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On Product Approximation Spaces

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Abstract:
The present work is a study of the properties of the product rough approximation space determined by two rough approximation spaces. It is found that the projection functions are continuous with respect to the induced topologies on the product space and the individual spaces. Further, the product rough topology is defined as the topology of rough sets generated by the family of the products of the rough open sets in the rough topologies on the individual spaces. As in the case of general topological spaces, the product rough topology is found to be the smallest topology which makes the projection functions rough continuous. The rough interior and rough closure with respect to the rough product topology are also presented.
***Key Words**: Approximation space; Product Topology; Rough Set; Topology; Rough Topology.*

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I. Introduction

During the early 1980's, the development of rough set theory by Z. Pawlak paved a new way to handle uncertainty in the information available [13]. The theory has found potential applications in various domains. Since its inception, the rough set theoretical concepts have been correlated with the topological concepts and a number of studies have been conducted in this regard. Many authors analyzed the topology on the approximation spaces induced by the lower and upper approximations [2,7,9,10,19,20]. Several others discussed about the rough set approximations on topological spaces [1,3,8,16].

The idea of rough topology on an approximation space is a relatively new concept. B. P. Mathew and S. J. John [11] considered a rough topology on a rough set to be an ordered pair of topologies of exact subsets of the lower and upper approximations of that rough set. The concepts like separation axioms, connectedness, and compactness on these rough topological spaces were studied by M. Ravindran and A. J. Divya [15]. But these topologies are actually topology of rough sets and not rough topology of rough sets. Just like a rough set which is a pair of approximations of a set in an approximation space, a rough topology must be a pair of approximations of the extended approximation space. Such a definition of a rough topology was proposed by the present authors and its properties were investigated in [17].

In the present work, the properties of the product of two approximation spaces are investigated. The approximations of the product of two sets are proved to be the product of their approximations. Also, the induced topology on the product approximation space is studied. Besides, the product rough topology, which is defined as the product topology determined by the rough open sets in the rough topologies on the individual spaces is discussed and it is verified that it is the smallest topology which makes the projection functions rough continuous. The rough interior and rough closure with respect to the rough product topology are also presented. The remaining paper is divided as follows: section 2 provides many of the basic concepts and formula needed for the paper, section 3 studies the topological properties of the product approximation space, section 4 presents the product rough topology and its properties and section 5 gives the conclusion.

II. Basic Concepts

Definition 2.1: Consider the approximation space (X, θ), where X is a non-empty set of objects and θ is an equivalence relation on X. The equivalence classes of θ are called *elementary sets* and sets which can be expressed as union of some equivalence classes are called *composed sets [13]*.

Definition 2.2: The *lower and upper approximations* of $A \subseteq X$, with respect to θ are defined respectively as

$$\underline{\theta}(A) = \{x \in U : [x]_\theta \subseteq A\} \quad (1)$$

$$\overline{\theta}(A) = \{x \in U : [x]_\theta \cap A \neq \emptyset\} \quad (2)$$

where, $[x]_\theta$is the equivalence class of θ containing x. In other words, $\underline{\theta}(A)$ is the union of equivalence classes that are contained in A and $\overline{\theta}(A)$ is the union of equivalence classes that have non-empty intersection with A. Thus, both lower and upper approximations are composed sets on (X, θ) [13].

Proposition 2.3: The properties of the rough set approximations are as follows [14]:

1) $\underline{\theta}(\emptyset) = \overline{\theta}(\emptyset) = \emptyset, \underline{\theta}(X) = \overline{\theta}(X) = X$
2) $\underline{\theta}(A) \subseteq A \subseteq \overline{\theta}(A)$
3) $\underline{\theta}\left(\underline{R}(A)\right) = \underline{\theta}(A) = \overline{\theta}\left(\underline{\theta}(A)\right)$
4) $\overline{\theta}\left(\overline{\theta}(A)\right) = \overline{\theta}(A) = \underline{\theta}\left(\overline{\theta}(A)\right)$
5) $\underline{\theta}(A \cap B) = \underline{\theta}(A) \cap \underline{\theta}(B), \overline{\theta}(A \cap B) \subseteq \overline{\theta}(A) \cap \overline{\theta}(B)$
6) $\underline{\theta}(A \cup B) \supseteq \underline{\theta}(A) \cup \underline{\theta}(B), \overline{\theta}(A \cup B) = \overline{\theta}(A) \cup \overline{\theta}(B)$
7) $\underline{\theta}(A^C) = \left(\overline{\theta}(A)\right)^C, \overline{\theta}(A^C) = (\underline{\theta}(A))^C$
8) $A \subseteq B \Rightarrow \underline{\theta}(A) \subseteq \underline{\theta}(B),\ \overline{\theta}(A) \subseteq \overline{\theta}(B)$
9) $\underline{\theta}([x]_\theta) = [x]_\theta = \overline{\theta}([x]_\theta)$ for all $x \in X$

Definition 2.4: Let (X, θ) be an approximation space. Then, the lower approximation operator satisfies the properties of an interior operator and hence it determines a topology $\mathcal{T}_\theta = \{A \subseteq X\colon \underline{\theta}(A) = A\}$, called the *induced topology* on X. A subset $A \in \mathcal{T}_\theta$ if and only if $\underline{\theta}(A) = A$. Hence, $\mathcal{T}_\theta$ consists of all composed sets on X. By property (7) of theorem 2.3, a set A will be closed iff $\overline{\theta}(A) = A$. Hence all open subsets are closed and $\mathcal{T}_\theta$ is the quasi-discrete topology. The family of all equivalence classes form a basis for $\mathcal{T}_\theta$ [14].

Definition 2.5: The *rough equality relation* Θ on $\mathcal{P}(X)$ is defined as

$$(A, B) \in \Theta \iff \underline{\theta}(A) = \underline{\theta}(A) \text{and} \overline{\theta}(A) = \overline{\theta}(A) \tag{3}$$

Then, Θ is an equivalence relation on $\mathcal{P}(X)$ and the pair $(\mathcal{P}(X),\ \Theta)$ is called the *extended approximation space* corresponding to (X, θ) [14].

Definition 2.6: A rough set on (X, θ) is given by a pair of approximations $\langle \underline{A}, \overline{A} \rangle$, where $\underline{A}$ and $\overline{A}$ are subsets of X such that $\underline{A} = \underline{\theta}(A)$ and $\overline{A} = \overline{\theta}(A)$, for some $A \subseteq X$ [14]. For convenience, a rough set may be denoted by $\theta(A) = \langle \underline{\theta}(A), \overline{\theta}(A) \rangle$, where $A \subseteq X$.

Definition 2.7: The *rough inclusion, rough union, rough intersection and rough complement* operations of rough sets are given in [5] as

$$\langle \underline{\theta}(A), \overline{\theta}(A) \rangle \subseteq \langle \underline{\theta}(B), \overline{\theta}(B) \rangle \Leftrightarrow \underline{\theta}(A) \subseteq \underline{\theta}(B),\ \overline{\theta}(A) \subseteq \overline{\theta}(B) \tag{4}$$

$$\langle \underline{\theta}(A), \overline{\theta}(A) \rangle \cup \langle \underline{\theta}(B), \overline{\theta}(B) \rangle = \langle \underline{\theta}(A) \cup \underline{\theta}(B), \overline{\theta}(A) \cup \overline{\theta}(B) \rangle \tag{5}$$

$$\langle \underline{\theta}(A), \overline{\theta}(A) \rangle \cap \langle \underline{\theta}(B), \overline{\theta}(B) \rangle = \langle \underline{\theta}(A) \cap \underline{\theta}(B), \overline{\theta}(A) \cap \overline{\theta}(B) \rangle \tag{6}$$

$$\langle \underline{\theta}(A), \overline{\theta}(A) \rangle^C = \langle (\overline{\theta}(A))^C, (\underline{\theta}(A))^C \rangle = \langle \underline{\theta}(A^C), \overline{\theta}(A^C) \rangle \tag{7}$$

respectively.

Definition 2.8: Let $\mathcal{T}$ be a subfamily of $\mathcal{P}(U)$. The pair $\langle \underline{\Theta}(\mathcal{T}), \overline{\Theta}(\mathcal{T}) \rangle$ is called a *rough topology* on X, if both $\underline{\Theta}(\mathcal{T})$ and $\overline{\Theta}(\mathcal{T})$ are topologies on X. The set X together with the rough topology is called a *rough topological space* [17].

Definition 2.9: Let $\langle \underline{\Theta}(\mathcal{T}), \overline{\Theta}(\mathcal{T}) \rangle$ be a rough topology on X. The rough set $\langle \underline{\theta}(A), \overline{\theta}(A) \rangle$ is called *rough open* if $\underline{\theta}(A)$ is an open set in $\underline{\Theta}(\mathcal{T})$ and $\overline{\theta}(A)$ is an open set in $\overline{\Theta}(\mathcal{T})$. It is called *rough closed* if its rough complement $\langle (\underline{\theta}(A))^C, (\overline{\theta}(A))^C \rangle = \langle \underline{\theta}(A^C), \overline{\theta}(A^C) \rangle$ is *rough open [17].*

Definition 2.10: The *rough interior* of a rough set $\langle \underline{\theta}(A), \overline{\theta}(A) \rangle$ is the largest rough open set contained in it. The *rough closure* of a rough set is the smallest *rough closed set* containing it.*[17]*.

Definition 2.11: The set $X_1 \times X_2$ together with the equivalence relation θ defined by

$$(x_1, x_2)\ \theta\ (y_1, y) \Leftrightarrow x_1 \theta_1 y_1, \quad x_2 \theta_2 y_2 \tag{8}$$

is called the *product approximation space* of (X_1, θ_1) and (X_2, θ_2) [12].

Proposition 2.12: For all elements $x \in X_1$ and $y \in X_2$, $[(x, y)]_\theta = [x]_{\theta_1} \times [y]_{\theta_2}$ [12].

Definition 2.13: Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be two topological spaces. The product topology is the topology $\mathcal{T}$ on $X_1 \times X_2$, generated by the basis $\mathfrak{B} = \{U \times V : U \in \mathcal{T}_1,\ V \in \mathcal{T}_2\}$ [6].

Proposition 2.14: The product topology on $X_1 \times X_2$ is the weakest topology that make each projection function $\Pi_i\colon X_1 \times X_2 \longrightarrow X_i$ defined by $\Pi_i(x_1 \times x_2) = x_i$ continuous [6].

III. Topology on Product Approximation Spaces

Let (X_1, θ_1) and (X_2, θ_2) be two approximation spaces where X_1 and X_2 are non-empty sets and θ_1 and θ_2 are equivalence relations on X_1 and X_2 respectively. Then, $(\mathcal{P}(X_1), \Theta_1)$and $(\mathcal{P}(X_2), \Theta_2)$ are the corresponding extended approximation spaces. Let $(X_1 \times X_2, \theta)$ be the product approximation space.

Theorem 3.1: For all $A_1 \subseteq X_1$ and $A_2 \subseteq X_2$,

i) $\underline{\theta_1}(A_1) \times \underline{\theta_2}(A_2) = \underline{\theta}(A_1 \times A_2)$

ii) $\overline{\theta_1}(A_1) \times \overline{\theta_2}(A_2) = \overline{\theta}(A_1 \times A_2)$.

Thus,$\langle \underline{\theta_1}(A_1) \times \underline{\theta_2}(A_2),\ \overline{\theta_1}(A_1) \times \overline{\theta_2}(A_2) \rangle = \langle \underline{\theta}(A_1 \times A_2),\ \overline{\theta}(A_1 \times A_2) \rangle$ is a rough set on $(X_1 \times X_2, \theta)$.

Proof:

i) $\forall\, x_1 \in A_1,\ x_2 \in A_2, (x_1, x_2) \in \underline{\theta}(A_1 \times A_2) \Leftrightarrow [(x_1, x_2)]_\theta \subseteq A_1 \times A_2 \Leftrightarrow [x_1]_{\theta_1} \times [x_2]_{\theta_2} \subseteq A_1 \times A_2$
$\Leftrightarrow [x_1]_{\theta_1} \subseteq A_1, [x_2]_{\theta_2} \subseteq A_2 \ \Leftrightarrow x_1 \in \underline{\theta_1}(A_1), x_2 \in \underline{\theta_2}(A_2) \ \Leftrightarrow (x_1, x_2) \in \underline{\theta_1}(A_1) \times \underline{\theta_2}(A_2)$.
Thus, $\underline{\theta}(A_1 \times A_2) = \underline{\theta_1}(A_1) \times \underline{\theta}(A_2)$.

ii) $(x_1, x_2) \in \overline{\theta}(A_1 \times A_2) \ \Leftrightarrow [(x_1, x_2)]_\theta \cap (A_1 \times A_2) \neq \varphi \ \Leftrightarrow [x_1]_{\theta_1} \times [x_2]_{\theta_2} \cap (A_1 \times A_2) \neq \varphi$
$\Leftrightarrow [x_1]_{\theta_1} \cap A_1 \neq \varphi, [x_2]_{\theta_2} \cap A_2 \neq \varphi \ \Leftrightarrow x_1 \in \overline{\theta_1}(A_1), x_2 \in \overline{\theta_2}(A_2) \Leftrightarrow (x_1, x_2) \in \overline{\theta_1}(A_1) \times \overline{\theta}(A_2)$.
Thus, $\overline{\theta}(A_1 \times A_2) = \overline{\theta_1}(A_1) \times \overline{\theta}(A_2)$.

Theorem 3.2: For all $A_1 \subseteq X_1$ and $A_2 \subseteq X_2$,

i) $\underline{\theta}((A_1 \times A_2)^C) = \left(\underline{\theta_1}({A_1}^C) \times X_2\right) \cup \left(X_1 \times \underline{\theta_2}({A_2}^C)\right)$

ii) $\overline{\theta}((A_1 \times A_2)^C) = \left(\overline{\theta_1}({A_1}^C) \times X_2\right) \cup \left(X_1 \times \overline{\theta_2}({A_2}^C)\right)$.

Proof:

i) $(x_1, x_2) \in \underline{\theta}((A_1 \times A_2)^C) \ \Leftrightarrow [(x_1, x_2)]_\theta \subseteq (A_1 \times A_2)^C \ \Leftrightarrow [(x_1, x_2)]_\theta \cap (A_1 \times A_2) = \emptyset$
$\Leftrightarrow [x_1]_{\theta_1} \times [x_2]_{\theta_2} \cap (A_1 \times A_2) = \emptyset \Leftrightarrow [x_1]_{\theta_1} \cap A_1 = \emptyset \ OR\ [x_2]_{\theta_2} \cap A_2 = \emptyset$
$\Leftrightarrow x_1 \notin \overline{\theta_1}(A_1)\ OR\ x_2 \notin \overline{\theta_2}(A_2) \Leftrightarrow x_1 \in \left(\overline{\theta_1}(A_1)\right)^C\ OR\ x_2 \in \left(\overline{\theta_2}(A_2)\right)^C$
$\Leftrightarrow x_1 \in \underline{\theta_1}({A_1}^C)\ OR\ x_2 \in \underline{\theta_2}({A_2}^C) \Leftrightarrow (x_1, x_2) \in \underline{\theta_1}({A_1}^C) \times X_2\ OR\ (x_1, x_2) \in X_1 \times \underline{\theta_2}(A_2)^C$
$\Leftrightarrow (x_1, x_2) \in \left(\underline{\theta_1}({A_1}^C) \times X_2\right) \cup \left(X_1 \times \underline{\theta_2}(A_2)^C\right)$
Thus, $\underline{\theta}((A_1 \times A_2)^C) = \left(\underline{\theta_1}({A_1}^C) \times X_2\right) \cup \left(X_1 \times \underline{\theta_2}(A_2)^C\right)$

ii) $(x_1, x_2) \in \overline{\theta}((A_1 \times A_2)^C) \Leftrightarrow [(x_1, x_2)]_\theta \cap (A_1 \times A_2)^C \neq \emptyset \Leftrightarrow [(x_1, x_2)]_\theta \nsubseteq A_1 \times A_2$
$\Leftrightarrow [x_1]_{\theta_1} \times [x_2]_{\theta_2} \nsubseteq A_1 \times A_2 \Leftrightarrow [x_1]_{\theta_1} \nsubseteq A_1\ OR\ [x_2]_{\theta_2} \nsubseteq A_2$
$\Leftrightarrow x_1 \notin \underline{\theta_1}(A_1)\ OR\ x_2 \notin \underline{\theta_2}(A_2) \Leftrightarrow x_1 \in \left(\underline{\theta_1}(A_1)\right)^C\ OR\ x_2 \in \left(\underline{\theta_2}(A_2)\right)^C$
$\Leftrightarrow x_1 \in \overline{\theta_1}({A_1}^C)\ OR\ x_2 \in \overline{\theta_2}({A_2}^C)$
$\Leftrightarrow (x_1, x_2) \in \left(\overline{\theta_1}({A_1}^C) \times X_2\right)\ OR\ (x_1, x_2) \in \left(X_1 \times \overline{\theta_2}({A_2}^C)\right)$
$\Leftrightarrow (x_1, x_2) \in \left(\overline{\theta_1}({A_1}^C) \times X_2\right) \cup \left(X_1 \times \overline{\theta_2}({A_2}^C)\right)$
Thus, $\overline{\theta}((A_1 \times A_2)^C) = \left(\overline{\theta_1}({A_1}^C) \times X_2\right) \cup \left(X_1 \times \overline{\theta_2}({A_2}^C)\right)$.

Theorem 3.3: Let $\mathcal{T}_{1_{\theta_1}}$ and $\mathcal{T}_{2_{\theta_2}}$ be the induced topologies on X_1 and X_2 respectively and $\mathcal{T}_\theta$ denote the induced topology on $X_1 \times X_2$. Then, the projection functions $\Pi_i: X_1 \times X_2 \longrightarrow X_i$ are continuous with respect to the induced topologies on $X_1 \times X_2$ and X_i for $i = 1,2$.

Proof:

We have, $\mathcal{T}_{1_{\theta_1}} = \left\{A_1 \subseteq X_1 : \underline{\theta_1}(A_1) = A_1\right\}$,$\mathcal{T}_{2_{\theta_2}} = \left\{A_2 \subseteq X_2 : \underline{\theta_1}(A_2) = A_2\right\}$ and $\mathcal{T}_\theta = \left\{A \subseteq X_1 \times X_2 : \underline{\theta}(A) = A\right\}$.
The projection function from $X_1 \times X_2$ to X_1 is given by $\Pi_1(x_1, x_2) = x_1,\ \forall\, (x_1, x_2) \in X_1 \times X_2$.
Let A_1 be an open set in $\left(X_1, \mathcal{T}_{1_{\theta_1}}\right)$. Then, $\underline{\theta_1}(A_1) = A_1$. Consider ${\Pi_1}^{-1}(A_1)$.
Clearly, $\underline{\theta}\left({\Pi_1}^{-1}(A_1)\right) \subseteq {\Pi_1}^{-1}(A_1)$.
Now, $(x_1, x_2) \in {\Pi_1}^{-1}(A_1) \Rightarrow \Pi_1(x_1, x_2) \in A_1 \Rightarrow x_1 \in A_1 \Rightarrow [x_1]_{\theta_1} \subseteq A_1$.
Also, $(y_1, y_2) \in [(x_1, x_2)]_\theta \Rightarrow (y_1, y_2) \in [x_1]_{\theta_1} \times [x_2]_{\theta_2} \Rightarrow y_1 \in [x_1]_{\theta_1} \Rightarrow y_1 \in A_1$.
So, $(y_1, y_2) \in {\Pi_1}^{-1}(A_1)$. Hence, $[(x_1, x_2)]_\theta \subseteq {\Pi_1}^{-1}(A_1)$. Therefore, $(x_1, x_2) \in \underline{\theta}\left({\Pi_1}^{-1}(A_1)\right)$.
Thus,${\Pi_1}^{-1}(A_1) \subseteq \underline{\theta}\left({\Pi_1}^{-1}(A_1)\right)$.
It follows that, $\underline{\theta}\left({\Pi_1}^{-1}(A_1)\right) = {\Pi_1}^{-1}(A_1)$. Therefore, ${\Pi_1}^{-1}(A_1)$ is an open set in $(X_1 \times X_2, \mathcal{T}_\theta)$.

That is, the inverse image of open sets in $\mathcal{T}_\theta$ are open. Hence Π_1 is continuous. Similarly, Π_2 is continuous.

Corollary 3.4: Let $\mathcal{T}_{1_{\theta_1}}$ and $\mathcal{T}_{2_{\theta_2}}$ be the induced topologies on X_1 and X_2 respectively and $\mathcal{T}_\theta$ denote the induced topology on $X_1 \times X_2$. Then, $\mathcal{T}_\theta$ is stronger than the product topology on $X_1 \times X_2$ determined by $\mathcal{T}_{1_{\theta_1}}$ and $\mathcal{T}_{2_{\theta_2}}$.

Proof:

The product topology on $X_1 \times X_2$ determined by $\mathcal{T}_{1_{\theta_1}}$ and $\mathcal{T}_{2_{\theta_2}}$ is the weakest topology on $X_1 \times X_2$ that makes each projection functions continuous. From theorem 3.3, each projection function is continuous with respect to the topology $\mathcal{T}_\theta$ on $X_1 \times X_2$. Therefore, $\mathcal{T}_\theta$ is stronger than the product topology on $X_1 \times X_2$ determined by $\mathcal{T}_{1_{\theta_1}}$ and $\mathcal{T}_{2_{\theta_2}}$.

IV. Product Rough Topology

Definition 4.1: The *product of the rough set* $\theta_1(A_1)$ on (X_1, θ_1) and the rough set $\theta_2(A_2)$ on (X_2, θ_2) is defined as the rough set $\theta(A_1 \times A_2) = \langle \underline{\theta}(A_1 \times A_2),\ \overline{\theta}(A_1 \times A_2)\rangle$ on the product approximation space $(X_1 \times X_2, \theta)$.

Theorem 4.2: Let $\langle \underline{\Theta_1}(\mathcal{T}_1),\ \overline{\Theta_1}(\mathcal{T}_1)\rangle$ and $\langle \underline{\Theta_2}(\mathcal{T}_2),\ \overline{\Theta_2}(\mathcal{T}_2)\rangle$ be rough topologies on X_1 and X_2 and $\mathfrak{T}_{\mathcal{T}_1}$ and $\mathfrak{T}_{\mathcal{T}_2}$ denote the underlying topologies of rough open sets on X_1 and X_2 respectively. Then, the family of rough sets $\mathfrak{B}_{\mathcal{T}} = \{\theta_1(A_1) \times \theta_2(A_2) :\ \theta_1(A_1) \in \mathfrak{T}_{\mathcal{T}_1},\ \theta_2(A_2) \in \mathfrak{T}_{\mathcal{T}_2}\}$ is a basis for a topology of rough sets on $X_1 \times X_2$.

Proof:

Consider $\theta_1(A_1) \times \theta_2(A_2),\ \theta_1(B_1) \times \theta_2(B_2) \in \mathfrak{B}_{\mathcal{T}}$, where $\theta_1(A_1) = \langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \in \mathfrak{T}_{\mathcal{T}_1}$,

$\theta_2(A_2) = \langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle \in \mathfrak{T}_{\mathcal{T}_2}$ and $\theta_1(B_1) = \langle \underline{\theta_1}(B_1), \overline{\theta_1}(B_1)\rangle \in \mathfrak{T}_{\mathcal{T}_1}$, $\theta_2(B_2) = \langle \underline{\theta_2}(B_2), \overline{\theta_2}(B_2)\rangle \in \mathfrak{T}_{\mathcal{T}_2}$.

Then, $\big(\theta_1(A_1) \times \theta_2(A_2)\big) \cap \big(\theta_1(B_1) \times \theta_2(B_2)\big)$

$$= \langle \underline{\theta_1}(A_1) \times \underline{\theta_2}(A_2), \overline{\theta_1}(A_1) \times \overline{\theta_2}(A_2)\rangle \cap \langle \underline{\theta_1}(B_1) \times \underline{\theta_2}(B_2), \overline{\theta_1}(B_1) \times \overline{\theta_2}(B_2)\rangle$$

$$= \langle \underline{\theta_1}(A_1) \times \underline{\theta_2}(A_2) \cap \underline{\theta_1}(B_1) \times \underline{\theta_2}(B_2), \overline{\theta_1}(A_1) \times \overline{\theta_2}(A_2) \cap \overline{\theta_1}(B_1) \times \overline{\theta_2}(B_2)\rangle$$

$$= \langle \underline{\theta_1}(A_1) \cap \underline{\theta_1}(B_1) \times \underline{\theta_2}(A_2) \cap \underline{\theta_2}(B_2),\ \overline{\theta_1}(A_1) \cap \overline{\theta_1}(B_1) \times \overline{\theta_2}(A_2) \cap \overline{\theta_2}(B_2)\rangle$$

$$= \langle \underline{\theta_1}(A_1) \cap \underline{\theta_1}(B_1), \overline{\theta_1}(A_1) \cap \overline{\theta_1}(B_1)\rangle \times \langle \underline{\theta_2}(A_2) \cap \overline{\theta_1}(B_1), \overline{\theta_2}(A_2) \cap \overline{\theta_2}(B_2)\rangle$$

$$= \Big(\langle \underline{\theta_1}(A_1),\ \overline{\theta_1}(A_1)\rangle \cap \langle \underline{\theta_1}(B_1),\ \overline{\theta_1}(B_1)\rangle\Big) \times \big(\langle \overline{\theta_1}(B_1), \overline{\theta_2}(B_2)\rangle \cap \langle \overline{\theta_1}(B_1), \overline{\theta_2}(B_2)\rangle\big)$$

Since both $\mathfrak{T}_{\mathcal{T}_1}$ and $\mathfrak{T}_{\mathcal{T}_2}$ are topologies of rough sets,

$\langle \underline{\theta_1}(A_1),\ \overline{\theta_1}(A_1)\rangle \cap \langle \underline{\theta_1}(B_1),\ \overline{\theta_1}(B_1)\rangle \in \mathfrak{T}_{\mathcal{T}_1}$ and $\langle \overline{\theta_1}(B_1), \overline{\theta_2}(B_2)\rangle \cap \langle \overline{\theta_1}(B_1), \overline{\theta_2}(B_2)\rangle \in \mathfrak{T}_{\mathcal{T}_2}$.

Therefore, $\theta_1(A_1) \times \theta_2(A_2) \cap \theta_1(B_1) \times \theta_2(B_2) \in \mathfrak{B}_{\mathcal{T}}$. Thus, $\mathfrak{B}_{\mathcal{T}}$ is closed under finite rough intersection.

Also, since $X_1 \in \mathfrak{T}_{\mathcal{T}_1}$ and $X_2 \in \mathfrak{T}_{\mathcal{T}_2}$, $X_1 \times X_2 \in \mathfrak{B}_{\mathcal{T}}$. Hence, $\mathfrak{B}_{\mathcal{T}}$ forms a cover for $X_1 \times X_2$.

Therefore, $\mathfrak{B}_{\mathcal{T}}$is a basis for a topology of rough sets on $X_1 \times X_2$.

Definition 4.3: Let $\langle \underline{\Theta_1}(\mathcal{T}_1),\ \overline{\Theta_1}(\mathcal{T}_1)\rangle$ and $\langle \underline{\Theta_2}(\mathcal{T}_2),\ \overline{\Theta_2}(\mathcal{T}_2)\rangle$ be rough topologies on X_1 and X_2 respectively and $\mathfrak{T}_{\mathcal{T}_1}$ and $\mathfrak{T}_{\mathcal{T}_2}$denote the underlying topologies of rough open sets on X_1 and X_2. Then, the topology generated by the basis $\mathfrak{B}_{\mathcal{T}} = \{\theta_1(A_1) \times \theta_2(A_2) : \theta_1(A_1) \in \mathfrak{T}_{\mathcal{T}_1}, \theta_2(A_2) \in \mathfrak{T}_{\mathcal{T}_2}\}$ is called the *product rough topology* on $X_1 \times X_2$.

Theorem 4.4: Let $\mathfrak{T}_{\mathcal{T}}$ be the product rough topology on $X_1 \times X_2$, generated by $\mathfrak{B}_{\mathcal{T}}$. Then, the product of two rough closed sets in $\mathfrak{T}_{\mathcal{T}_1}$ and $\mathfrak{T}_{\mathcal{T}_2}$ is a rough closed set in $\mathfrak{T}_{\mathcal{T}}$.

Proof:

$\langle \underline{\theta_1}(D_1), \overline{\theta_1}(D_1)\rangle$and $\langle \underline{\theta_2}(D_2), \overline{\theta_2}(D_2)\rangle$ are rough closed sets in $\mathfrak{T}_{\mathcal{T}_1}$ and $\mathfrak{T}_{\mathcal{T}_2}$ respectively

$\Rightarrow \langle \underline{\theta_1}(D_1{}^C), \overline{\theta_1}(D_1{}^C)\rangle$ and $\langle \underline{\theta_2}(D_2{}^C), \overline{\theta_2}(D_2{}^C)\rangle$ are rough open sets in $\mathfrak{T}_{\mathcal{T}_1}$ and $\mathfrak{T}_{\mathcal{T}_2}$ respectively

$\Rightarrow \langle \underline{\theta_1}(D_1{}^C), \overline{\theta_1}(D_1{}^C)\rangle \times \langle X_2, X_2\rangle$ and $\langle X_1, X_1\rangle \times \langle \underline{\theta_2}(D_2{}^C), \overline{\theta_2}(D_2{}^C)\rangle$ are rough open sets in $\mathfrak{T}_{\mathcal{T}}$, since $\langle X_1, X_1\rangle$ and $\langle X_2, X_2\rangle$ are rough open sets in $\mathfrak{T}_{\mathcal{T}_1}$ and $\mathfrak{T}_{\mathcal{T}_2}$ respectively

$\Rightarrow \langle \underline{\theta_1}(D_1{}^C) \times X_2, \overline{\theta_1}(D_1{}^C) \times X_2\rangle$ and $\langle X_1 \times \underline{\theta_2}(D_2{}^C), X_1 \times \overline{\theta_2}(D_2{}^C)\rangle$are rough open sets in $\mathfrak{T}_{\mathcal{T}}$

$\Rightarrow \langle \underline{\theta_1}(D_1{}^C) \times X_2, \overline{\theta_1}(D_1{}^C) \times X_2\rangle \cup \langle X_1 \times \underline{\theta_2}(D_2{}^C), X_1 \times \overline{\theta_2}(D_2{}^C)\rangle$ is a rough open set in $\mathfrak{T}_{\mathcal{T}}$

$\Rightarrow \langle \big(\underline{\theta_1}(D_1{}^C) \times X_2\big) \cup \Big(X_1 \times \underline{\theta_2}(D_2{}^C)\Big), \big(\overline{\theta_1}(D_1{}^C) \times X_2\big) \cup \Big(X_1 \times \overline{\theta_2}(D_2{}^C)\Big)\rangle$ is rough openin $\mathfrak{T}_{\mathcal{T}}$

$\Rightarrow \langle \underline{\theta}((D_1 \times D_2)^C), \overline{\theta}((D_1 \times D_2)^C)\rangle$ is rough openin $\mathfrak{T}_{\mathcal{T}}$

$\Rightarrow \langle \underline{\theta}\big((D_1 \times D_2)\big), \overline{\theta}\big((D_1 \times D_2)\big)\rangle$ is rough closed in $\mathfrak{T}_{\mathcal{T}}$

$\Rightarrow \langle \underline{\theta_1}(D_1), \overline{\theta_1}(D_1)\rangle \times \langle \underline{\theta_2}(D_2), \overline{\theta_2}(D_2)\rangle$ is rough closed in $\mathfrak{T}_{\mathcal{T}}$

Thus, the product of two rough closed sets in $\mathfrak{T}_{\mathcal{T}_1}$ and $\mathfrak{T}_{\mathcal{T}_2}$ is a rough closed set in $\mathfrak{T}_{\mathcal{T}}$.

Lemma 4.5: Let $\langle \underline{\Theta_1}(\mathcal{J}_1),\ \overline{\Theta_1}(\mathcal{J}_1)\rangle$ and $\langle \underline{\Theta_2}(\mathcal{J}_2),\ \overline{\Theta_2}(\mathcal{J}_2)\rangle$ be rough topologies on X_1 and X_2 respectively and let $\mathfrak{T}_{\mathcal{J}}$ be the product rough topology on $X_1 \times X_2$. Then, the projections $\Pi_i: X_1 \times X_2 \longrightarrow X_i$ are rough continuous functions with respect to $\mathfrak{T}_{\mathcal{J}}$ and $\mathfrak{T}_{\mathcal{J}_i}$ for $i = 1, 2$.
Proof:
Let $\theta_1(A_1) = \langle \underline{\theta_1}(A_1),\ \overline{\theta_1}(A_1)\rangle$ be a rough open set in $\mathfrak{T}_{\mathcal{J}_1}$.
We have, ${\Pi_1}^{-1}\left(\underline{\theta_1}(A_1)\right) = \underline{\theta_1}(A_1) \times X_2$ and ${\Pi_1}^{-1}\left(\overline{\theta_1}(A_1)\right) = \overline{\theta_1}(A_1) \times X_2$.
Hence, ${\Pi_1}^{-1}\left(\langle \underline{\theta_1}(A_1),\ \overline{\theta_1}(A_1)\rangle\right) = \langle \underline{\theta_1}(A_1) \times X_2,\ \overline{\theta_1}(A_1) \times X_2\rangle = \langle \underline{\theta_1}(A_1),\ \overline{\theta_1}(A_1)\rangle \times \langle X_2,\ X_2\rangle$.
Also, $\langle \underline{\theta_1}(A_1),\ \overline{\theta_1}(A_1)\rangle$ is a rough open set in $\mathfrak{T}_{\mathcal{J}_1}$ and $\langle X_2,\ X_2\rangle$ is a rough open set in $\mathfrak{T}_{\mathcal{J}_2}$.
Hence, $\langle \underline{\theta_1}(A_1),\ \overline{\theta_1}(A_1)\rangle \times \langle X_2,\ X_2\rangle$ is a rough open set in the rough product topology $\mathfrak{T}_{\mathcal{J}}$.
That is, ${\Pi_1}^{-1}\left(\langle \underline{\theta_1}(A_1),\ \overline{\theta_1}(A_1)\rangle\right)$ is a rough open set in the rough product topology $\mathfrak{T}_{\mathcal{J}}$.
Hence Π_1 is continuous and similarly, Π_2 is continuous.

Theorem 4.6: The product rough topology on $X_1 \times X_2$ is the weakest topology that make the projection functions $\Pi_i: X_1 \times X_2 \longrightarrow X_i$, rough continuous functions with respect to $\mathfrak{T}_{\mathcal{J}}$ and $\mathfrak{T}_{\mathcal{J}_i}$ for $i = 1, 2$.
Proof:
From lemma 4.5, the projections are rough continuous functions with respect to $\mathfrak{T}_{\mathcal{J}}$ and $\mathfrak{T}_{\mathcal{J}_i}$ for $i = 1, 2$.
Let $\mathfrak{T}_{\mathcal{J}}'$ represent any topology of rough sets on $X_1 \times X_2$ that make both the projections Π_1 and Π_2 rough continuous. We will prove that the basis of the product rough topology $\mathfrak{B}_{\mathcal{J}} \subseteq \mathfrak{T}_{\mathcal{J}}'$.
Consider a basic rough open set $\langle \underline{\theta}(A),\ \overline{\theta}(A)\rangle \in \mathfrak{B}_{\mathcal{J}}$.
Then, $\langle \underline{\theta}(A), \overline{\theta}(A)\rangle = \langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times \langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$, where $\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle$ and $\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$ are rough open sets in $\mathfrak{T}_{\mathcal{J}_i}$ and $\mathfrak{T}_{\mathcal{J}_i}$ respectively.
We have, ${\Pi_1}^{-1}\left(\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle\right) = \langle \underline{\theta_1}(A_1) \times X_2, \overline{\theta_1}(A_1) \times X_2\rangle$ and
${\Pi_2}^{-1}\left(\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle\right) = \langle X_1 \times\ \underline{\theta_2}(A_2) X_2, X_1 \times \overline{\theta_2}(A_2)\rangle$.
Both Π_1 and Π_2 are rough continuous functions with respect to the product rough topology $\mathfrak{T}_{\mathcal{J}}'$.
Hence ${\Pi_1}^{-1}\left(\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle\right)$ and ${\Pi_1}^{-1}\left(\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle\right)$ are rough open sets in $\mathfrak{T}_{\mathcal{J}}'$.
Thus, $\langle \underline{\theta_1}(A_1) \times X_2, \overline{\theta_1}(A_1) \times X_2\rangle$ and $\langle X_1 \times \underline{\theta_2}(A_2) X_2, X_1 \times \overline{\theta_2}(A_2)\rangle$ are rough open sets in $\mathfrak{T}_{\mathcal{J}}'$.
Since, $\mathfrak{T}_{\mathcal{J}}'$ is a topology, $\langle \underline{\theta_1}(A_1) \times X_2,\ \overline{\theta_1}(A_1) \times X_2\rangle \cap \langle X_1 \times\ \underline{\theta_2}(A_2),\ X_1 \times \overline{\theta_2}(A_2)\rangle$ is a rough open set in $\mathfrak{T}_{\mathcal{J}}'$. That is, $\langle \underline{\theta_1}(A_1) \times \underline{\theta_2}(A_2),\ \overline{\theta_1}(A_1) \times \overline{\theta_2}(A_2)\rangle$ is a rough open set in $\mathfrak{T}_{\mathcal{J}}'$.
Thus, $\langle \underline{\theta}(A),\ \overline{\theta}(A)\rangle \in \mathfrak{T}_{\mathcal{J}}'$. Hence, $\mathfrak{B}_{\mathcal{J}} \subseteq \mathfrak{T}_{\mathcal{J}}'$. It follows that $\mathfrak{T}_{\mathcal{J}} \subseteq \mathfrak{T}_{\mathcal{J}}'$.
Thus, the product rough topology on $X_1 \times X_2$ is the weakest topology that make the the projection functions $\Pi_i: X_1 \times X_2 \longrightarrow X_i$, rough continuous functions with respect to $\mathfrak{T}_{\mathcal{J}}$ and $\mathfrak{T}_{\mathcal{J}_i}$ for $i = 1, 2$.

Theorem 4.7: If $\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle$ and $\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$ are rough sets on (X_1, θ_1) and (X_2, θ_2) respectively, then

i) $int\left(\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times \langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle\right) = int\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times int\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$

ii) $cl\left(\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times \langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle\right) = cl\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times cl\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$

Proof:

i) The rough interior of a rough set is the largest rough open set contained in it.
So, $int\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \subseteq \langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle$ and $int\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle \subseteq \langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$.
Hence, $int\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times int\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle \subseteq \left(\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times \langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle\right)$.
Also, $int\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle$ and $int\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$ are rough open sets in $\mathfrak{T}_{\mathcal{J}_1}$ and $\mathfrak{T}_{\mathcal{J}_2}$ respectively.
Hence, $int\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times int\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$ is a rough open set in $\mathfrak{T}_{\mathcal{J}}$.
Therefore, $int\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times int\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle \subseteq int\left(\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times \langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle\right)$.
Now if $\langle \underline{\theta}(B), \overline{\theta}(B)\rangle$ is an arbitrary rough open set contained in $\langle \underline{\theta_1}(A_1),\ \overline{\theta_1}(A_1)\rangle \times \langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$, then
$$\langle \underline{\theta}(B), \overline{\theta}(B)\rangle = \cup_i \left(\langle \underline{\theta_1}(U_i), \overline{\theta_1}(U_i)\rangle \times \langle \underline{\theta_2}(V_i), \overline{\theta_2}(V_i)\rangle\right)$$
where $\langle \underline{\theta_1}(U_i), \overline{\theta_1}(U_i)\rangle \times \langle \underline{\theta_2}(V_i), \overline{\theta_2}(V_i)\rangle \in \mathfrak{B}_{\mathcal{J}}, \forall i$.
So, $\cup_i \left(\langle \underline{\theta_1}(U_i), \overline{\theta_1}(U_i)\rangle \times \langle \underline{\theta_2}(V_i), \overline{\theta_2}(V_i)\rangle\right) \subseteq\ \langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times \langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$.

Hence, for each i, $\langle \underline{\theta_1}(U_i), \overline{\theta_1}(U_i)\rangle \times \langle \underline{\theta_2}(V_i), \overline{\theta_2}(V_i)\rangle \subseteq \langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times \langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$.

That is, $\langle \underline{\theta_1}(U_i), \overline{\theta_1}(U_i)\rangle \subseteq \langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle$ and $\langle \underline{\theta_2}(V_i), \overline{\theta_2}(V_i)\rangle \subseteq \langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle, \forall i$.

Also, $\langle \underline{\theta_1}(U_i), \overline{\theta_1}(U_i)\rangle$ and $\langle \underline{\theta_2}(V_i), \overline{\theta_2}(V_i)\rangle$ are rough open sets, $\forall i$.

Therefore, $\langle \underline{\theta_1}(U_i), \overline{\theta_1}(U_i)\rangle \subseteq int\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle$ and $\langle \underline{\theta_2}(V_i), \overline{\theta_2}(V_i)\rangle \subseteq int\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle,\ \forall i$.

Thus, $\langle \underline{\theta_1}(U_i), \overline{\theta_1}(U_i)\rangle \times \langle \underline{\theta_2}(V_i), \overline{\theta_2}(V_i)\rangle \subseteq int\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times int\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle,\ \forall i$.

Hence, $\langle \underline{\theta}(B), \overline{\theta}(B)\rangle \subseteq int\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times int\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$.

It follows that $int\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times int\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$ is the largest open set in $\mathfrak{T}_{\mathcal{T}}$ contained in $\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times \langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$.

Thus, $int\left(\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times \langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle\right) = int\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times int\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$.

ii) The rough closure of a rough set is the smallest rough closed set containing it.

So, $cl\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \supseteq \langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle$ and $cl\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle \supseteq \langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$.

Hence, $cl\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times cl\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle \supseteq \left(\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times \langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle\right)$.

Also, $cl\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle$ and $cl\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$ are rough closed sets in $\mathfrak{T}_{\mathcal{T}_1}$ and $\mathfrak{T}_{\mathcal{T}_2}$ respectively.

Hence, $cl\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times cl\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$ is a rough closed set in $\mathfrak{T}_{\mathcal{T}}$.

Therefore, $cl\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times cl\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle \supseteq cl\left(\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times \langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle\right)$.

Now if $\langle \underline{\theta}(C), \overline{\theta}(C)\rangle$ is an arbitrary rough closed set containing $\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times \langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$,

Then $\langle \underline{\theta}(C^C), \overline{\theta}(C^C)\rangle$ is a rough open set contained in $\langle \underline{\theta_1}(A_1{}^C), \overline{\theta_1}(A_1{}^C)\rangle \times \langle \underline{\theta_2}(A_2{}^C), \overline{\theta_2}(A_2{}^C)\rangle$. Hence,

$\langle \underline{\theta}(C^C), \overline{\theta}(C^C)\rangle = \cup_i \left(\langle \underline{\theta_1}(U_i), \overline{\theta_1}(U_i)\rangle \times \langle \underline{\theta_2}(V_i), \overline{\theta_2}(V_i)\rangle\right)$ for $\langle \underline{\theta_1}(U_i), \overline{\theta_1}(U_i)\rangle \times \langle \underline{\theta_2}(V_i), \overline{\theta_2}(V_i)\rangle \in \mathfrak{B}_{\mathcal{T}}$.

Also, $\langle \underline{\theta_1}(U_i), \overline{\theta_1}(U_i)\rangle \times \langle \underline{\theta_2}(V_i), \overline{\theta_2}(V_i)\rangle \subseteq \langle \underline{\theta_1}(A_1{}^C), \overline{\theta_1}(A_1{}^C)\rangle \times \langle \underline{\theta_2}(A_2{}^C), \overline{\theta_2}(A_2{}^C)\rangle$ for each i.

Hence, $\langle \underline{\theta_1}(U_i), \overline{\theta_1}(U_i)\rangle \subseteq \langle \underline{\theta_1}(A_1{}^C), \overline{\theta_1}(A_1{}^C)\rangle$ and $\langle \underline{\theta_2}(V_i), \overline{\theta_2}(V_i)\rangle \subseteq \langle \underline{\theta_2}(A_2{}^C), \overline{\theta_2}(A_2{}^C)\rangle$ for each i.

Therefore, $\langle \underline{\theta_1}(U_i{}^C), \overline{\theta_1}(U_i{}^C)\rangle \supseteq \langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle$ and $\langle \underline{\theta_2}(V_i{}^C), \overline{\theta_2}(V_i{}^C)\rangle \supseteq \langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$ for each i.

Also, $\langle \underline{\theta_1}(U_i{}^C), \overline{\theta_1}(U_i{}^C)\rangle$ and $\langle \underline{\theta_2}(V_i{}^C), \overline{\theta_2}(V_i{}^C)\rangle$ are rough closed sets.

So, $\langle \underline{\theta_1}(U_i{}^C), \overline{\theta_1}(U_i{}^C)\rangle \supseteq cl\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle$ and $\langle \underline{\theta_2}(V_i{}^C), \overline{\theta_2}(V_i{}^C)\rangle \supseteq cl\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$ for each i.

Hence, $\langle \underline{\theta_1}(U_i{}^C), \overline{\theta_1}(U_i{}^C)\rangle \times \langle X_2, X_2\rangle \supseteq cl\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times cl\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$ and

$\langle X_1, X_1\rangle \times \langle \underline{\theta_2}(V_i{}^C), \overline{\theta_2}(V_i{}^C)\rangle \supseteq cl\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times cl\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$ for each i.

Thus, $\cap_i \left(\langle \underline{\theta_1}(U_i{}^C), \overline{\theta_1}(U_i{}^C)\rangle \times \langle X_2, X_2\rangle\right) \supseteq cl\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times cl\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$ and

$\cap_i \left(\langle X_1, X_1\rangle \times \langle \underline{\theta_2}(V_i{}^C), \overline{\theta_2}(V_i{}^C)\rangle\right) \supseteq cl\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times cl\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$.

So, $\cap_i \left(\langle \underline{\theta_1}(U_i{}^C), \overline{\theta_1}(U_i{}^C)\rangle \times \langle X_2, X_2\rangle\right) \cup \cap_i \left(\langle X_1, X_1\rangle \times \langle \underline{\theta_2}(V_i{}^C), \overline{\theta_2}(V_i{}^C)\rangle\right)$

$$\supseteq cl\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times cl\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$$

Now, $\cap_i \left(\langle \underline{\theta_1}(U_i{}^C), \overline{\theta_1}(U_i{}^C)\rangle \times \langle X_2, X_2\rangle\right) \cup \cap_i \left(\langle X_1, X_1\rangle \times \langle \underline{\theta_2}(V_i{}^C), \overline{\theta_2}(V_i{}^C)\rangle\right)$

$$= \cap_i \left(\langle \underline{\theta_1}(U_i{}^C), \overline{\theta_1}(U_i{}^C)\rangle \times \langle X_2, X_2\rangle \cup \langle X_1, X_1\rangle \times \langle \underline{\theta_2}(V_i{}^C), \overline{\theta_2}(V_i{}^C)\rangle\right)$$
$$= \cap_i \left(\langle \underline{\theta_1}(U_i{}^C) \times X_2,\ \overline{\theta_1}(U_i{}^C) \times X_2\rangle \cup \langle X_1 \times \underline{\theta_2}(V_i{}^C), X_1 \times \overline{\theta_2}(V_i{}^C)\rangle\right)$$
$$= \cap_i \left(\langle \underline{\theta_1}(U_i{}^C) \times X_2 \cup X_1 \times \underline{\theta_2}(V_i{}^C), \overline{\theta_1}(U_i{}^C) \times X_2 \cup X_1 \times \overline{\theta_2}(V_i{}^C)\rangle\right)$$
$$= \cap_i\left(\langle \underline{\theta}((U_i \times V_i)^C), \overline{\theta}((U_i \times V_i)^C)\rangle\right) = \cap_i\left(\langle \underline{\theta}(U_i \times V_i), \overline{\theta}(U_i \times V_i)\rangle\right)^C$$
$$= \left(\cup_i \langle \underline{\theta}(U_i \times V_i), \overline{\theta}(U_i \times V_i)\rangle\right)^C = \left(\langle \underline{\theta}(C^C), \overline{\theta}(C^C)\rangle\right)^C = \langle \underline{\theta}(C), \overline{\theta}(C)\rangle$$

Thus, $\langle \underline{\theta}(C), \overline{\theta}(C)\rangle \supseteq cl\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times cl\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$.

Therefore, $cl\,\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times cl\,\langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle$ is the smallest closed set containing (rough closure) $\left(\langle \underline{\theta_1}(A_1), \overline{\theta_1}(A_1)\rangle \times \langle \underline{\theta_2}(A_2), \overline{\theta_2}(A_2)\rangle\right)$.

V. Conclusion

The relationship of rough set theory with topology has been of great interest for researchers, from the very beginning of this theory. In this paper, the properties of the product approximation space determined by two approximation spaces have been studied. It has been found that the projection functions are continuous with respect to the induced topologies on the product space and the individual spaces. Also, the induced topology of the product approximation space is proved to be stronger than the product topology of the induced topologies on the individual spaces. Further, the product rough topology has been defined as the product topology determined by the rough open sets in the rough topologies on the individual spaces. It has been verified that as in the case of general topological spaces, the product rough topology is the smallest topology which makes the projection functions rough continuous. The rough interior and rough closure with respect to the rough product topology have also been discussed. The results presented in this paper can be extended to the product of a finite number of approximation spaces.

References

[1]. M. E. Abd El-Monsef, O. A. Embaby and M. K. El-Bably, Comparison between Rough Set Approximations Based on Different Topologies, International Journal of Granular Computing, Rough Sets and Intelligent Systems, 2014;3(4);292-305.
[2]. K. Anitha, Rough Set Theory on Topological Spaces, Rough Sets and Knowledge Technology LNAI 8818, Springer 2014;69-74.
[3]. M. J. Iqlan, M. S. Marouf and M. S. Shulaq, Approximations of Sets in Topological Spaces, International Research Journal of Pure Algebra, 2014;4(12);636-643.
[4]. T. B. Iwinski, Algebraic Approach to Rough Sets, Bulletin of the Polish Academy of Science and Mathematics, 1987;35;673-683.
[5]. J. Jarvinen, The Ordered Set of Rough Sets. in RSCTC 2004, LNAI 3066, eds S. Tsumoto et al., Springer- Verlag, Berlin, 2004;49–58.
[6]. J. Kelley, General Topology, Van Nostrand Company, 1955.
[7]. M. Kondo and W. A. Dudek, Topological Structures of Rough Sets Induced by Equivalence Relations, Journal of advanced computational Intelligence and Intelligent Informatics, 2006;10(5);621-624.
[8]. E. F. Lashin, A. M. Kozae, A. A. Abo Khadra and T. Medhat, Rough Set Theory for Topological Spaces, International Journal of Approximate Reasoning, 2005;40;35-43.
[9]. Z. Li, Topological Properties of Generalized Rough Sets, Proceedings of the Seventh International Conference on Fuzzy Systems and Knowledge, 2010; 2067-2070.
[10]. W. J. Liu, Topological Space Properties of Rough Sets, Proceedings of the third IEEE International Conference on Machine Learning and Cybernetics, Shanghai, 2004;2353-2355.
[11]. B. P. Mathew and S. J. John, On Rough Topological Spaces, International Journal of Mathematical Archive, 2012;3(9);3413-3421.
[12]. M. Nicoletti, J. Uchôa, M. Baptistini, Rough relation properties, International Journal of Applied Mathematics and Computer Science, 2001;11(3);621-635.
[13]. Z. Pawlak, Rough Sets, International Journal of Computer and Information Sciences, 1982;11(5);341-356.
[14]. Z. Pawlak, Rough Sets - Theoretical Aspect of Reasoning About Data, Kluwer Academic Publishers, The Netherlands, 1991.
[15]. M. Ravindran and A. J. Divya, "A Study of Compactness and Connectedness in Rough Topological Spaces", International Journal of Mathematics, 2016;12(6);01-07.
[16]. A. S. Salama, Some Topological Properties of Rough Sets with Tools for Data Mining, International Journal of Computer Mathematics, 2011;8(3)(2);588 595.
[17]. T. K. Sheeja, A. Sunny Kuriakose, Rough Topology on Approximation Spaces, International Journal of Advanced Research in Computer Science,2017;8(9);379-384.
[18]. B. K. Tripathy and A. Mitra, Some Topological Properties of Rough sets and their applications, International Journal of Granular Computing, Rough Sets and Intelligent Systems, 2010;1(4);355-369.
[19]. Q. Wu, T. Wang, Y. X. Huang and J. S. Li, Topology Theory on Rough Sets, IEEE Transactions on Systems, Man and Cybernetics - part B. Cybernetics, 2008;38(1);68 -77.
[20]. H. Yu and W. R. Zhan, On the Topological Properties of Generalized Rough Sets, Information Sciences, 2014;263;141-152.

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