{b_7 \tilde{u}_s^2 (1-\hat{\lambda})}{\left[\tilde{u}_s(1-\hat{\lambda}) + K_1\tilde{v}_2\right]^2} - b_8$$

The characteristic equation at $E_4(\tilde{u}_s, \tilde{u}_I, \tilde{v}_1, \tilde{v}_2)$ is given by

$$\left(\lambda^4 + H_1\lambda^3 + H_2\lambda^2 + H_3\lambda + H_4\right) = 0 \tag{5.20}$$

where

$$H_1 = -(\gamma_0 + \gamma_1)$$

$$H_2 = \gamma_0\gamma_1 + \gamma_2 - (\gamma_3 + \gamma_4 + \gamma_5)$$

$$H_3 = -\left[\gamma_0(\gamma_2 - \gamma_3) - \gamma_1\gamma_4 - m_{14}(-\gamma_7 + \gamma_6)\right]$$

$$H_4 = -(\gamma_2 - \gamma_3)\gamma_4$$

and

$\gamma_0 = c_{11} + c_{22}$; $\gamma_1 = c_{33} + c_{44}$; $\gamma_2 = c_{33}c_{44}$; $\gamma_3 = c_{34}c_{43}$; $\gamma_4 = c_{12}c_{21}$; $\gamma_5 = c_{14}c_{41}$; $\gamma_6 = c_{31}c_{43}$; $\gamma_7 = c_{33}c_{41}$

By Routh Hurwitz criterion if $H_1 > 0,\ H_3 > 0,\ H_4 > 0$ and $\Delta = (H_1H_2 - H_3)H_3 - H_1^2H_4 > 0$, then all the eigen values of (5.20) have negative real parts. $H_i > 0$, i=1,3,4 provided that

$$\frac{[1-(1+b_1)\tilde{u}_I][\tilde{u}_s(1-\lambda)+K_1\tilde{v}_2]^2 - b_2K_1\tilde{v}_2^2}{2[\tilde{u}_s(1-\lambda)+K_1\tilde{v}_2]^2} < \tilde{u}_s;\ b_8 > \frac{b_7\tilde{u}_s^2(1-\lambda)}{[\tilde{u}_s(1-\lambda)+K_1\tilde{v}_2]^2};$$

$$(b_5+b_6)\left(b_8 - \frac{b_7\tilde{u}_s^2(1-\lambda)}{[\tilde{u}_s(1-\lambda)+K_1\tilde{v}_2]^2}\right) > \frac{b_4b_5\tilde{u}_s^2(1-\lambda)}{[\tilde{u}_s(1-\lambda)+K_1\tilde{v}_2]^2}$$

and

$\Delta = q_1 - q_2$ where

$$\begin{aligned} q_1 = &(\gamma_0+\gamma_1)\left[\gamma_0\gamma_1 + (\gamma_2-\gamma_3) - (\gamma_4+\gamma_5)\right]\left[\gamma_0(\gamma_2-\gamma_3) - \gamma_1\gamma_4 - c_{14}(\gamma_7-\gamma_6)\right] \\ &+ c_{14}(\gamma_7-\gamma_6)(\gamma_2-\gamma_3)(\gamma_0-\gamma_1) - c_{14}\gamma_4(\gamma_7-\gamma_6)(\gamma_0-\gamma_1) \\ &+\gamma_0\gamma_1(\gamma_2-\gamma_3)^2 + \gamma_0\gamma_1\gamma_4^2 \end{aligned} \tag{5.21}$$

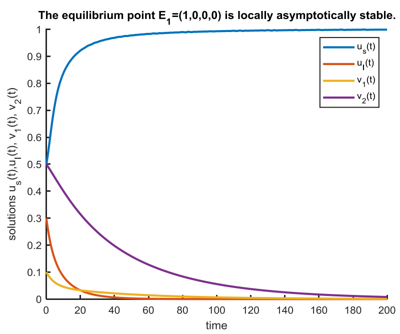
$$q_2 = \left[\gamma_0(\gamma_2-\gamma_3) - \gamma_1\gamma_4 - c_{14}(\gamma_7-\gamma_6)\right]^2 + (\gamma_0+\gamma_1)^2(\gamma_2-\gamma_3)\gamma_4 \tag{5.22}$$

Δ will be positive if $q_1 > q_2$ in addition to the above conditions.

VI. Numerical Simulation

Example 1: In (2.2) let

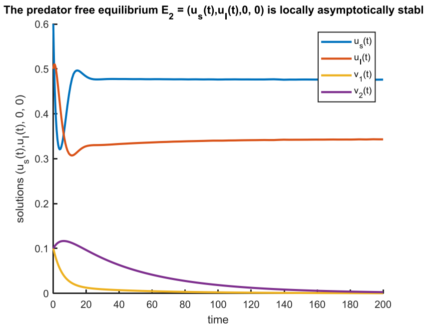
$K_1 = 0.3,\ b_1 = 0.175,\ b_2 = 0.0625,\ b_3 = 0.25,\ b_4 = 0.00938,\ b_5 = 0.1,\ b_6 = 0.075,\ b_7 = 0.00938,\ b_8 = 0.05$ and $\lambda = 0.5$. For the equilibrium point $E_1(1,0,0,0)$, the conditions (5.3), (5.4) and (5.5) (0.175 < 0.25, 0.01875 < 0.225 and 0.00875 > 0.00516) are satisfied. Hence by theorem (5.2), $E_1(1,0,0,0)$ is locally asymptotically stable (see Figure(1)).



Figure(1)

Example 2: In (2.2) let

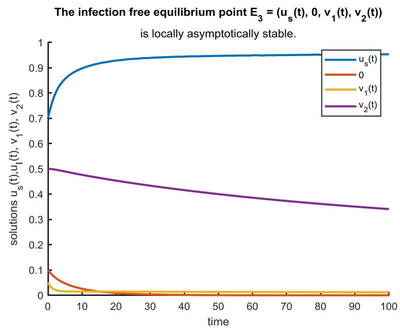
$K_1 = 0.3,\ b_1 = 0.525,\ b_2 = 0.0625,\ b_3 = 0.25,\ b_4 = 0.00938,\ b_5 = 0.1,\ b_6 = 0.075,\ b_7 = 0.00938,\ b_8 = 0.05$ and $\lambda = 0.5$. For the equilibrium point $E_2(0.47619, 0.34348, 0, 0)$, the conditions (4.1), (5.4) and (5.5) (0.525 > 0.25, 0.01875 < 0.225 and 0.00875 > 0.00516) are satisfied. Hence by theorem (5.3), $E_2(0.47619, 0.34348, 0, 0)$ is locally asymptotically stable (see Figure(2)).



Figure(2)

Example 3: In (2.2) let

$K_1 = 0.3,\ b_1 = 0.375,\ b_2 = 0.0525,\ b_3 = 0.45,\ b_4 = 0.01103,\ b_5 = 0.12,\ b_6 = 0.65,\ b_7 = 0.00735, b_8 = 0.025$ and $\lambda = 0.7$. For the equilibrium point $E_3(1, 0, 0.00825, 0.20909)$, all the conditions stated in the theorem (5.4) (1 < 1.2, 2.03738 > 1, 0.025 > 0.01676, 0.00635 > 0.00302, 2.92E-06>0) are satisfied. Hence $E_3(.96974, 0, 0.00825, 0.20286)$ locally asymptotically stable (see Figure(3)).

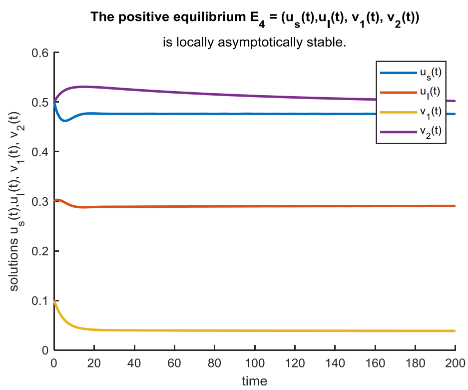


Figure(3)

Example 4: In (2.2) let

$K_1 = 0.3,\ b_1 = 0.525,\ b_2 = 0.0625,\ b_3 = 0.25,\ b_4 = 0.0125,\ b_5 = 0.125,\ b_6 = 0.075,\ b_7 = 0.0125,\ b_8 = .025,$ and $\lambda = 0.5$

For the equilibrium point $E_4(0.47619, 0.290941, 0.03816, 0.49603)$, the conditions (4.2), (4.5), (5.16), (5.17), (5.18) and (5.19) (0.02031 > 0.0125, 1> 0.55632, 0.47619 > 0.26275, 0.025 > 0.00947, 0.00311 > 0.00118, 0.00391 > .00051) are satisfied. Hence by theorem (5.5), $E_4(0.47619, 0.290941, 0.03816, 0.49603)$ is locally asymptotically stable (see Figure(4)).



Figure(4)

VII. Conclusion

The considered prey-predator model with Michaelis Menten Holling Type II predation exhibits five equilibrium points. The trivial equilibrium point is unstable. All the other equilibrium points maintains the local stability property when necessary conditions are met as mentioned in the respective theorems.

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