IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 17, Issue 2 Ser. IV (Mar. – Apr. 2021), PP 01-09
www.iosrjournals.org

On Metric Geometry of Weighted Groups

Shreema S. Bhatt and H. V. Dedania

ABSTRACT. Let $(G, \cdot)$ be any group and let $\omega : G \longrightarrow (0, \infty)$ such that $\omega(xy) \leq \omega(x)\omega(y)$ $(x, y \in G)$; so-called a *weight* on G and the pair (G, ω) is called a *weighted group*. In this article, we explore the geometric and topological structure of (G, ω). We associate an invariant metric d_ω with ω and vice-versa. The completion $(\widetilde{G}, \widetilde{\omega})$ of (G, ω) is constructed and the structure theorem for topological groups with an invariant topology is also proved. Several weights on groups of number fields arising from valuations are exhibited.

Date of Submission: 10-04-2021 Date of Acceptance: 26-04-2021

1. INTRODUCTION

Let G be a group with binary operation *multiplicative*. A *weight* on G is a map $\omega : G \longrightarrow (0, \infty)$ such that $\omega(xy) \leq \omega(x)\omega(y)$ $(x, y \in G)$. Looking to (G, ω) as a single object, the analysis on the weighted group (G, ω) is an important aspect of harmonic analysis, especially in the context of the associated Beurling algebra $L^1(G, \omega)$ and the weighted measure algebra $M(G, \omega)$ [BhDe2]. The harmonic analysis on such a weighted group has at least three exciting novel aspects, which are not found in the absence of a weight: (i) the structure of the Beurling algebra $L^1(G, \omega)$ is closely tied up with the structure of the weight ω [BhDe1]; (ii) the $L^p(G, \omega)$ becomes a convolution Banach algebra for a large class of weights ω [Ku]; (iii) the presence of a weight ω tends to impose an analytic structure on the Gel'fand spaces of $L^1(\mathbb{R}^n, \omega)$ and $\ell^1(\mathbb{Z}^n, \omega)$ making Gel'fand transforms holomorphic thereby bringing in the theory of complex analysis [RS].

The positive definiteness and submultiplicativity of ω suggest that it is an analogue of a norm on an algebra and $\omega(x)$ can be thought of as representing the size of x in (G, ω) [BDD]. Thus, like a norm, it should be related with a topological or geometric structure of G. The purpose of this article is to develop this idea. In Section-2, we discuss the relation between weights and metrics. We also discuss the

2010 *Mathematics Subject Classification.* 43A99, and 11S31.
Key words and phrases. Group, Weight, G-invariant weight, Invariant metric, Topological Group, inverse limit, Valuation rings, p-adic valuation.
$*$ Corresponding author
This research work is supported by the UGC-SAP-DRS-III; The grant number is F.510/1/DRS-III/2015(SAP-I), given to the Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar.

completion of the weighted group (G,ω). In Section-3, these results are applied to show that a topological group with an inner invariant topology is dense in the inverse limit of a family of weighted groups. In Section-4, we discuss several weights on groups of number fields such as algebraic numbers and p-adic numbers in the number theory.

2. Weights and Metrics

Throughout G is a group with the identity e. In this section, we establish the relation between weights and metrics. For this purpose, we classify strictly positive functions on G as follow.

Definition 2.1. A map $\omega : G \longrightarrow (0,\infty)$ is: (i) *unitary* if $\omega(x) \geq 1$ $(x \in G)$; (ii) *unitary definite* if ω is unitary and $\omega(x) = 1$ iff $x = e$; (iii) *symmetric* if $\omega(x) = \omega(x^{-1})$ $(x \in G)$; (iv) *G-invariant* if $\omega(sxs^{-1}) = \omega(x)$ $(x, s \in G)$.

Since the next two results are easy to prove, we skip their proofs. They are converse of each other.

Proposition 2.2. *Let $\omega : G \longrightarrow (0,\infty)$ be any function. Define $d_\omega : G \times G \longrightarrow \mathbb{R}$ as $d_\omega(x,y) = \log\omega(xy^{-1})$. Then*

(1) $\omega(x) = e^{d_\omega(x,e)}$ $(x \in G)$.
(2) *d_ω is right invariant.*
(3) *ω is unitary iff d_ω is non-negative.*
(4) *ω is unitary definite iff d_ω is positive definite.*
(5) *ω is symmetric iff d_ω is symmetric.*
(6) *ω is G-invariant iff d_ω is invariant.*
(7) *ω is a weight iff d_ω satisfies the triangle inequality.*

Proposition 2.3. *Let $d : G \times G \longrightarrow \mathbb{R}$ be any right invariant map. Define $\omega_d : G \longrightarrow (0,\infty)$ as $\omega_d(x) = e^{d(x,e)}$. Then*

(1) $d(x,e) = \log\omega_d(x)$ $(x \in G)$.
(2) $d(x,y) = \log\omega_d(xy^{-1})$ $(x, y \in G)$.
(3) *d is non-negative iff ω_d is unitary.*
(4) *d is positive definite iff ω_d is unitary definite.*
(5) *d is symmetric iff ω_d is symmetric.*
(6) *d is invariant iff ω_d is G-invariant.*
(7) *d satisfies the triangle inequality iff ω_d is a weight.*

Next corollary establishes the relation between weights and metrics.

Corollary 2.4. *Let $\omega : G \longrightarrow (0,\infty)$ be any function. Then ω is a unitary definite, symmetric weight on G iff the induced map d_ω on $G \times G$ is a right invariant metric on G.*

Proof. This follows from Proposition 2.2. □

Theorem 2.5. *Let G be any group.*

(1) *If d is an invariant metric, then (G,d) is a topological group.*

(2) *If ω is a unitary definite, symmetric, G-invariant weight, then d_ω is an invariant metric and so (G, d_ω) is a topological group.*
(3) *If d is an invariant metric on G, then ω_d is a unitary definite, symmetric, G-invariant weight on G.*

Proof. (1) Let $(x_n, y_n) \longrightarrow (x, y)$ in $G \times G$. Then

$$\begin{aligned} d(x_n y_n^{-1}, xy^{-1}) &\leq d(x_n y_n^{-1}, x_n y^{-1}) + d(x_n y^{-1}, xy^{-1}) \\ &= d(y_n^{-1}, y^{-1}) + d(x_n, x) \ (\because d \text{ is invariant}) \\ &= d(y_n y_n^{-1} y, y_n y^{-1} y) + d(x_n, x) \ (\because d \text{ is invariant}) \\ &= d(x_n, x) + d(y_n, y). \end{aligned}$$

Since $d(x_n, x) \longrightarrow 0$ and $d(y_n, y) \longrightarrow 0$, we have $d(x_n y_n^{-1}, xy^{-1}) \longrightarrow 0$. Thus the map $f : G \times G \longrightarrow G$ defined as $f(x, y) = xy^{-1}$ is continuous. Hence (G, d) is a topological group.
(2) This follows from Proposition 2.2 and Statement (1) above.
(3) This follows from Proposition 2.3. □

Corollary 2.6. *Let G be an abelian group and ω be a unitary definite weight on G. Then there exists an invariant metric d on G such that (G, d) is a topological group.*

Proof. Define $\omega_0(x) = \omega(x)\omega(x^{-1})$ $(x \in G)$. Then ω_0 is a symmetric weight on G. Since ω is unitary definite and G is abelian, ω_0 is a unitary definite, G-invariant weight on G. Define $d(x, y) = \log \omega_0(xy^{-1})$. Then, by Corollary 2.4, d is an invariant metric on G. By Theorem 2.5(1), (G, d) is a topological group. □

Corollary 2.7. *Let G be a compact group and ω be a unitary definite, symmetric, G-invariant, continuous weight on G. Then G is metrizable.*

Proof. Let τ be the compact topology of G. Define a metric d on G as $d(x, y) = \log \omega(xy^{-1}) + \log \omega(x^{-1}y)$ $(x, y \in G)$. Then d is an invariant metric on G. By Theorem 2.5(1), (G, d) is a topological group. Consider $f : (G, \tau) \longrightarrow (G, d)$ being the identity map $f(x) = x$. First we show that f is continuous. Let $\{x_\alpha\}$ be a net in G such that $x_\alpha \longrightarrow x$ in (G, τ). As (G, τ) is a topological group, $x_\alpha x^{-1} \longrightarrow e$ and $x_\alpha^{-1} x \longrightarrow e$. Since ω is τ-continuous, $\omega(x_\alpha x^{-1}) \longrightarrow \omega(e) = 1$ and $\omega(x_\alpha^{-1} x) \longrightarrow 1$. This implies that $d(x_\alpha, x) = \log \omega(x_\alpha x^{-1}) + \log \omega(x_\alpha^{-1} x) \longrightarrow 0$. Thus f is continuous. Since f is an injective, continuous function from a compact space to a Hausdorff space, it is a homeomorphism. □

Theorem 2.8. *Let (G, τ) be a σ-compact, locally compact group. Let ω be a unitary definite, symmetric, G-invariant, continuous weight on G. If (G, d_ω) is a locally compact group, then (G, τ) is metrizable and it is determined by d_ω.*

Proof. As in the proof of Corollary 2.7, we can show that the identity map $f : (G, \tau) \longrightarrow (G, d_\omega)$ is continuous. Now the result follows from a known result [CoHa, Corollary 2.D.6, P. 46] that an injective, continuous homomorphism from a σ-compact, locally compact group G_1 onto a locally compact group G_2 is a homeomorphism. $\square$

Definition 2.9. Let $\omega : G \longrightarrow (0, \infty)$ be a unital, symmetric weight on G. A net $\{x_\alpha\} \subset G$ is ω-*Cauchy* if it is d_ω-Cauchy. The net $\{x_\alpha\}$ is ω-*convergent* to $x \in G$ if $d_\omega(x_\alpha, x) \longrightarrow 0$. The weight ω is complete if the pseudo-metric d_ω is complete in G.

Theorem 2.10. *Let ω be any unitary definite, symmetric, G-invariant weight on G. Then there exists a weighted group $(\widetilde{G}, \widetilde{\omega})$ such that*

(1) *$\widetilde{\omega}$ is a unitary definite, symmetric, G-invariant weight on $\widetilde{G}$;*
(2) *G is a dense subgroup of $(\widetilde{G}, d_{\widetilde{\omega}})$;*
(3) *$\widetilde{\omega} = \omega$ on G.*

The pair $(\widetilde{G}, \widetilde{\omega})$ is called the completion *of (G, ω).*

Proof. Let d_ω be the invariant metric on G induced by ω. Let $(\widetilde{G}, \widetilde{d_\omega})$ be the completion of the metric space (G, d_ω). As (G, d_ω) is a topological group and d_ω is invariant, $(\widetilde{G}, \widetilde{d_\omega})$ is also a topological group. Let $\widetilde{\omega}$ be the weight on $\widetilde{G}$ induced by $\widetilde{d_\omega}$ as in Proposition 2.3. Then $\widetilde{\omega}$ is a unitary definite, symmetric, $\widetilde{G}$-invariant weight on $\widetilde{G}$. This proves (1). Clearly, G is dense in $(\widetilde{G}, \widetilde{d_\omega})$ which proves (2). It is routine to prove that the weighted group $(\widetilde{G}, \widetilde{\omega})$ is unique satisfying (1) and (2) above. $\square$

Examples 2.11. Following are examples of complete weighted groups.
(1) Let $(X, \|\cdot\|)$ be a Banach space. Then $\omega(x) = e^{\|x\|}$ is a complete weight on the additive group $(X, +)$.
(2) Let $\delta : \mathbb{R}^n \longrightarrow [0, \infty)$ be a function satisfying: (i) $\delta(x) = 0$ iff $x = 0$; (ii) $\delta(x+y) \leq \delta(x)+\delta(y)$; (iii) $\delta(-x) = \delta(x)$. Then $\omega_{\delta,\alpha}(x) = (1+\delta(x))^\alpha$ defines a unitary definite, symmetric weight on $(\mathbb{R}^n, +)$. It is complete if $\delta(x) = \|x\|_p, 1 \leq p \leq \infty$.

3. Structure Theorem for Invariant Topological Groups

Our main goal in this section is to prove the structure theorem for topological groups in terms of weighted groups. We begin with the following standard definitions.
Definition 3.1. [HR] Let (G, τ) be a (Hausdorff) topological group. A net $\{x_\alpha\}_{\alpha\in\Delta}$ is called a *left (resp.,* right*) Cauchy net* if, for each neighborhood V of e, there exists $\alpha_0 \in \Delta$ such that $x_\alpha^{-1}x_\beta \in V$ (resp., $x_\alpha x_\beta^{-1} \in V$) for all $\alpha, \beta \geq \alpha_0$.

By the continuity of the map $x \longrightarrow x^{-1}$ on G, it follows that a net in G is left Cauchy iff it is right Cauchy. This justifies the following.

Definition 3.2. A topological group G is *complete* if every (left) Cauchy net is convergent in G.

Definition 3.3. The topology τ of a topological group (G, τ) is *inner invariant* if the family of all open neighbourhoods V of e satisfying $xVx^{-1} = V$ $(x \in G)$ forms a base for e.

Definition 3.4. Let $(\Delta, \leq)$ be a directed set. Let $\{G_\alpha : \alpha \in \Delta\}$ be a family of topological groups. Suppose that, for each pair $\alpha \leq \beta$ in Δ, there exists a continuous homomorphism $\pi_{\alpha\beta} : G_\beta \longrightarrow G_\alpha$ such that $\pi_{\alpha\alpha}$ is the identity map for all α and $\pi_{\alpha\beta} \circ \pi_{\beta\gamma} = \pi_{\alpha\gamma}$ $(\alpha \leq \beta \leq \gamma)$. Then the family $\{(G_\alpha, \pi_{\alpha\beta}) : \alpha \leq \beta \text{ in } \Delta\}$ is called an *inverse system of topological groups*. Its *inverse limit* is the topological group defined as

$$\lim_{\longleftarrow} G_\alpha = \{x = (x_\alpha) \in \Pi_\alpha G_\alpha : \pi_{\alpha\beta}(x_\beta) = x_\alpha \text{ for all } \alpha \leq \beta \text{ in } \Delta\}$$

with the relative product topology. It is a closed subgroup of the product topological group $\Pi_\alpha G_\alpha$.

Let $\{(G_\alpha, \omega_\alpha) : \alpha \in \Delta\}$ be a family of weighted groups with unitary definite, symmetric, G_α-invariant weights ω_α on G_α inducing the metric d_α. Then the product topology on $G = \prod(G_\alpha, d_\alpha)$ is inner invariant. We know that the relative topology on a subgroup of a group with inner invariant topology is inner invariant. Thus the inverse limit of a family $\{(G_\alpha, \omega_\alpha) : \alpha \in \Delta\}$ of weighted groups with unitary definite, G-invariant weights ω_α is a group with inner invariant topology. The following is its converse.

Theorem 3.5. *Let G be a topological group whose topology is inner invariant. Then there exists a family $\{(\widetilde{G_\alpha}, \widetilde{\omega_\alpha}) : \alpha \in \Delta\}$ of weighted groups such that the following hold.*

1. *Each $\widetilde{\omega_\alpha}$ is a unitary definite, symmetric, $\widetilde{G_\alpha}$-invariant weight.*
2. *Each $(\widetilde{G_\alpha}, \widetilde{\omega_\alpha})$ is a complete weighted group.*
3. *G is homeomorphic to a dense subgroup of $\lim\limits_{\longleftarrow} \widetilde{G_\alpha}$.*
4. *If G is complete, then $G \simeq \lim\limits_{\longleftarrow} \widetilde{G_\alpha}$.*

Proof. We know that the topology of a topological group G is inner invariant iff the family of all invariant pseudo-metrics on G which are continuous and generate the topology on G. Let $D = \{d_\alpha : \alpha \in \Delta\}$ be the family of all such pseudo-metrics on G. For each $\alpha \in \Delta$, let $w_\alpha(x) = e^{d_\alpha(x,e)}$ $(x \in G)$. Then w_α is a unital, unitary, symmetric, G-invariant weight on G. Since the topology τ on G is Hausdorff, $d_\alpha(x, y) = 0$ $(\alpha \in \Delta)$ implies $x = y$. So it follows that $w_\alpha(x) = 1$ $(\alpha \in \Delta)$ implies $x = e$. Let $N_\alpha = \{x \in G : w_\alpha(x) = 1\}$ which is a normal subgroup of G. Consider the quotient group $G_\alpha = G/N_\alpha$ and $\omega_\alpha(xN_\alpha) = w_\alpha(x)$ $(xN_\alpha \in G_\alpha)$. Then each ω_α is a unitary definite, symmetric, G_α-invariant weight on G_α. Let $(\widetilde{G_\alpha}, \widetilde{\omega_\alpha})$ be the completion of the weighted group $(G_\alpha, \omega_\alpha)$. Then $\widetilde{\omega_\alpha}$ is a unitary definite, symmetric, $\widetilde{G_\alpha}$-invariant weight on $\widetilde{G_\alpha}$. This proves (1).

Let $\widetilde{d_\alpha}(x,y) = \log \widetilde{\omega_\alpha}(xy^{-1})$ $(x, y \in \widetilde{G_\alpha})$. Then $(\widetilde{G_\alpha}, \widetilde{d_\alpha})$ is a complete topological group. This proves (2).

Let $G^1 = \Pi_{\alpha\in\Delta}(\widetilde{G_\alpha}, \widetilde{d_\alpha})$. Then G^1 is a complete topological group equipped with the product topology. Let $\pi_\beta : \Pi_{\alpha\in\Delta}G_\alpha \longrightarrow G_\beta$ be the projection map $\pi_\beta((xN_\alpha)_{\alpha\in\Delta}) = xN_\beta$ and extended continuously to G^1. These extensions are denoted by same notation $\pi_\alpha : G^1 \longrightarrow \widetilde{G_\alpha}$. The maps π_α are continuous, surjective homomorphisms. Consider the map $\phi : G \longrightarrow G^1$ defined as $\phi(x) = (xN_\alpha)_{\alpha\in\Delta}$. Then ϕ is an injective homeomorphism. Next we construct a subgroup G_ω of G^1 such that G is isomorphic to a dense subgroup of G_ω and G_ω is defined independently of ϕ. For $\alpha, \beta \in \Delta$, define $\alpha \leq \beta$ if $\omega_\alpha(x) \leq \omega_\beta(x)$ $(x \in G)$. Then Δ is a directed set. If $\alpha \leq \beta$, then $N_\beta \subset N_\alpha$ and hence $\pi_{\alpha\beta} : G_\beta \longrightarrow G_\alpha$ defined as $\pi_{\alpha\beta}(xN_\beta) = xN_\alpha$ is a uniformly continuous, onto homomorphism. Hence it can be extended as a continuous onto homomorphism $\widetilde{\pi_{\alpha\beta}} : \widetilde{G_\beta} \longrightarrow \widetilde{G_\alpha}$. In fact, $\widetilde{\omega_\alpha}(\pi_{\alpha\beta}(\pi_\beta(x))) \leq \widetilde{\omega_\beta}(\pi_\beta(x))$ $(x \in G)$. Let

$$G_\omega = \varprojlim \widetilde{G_\alpha} = \{(z_\alpha)_{\alpha\in\Delta} \in G^1 : \pi_{\alpha\beta}(z_\beta) = z_\alpha \text{ for } \alpha \leq \beta \in \Delta\}.$$

Then G_ω is a closed subgroup of G^1 and the restriction of projection map π_α to G_ω is a homomorphism from G_ω onto $\widetilde{G_\alpha}$. Next we show that $\phi(G)$ is dense in G_ω. Let $z = (z_\alpha)_{\alpha\in\Delta} \in G_\omega$. Let U be any neighbourhood of the identity element in G_ω. Then U contains a neighbourhood V of the form $V = \Pi_\alpha V_\alpha$, where each V_α is a neighbourhood of the identity in $\widetilde{G_\alpha}$ and there exists $\beta \in \Delta$ such that $V_\beta \neq \widetilde{G_\beta}$. Let $x \in G$ such that $\pi_\beta(x)z_\beta^{-1} \in V_\beta$. Then we have $\pi_\alpha(x)z_\alpha^{-1} \in V_\alpha$ $(\alpha \in \Delta)$. Hence $\phi(x) = (\pi_\alpha(x))_{\alpha\in\Delta} \in zU$. This shows that $\phi(G)$ is dense in G_ω. This proves (3).

Finally, let G be complete. Then $\phi(G)$ is complete, and hence closed in $\Pi_\alpha\widetilde{G_\alpha}$. Being dense in $\varprojlim \widetilde{G_\alpha}$, it follows that $G \simeq \varprojlim \widetilde{G_\alpha}$. This proves (4). □

Theorem 3.6. *Let G be a topological group with an inner invariant topology. Let $P = \{\omega_\alpha : \alpha \in \Delta\}$ be a family of unitary, symmetric G-invariant weights on G which determines the topology of G. Let $\omega(x) = \sup_\alpha \omega_\alpha(x)$ $(x \in G)$ and $P_b(G) = \{x \in G : \omega(x) < \infty\}$. Then*

1. *ω is a unitary definite, symmetric, G-invariant weight on $P_b(G)$;*
2. *$P_b(G)$ is a normal subgroup of G;*
3. *If G is complete, then $(P_b(G), d_\omega)$ is complete.*

Proof. (1) Since each ω_α is a symmetric weight on G, it is clear that $P_b(G)$ is a subgroup of G and ω is a symmetric weight on $P_b(G)$. Since each ω_α is a unitary, symmetric, G-invariant weight on G, the weight ω on $P_b(G)$ has same properties on $P_b(G)$. Further the family P is separating as G is Hausdorff. This implies that ω is unitary definite on $P_b(G)$.

(2) Let $x \in P_b(G)$ and $s \in G$. Then $\omega(sxs^{-1}) = \sup_\alpha \omega_\alpha(sxs^{-1}) = \sup_\alpha \omega(x) < \infty$. So $sxs^{-1} \in P_b(G)$. Thus $P_b(G)$ is a normal subgroup of G.

(3) Let d_α be the metric induced by ω_α on G and d be the metric induced by ω on $P_b(G)$. Let $\{x_n\}$ be a Cauchy sequence in $(P_b(G), d_\omega)$. Since the d_ω-metric topology on $P_b(G)$ is finer than the relative topology τ' on $P_b(G)$ inherited from the topology τ on G, the $\{x_n\}$ is a Cauchy sequence in (G, τ). Since (G, τ) is a complete topological group, there exists $x \in G$ such that $x_n \longrightarrow x$ in (G, τ). Since $\{x_n\}$ is d_ω-Cauchy, there exists $M > 0$ such that $d_\omega(x_n, x_m) < M$ $(m, n \in \mathbb{N})$. Hence $\omega(x_n) = \omega(x_n x_1^{-1} x_1) \leq \omega(x_n x_1^{-1})\omega(x_1) = e^{d_\omega(x_n, x_1)}\omega(x_1) \leq \omega(x_1)e^M = C$ (say). Now, for any $\alpha \in \Delta$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
\omega_\alpha(x) &= \omega_\alpha(x^{-1}) = \omega_\alpha(x_n^{-1} x_n x^{-1}) \\
&\leq \omega_\alpha(x_n^{-1})\omega_\alpha(x_n x^{-1}) = \omega_\alpha(x_n)e^{d_\alpha(x_n, x)} \\
&\leq \omega(x_n)e^{d_\alpha(x_n, x)} \leq Ce^{d_\alpha(x_n, x)}.
\end{aligned}$$

Let $\varepsilon > 0$. Since P determines the topology τ on G and $x_n \longrightarrow x$ in (G, τ), there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
& d_\alpha(x_n, x) < \varepsilon \ (n \geq n_0; \alpha \in \Delta) \\
\Longrightarrow\ & d_\alpha(x_{n_0}, x) < \varepsilon \ (\alpha \in \Delta) \\
\Longrightarrow\ & \omega_\alpha(x) \leq Ce^\varepsilon \ (\alpha \in \Delta) \\
\Longrightarrow\ & \omega(x) \leq Ce^\varepsilon < \infty.
\end{aligned}$$

Thus $x \in P_b(G)$. Finally, we show that $x_n \longrightarrow x$ in $(P_b(G), d)$. In fact, $d_\alpha(x_n, x) < \varepsilon$ $(\alpha \in \Delta; n \geq n_0)$. This, in turn, implies $d_\omega(x_n, x) = \sup_\alpha d_\alpha(x_n, x) \leq \varepsilon$ $(n \geq n_0)$. Thus $x_n \longrightarrow x$ in $(P_b(G), d)$. So $(P_b(G), d)$ is complete. □

4. Weighted Groups from Valuation Rings

We have shown in [ShDe] that several well-known functions such as $c\cosh(n), c\sinh(n), n^k + c, (n+c)^k, e^{n^c}, \log(n^k) + c, [\log(n) + c]^k$, and others, are weights for some special values of $k \in \mathbb{N}$ and $c > 0$. In this section, we supply some more weights ω on additive groups arising from valuation rings occurring in Number Theory and Algebra. The objective is to search weighted groups from the areas of mathematics other than Harmonic Analysis.

Definition 4.1. [Ba, P. 139] Let R be a ring. A *valuation* on R is a function $\nu : R \longrightarrow \mathbb{R}$ such that, for all $x, y \in R$,

(1) $\nu(x) \geq 0$; and $\nu(x) = 0$ iff $x = 0$;
(2) $\nu(xy) = \nu(x)\nu(y)$;
(3) $\nu(x \pm y) \leq \nu(x) + \nu(y)$

Proposition 4.2. *Let ν be a valuation on a unital ring R. Define $\omega_1(x) = 1 + \nu(x)$, $\omega_2(x) = e^{\nu(x)}$, and $\omega_3(x) = \max\{1, \nu(x)\}$ for all $x \in R$. Then*

(1) *ω_1 and ω_2 are unitary definite, symmetric, R-invariant weight on the abelian group $(R, +)$.*
(2) *If R is a field, then each ω_i is a unitary, R^*-invariant weight on the multiplicative group $(R^*, \times)$. Further ω_1 and ω_2 are unitary definite.*

Proof. (1) Notice that $0 < \nu(1) = \nu(1.1) = \nu(1)^2 = \nu(-1)^2$ implies $\nu(1) = \nu(-1) = 1$. So that $\nu(-x) = \nu(-1)\nu(x) = \nu(x)$. Hence ω_1 and ω_2 are symmetric. The rest is clear.
(2) Note that $1 = \nu(1) = \nu(s^{-1}s) = \nu(s^{-1})\nu(s)$ and so $\nu(s^{-1}) = \nu(s)^{-1}$ $(s \in R^*)$. Thus each ω_i is R^*-invariant. The rest is clear. □

Examples 4.3. We exhibit some examples of valuations on fields. Hence, by Proposition 4.2, we get variety of weights.
(1) Let p be a prime number. For $x \in \mathbb{Q} \setminus \{0\}$, let $x = p^k \frac{m}{n}$ for some $k \in \mathbb{Z}$ and some integers m and n which are not divisible by p. Define $\nu_p(x) = e^{-k}$ and $\nu_p(0) = 0$. Then ν_p is a valuation on $\mathbb{Q}$ [Ba, P. 139].
(2) Let $\mathbb{Q}(\sqrt{d}) = \{u + v\sqrt{d} : u, v \in \mathbb{Q}\}$, where $d \in \mathbb{Z} \setminus \{0\}$ such that $\sqrt{d} \notin \mathbb{Z}$. Now define $\nu_d(x) = |u^2 - dv^2|$ $(x = u + v\sqrt{d} \in \mathbb{Q}(\sqrt{d}))$ and $\nu_d(0) = 0$. Then $\mathbb{Q}(\sqrt{d})$ is a field with usual addition and multiplication, and ν_d is a valuation on $\mathbb{Q}(\sqrt{d})$. If we take $d = -1$, then we get the Gaussian field [Ba, P. 62].
(3) Let $\Omega = \{\alpha \in \mathbb{C} : \alpha \text{ is an algebraic number}\}$. Then Ω is a field with usual addition and multiplication. Let d_α denote the degree of the minimal polynomial $p_\alpha(x)$ for α. Let $\{\alpha_1, \ldots, \alpha_{d_\alpha}\}$ be the roots of $p_\alpha(x)$. Define $\nu_n(\alpha) = |\alpha_1 \cdots \alpha_{d_\alpha}|$ and $\nu_t(\alpha) = |\alpha_1| + \cdots + |\alpha_{d_\alpha}|$. Then both ν_n and ν_t are valuations on the algebraic field Ω [Ba, P. 102].
(4) Let F be a field. Let

$$F[x] = \left\{ \sum_{k=0}^{n} a_k x^k : n \in \mathbb{Z}^+; \, a_k \in F \ (0 \le k \le n) \right\}$$
$$F(x) = \left\{ \frac{p(x)}{q(x)} : p(x), q(x) \in F[x] \text{ and } q(x) \neq 0 \right\}.$$

Then $F(x)$ is a field with usual addition and multiplication operations. Let $r(x) = \frac{p(x)}{q(x)} = x^m \frac{a(x)}{b(x)} \in F(x) \setminus \{0\}$ such that $a(0)$ and $b(0)$ are non-zero. Define $\nu(r(x)) = m$ and $\nu(0) = 0$. Then one can check that ν is a valuation on $F(x)$.

References

[Ba] A. Baker, *A Comprehensive Course in Number Theory*, Cambridge, 2012.
[BDD] S. J. Bhatt, P. A. Dabhi and H. V. Dedania, *Multipliers of Weighted Semigroups and Associated Beurling Algebras*, Proc. Indian Acad. Sci.(Math.Sci.), 121(4)(2011)417-433.
[ShDe] Shreema S. Bhatt, and H. V. Dedania, *Weights on the Semigroup* $(\mathbb{N}, +)$, Communicated.
[BhDe1] S. J. Bhatt, and H. V. Dedania, *Beurling Algebras and Uniform Norms*, Studia Mathematica, 160(2)(2004)179-183.
[BhDe2] S. J. Bhatt, and H. V. Dedania, *Weighted Measure Algebras and Uniform Norms*, Studia Mathematica, 177(2)(2006)133-139.
[CoHa] Y. Cornulius and Pierre de la Harpe, *Metric Geometry of Locally Compact Groups*, arxiv:1403.3796v4[math.GR] 4 August 2016.

[DaDe] H. G. Dales and H. V. Dedania, *Weighted Convolution Algebras on Subsemigroups of the Real Line*, Dissertationes Mathematicae, 459(2009)1-60.

[HR] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, Vol-I, Springer, Berlin-Heidelberg-New York, 1963.

[Ku] Yu. N. Kuznetsova, *Example of a Weighted Algebra $L^p(G,\omega)$*, J. Math. Anal. Appl., 353(2009)660-665.

[RS] H. Reiter and J. D. Stegeman, *Classical Harmonic Analysis and Locally Compact Groups*, London Math. Soc. Monographs, Vol 22, Oxford Science Publications, 2000.

(Shreema S. Bhatt) Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar, 388120 (India)
E-mail address: shreemabhatt3@gmail.com

(H. V. Dedania) Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar, 388120 (India)
E-mail address: hvdedania@yahoo.com

Shreema S. Bhatt, et. al. "On Metric Geometry of Weighted Groups." *IOSR Journal of Mathematics (IOSR-JM)*, 17(2), (2021): pp. 01-09.