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Zeros of the Solutions in Some Cases of the Linear Nonhomogenous Vekua Equation

Slagjana Brsakoska
(Faculty for Natural Sciences and Mathematics, Sts. Cyril and Methodius University, Skopje, Republic of North Macedonia)

Abstract:
Background: *Vekua equation is an areolar equation from a complex function, which cannot be solved in general case. Its origin is from a practice problem from the theory of elasticity.*
Results: *In the paper zeros of the solutions of some special cases of Vekua equation are considered and the results are formulated in theorems.*
Key Word: *Areolar derivative, Areolar equation, Vekua equation, nonhomogeneous linear differential equation*.

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I. Introduction

The equation

$$\frac{\hat{d}W}{d\bar{z}} = AW + B\overline{W} + F \tag{1}$$

where $A = A(z)$, $B = B(z)$ and $F = F(z)$ are given complex functions from a complex variable $z \in D \subseteq \mathbb{C}$ is the well known Vekua equation [1] according to the unknown function $W = W(z) = u + iv$. The derivative on the left side of this equation has been introduced by G.V. Kolosov in 1909 [2]. During his work on a problem from the theory of elasticity, he introduced the expressions

$$\frac{1}{2}[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})] = \frac{\hat{d}W}{dz} \tag{2}$$

and

$$\frac{1}{2}[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})] = \frac{\hat{d}W}{d\bar{z}} \tag{3}$$

known as operator derivatives of a complex function $W = W(z) = u(x,y) + iv(x,y)$ from a complex variable $z = x + iy$ and $\bar{z} = x - iy$ corresponding. The operating rules for this derivatives are completely given in the monograph of Г. Н.Положий [3] (page18-31). In the mentioned monograph are defined so cold operator integrals

$$\hat{\int} f(z)dz \text{ and } \hat{\int} f(z)d\bar{z}$$

from $z = x + iy$ and $\bar{z} = x - iy$ corresponding (page 32-41). As for the complex integration in the same monograph is emphasized that it is assumed that all operator integrals can be solved in the area D.

In the Vekua equation (1) the unknown function $W = W(z)$ is under the sign of a complex conjugation which is equivalent to the fact that $B = B(z)$ is not identically equaled to zero in D. That is why for (1) the quadratures that we have for the equations where the unknown function $W = W(z)$ is not under the sign of a complex conjugation, stop existing.

This equation is important not only for the fact that it came from a practical problem, but also because depending on the coefficients A, B and F the equation (1) defines different classes of generalized analytic functions. For example, for $F = F(z) \equiv 0$ in D the equation (1) i.e.

$$\frac{\hat{d}W}{d\bar{z}} = AW + B\overline{W} \tag{4}$$

which is called canonical Vekua equation, defines so cold generalized analytic functions from fourth class; and for $A \equiv 0$ and $F \equiv 0$ in D, the equation (1) i.e. the equation $\frac{\hat{d}W}{d\overline{z}} = B\overline{W}$ defines so cold generalized analytic functions from third class or the (r+is)-analytic functions [3], [4].

Those are the cases when $B \neq 0$. But if we put $B \equiv 0$, we get the following special cases. In the case $A \equiv 0$, $B \equiv 0$ and $F \equiv 0$ in the working area $D \subseteq$ the equation (1) takes the following expression $\frac{\hat{d}W}{d\overline{z}} = 0$ and this equation, in the class of the functions $W = u(x,y) + iv(x,y)$ whose real and imaginary parts have unbroken partial derivatives u'_x, u'_y, v'_x and v'_y in D, is a complex writing of the Cauchy - Riemann conditions. In other words it defines the analytic functions in the sense of the classic theory of the analytic functions. In the case $B \equiv 0$ in D is the so cold areolar linear differential equation [3] (page 39-40) and it can be solved with quadratures.

II. Main result and Discussion

We will consider the Vekua equation (1) where the coefficient $B \equiv 0$, i.e. the areolar linear differential equation

$$\frac{\hat{d}W}{d\overline{z}} = AW + F \ . \qquad (5)$$

The solution of this equation is given with the formula

$$W(z,\overline{z}) = e^{\hat{\int} A(z,\overline{z})d\overline{z}} \left[C(z) + \hat{\int} F(z,\overline{z}) e^{-\hat{\int} A(z,\overline{z})d\overline{z}} d\overline{z} \right] \qquad (6)$$

In [5] and [6] some approaches of this thematic are considered and some theorems are formulated. These cases continues that discussion.

First case: $A(z,\overline{z}), F(z,\overline{z})$ **are analytic functions from** z **, i.e.** $A = f(z), F = g(z)$

In this case the equation (5) is in the following form

$$\frac{\partial W}{\partial \overline{z}} = f(z)W + g(z) \qquad (7)$$

Having in mind the solution (6) we have

$$W = e^{\hat{\int} f(z)d\overline{z}} \left[C(z) + \hat{\int} e^{-\hat{\int} f(z)d\overline{z}} g(z) d\overline{z} \right] = e^{f(z)\hat{\int} d\overline{z}} \left[C(z) + g(z) \hat{\int} e^{-f(z)\hat{\int} d\overline{z}} d\overline{z} \right] =$$

$$= e^{\overline{z}f(z)} \left[C(z) + g(z) \hat{\int} e^{-\overline{z}f(z)} d\overline{z} \right] = e^{\overline{z}f(z)} \left[C(z) + g(z) \frac{e^{-\overline{z}f(z)}}{-f(z)} \right]$$

$$W(z,\overline{z}) = C(z) e^{\overline{z}f(z)} - \frac{g(z)}{f(z)} \qquad (8)$$

This is the solution in this case and its zeroes are the zeroes of the equation

$$e^{\overline{z}f(z)} = \frac{g(z)}{f(z)C(z)} \qquad (9)$$

Since on the right side of the equation is some analytic function from z, and on the left side is not, we can formulate the conclusion in the following

Theorem. The zeros of the solutions of the equation (7), given with the functions (8), are at the same time solutions of (9) and it depends from the functions $f, g, C(z)$.

Example 1. Let us consider the equation $\frac{\partial W}{\partial \overline{z}} = zW + z^2$ and $C(z) = z$. This means that $f = z, g = z^2$.

For (9) we have that $e^{z\overline{z}} = \frac{z^2}{zz} = 1 \Rightarrow e^{|z|^2} = 1$ and the only solution is an isolated zero in the point $z = 0$.

Example 2. Let us consider the same equation with different C, i.e. $\frac{\partial W}{\partial \overline{z}} = zW + z^2$ $f = z, g = z^2, C(z) = z^2$

$$e^{z\bar{z}} = \frac{z^2}{z\bar{z}^2} = \frac{1}{\bar{z}} = \frac{x-iy}{x^2+y^2}$$

$$e^{x^2+y^2} = \frac{x}{x^2+y^2}, \quad 0 = \frac{y}{x^2+y^2}$$

$$y = 0, \quad e^{x^2} = \frac{1}{x}$$

The only zero is the point: $M_0\left(e^{1/\xi^2}, 0\right)$.

Example 3. Let us consider the equation with $f = g = z, C(z) = e^{-z}$

$$e^{z\bar{z}} = \frac{z}{ze^{-z}} = e^z$$

$$e^{|z|^2} = e^z; \quad e^{x^2+y^2} = e^x(\cos y + i\sin y)$$

$$\begin{cases} e^x \cos y = e^{x^2+y^2} \\ e^x \sin y = 0 \end{cases} \Rightarrow y = n\pi; e^x(\pm 1) = e^{x^2+(n\pi)^2}$$

there are not zeros except for $n = 0, M_0(0,1)$.

Second case: $A(z,\bar{z}), F(z,\bar{z})$ **are analytic functions from** $\bar{z}$ **, i.e.** $A = A(\bar{z}), F = F(\bar{z})$

In this case the equation (5) is in the following form

$$\frac{\partial W}{\partial \bar{z}} = A(\bar{z})W + F(\bar{z}) \tag{10}$$

and its solution according to (6) is

$$W(z,\bar{z}) = C(z)e^{\hat{\int} A(\bar{z})d\bar{z}} + e^{\hat{\int} A(\bar{z})d\bar{z}} \hat{\int} F(\bar{z}) e^{-\hat{\int} A(\bar{z})d\bar{z}} d\bar{z} \tag{11}$$

If these integrals can be solved with quadratures, i.e.

$$\hat{\int} A(\bar{z})d\bar{z} = I_1, \hat{\int} F(\bar{z}) I_2(\bar{z})d\bar{z} = I_3 \tag{12}$$

for the zeros we have

$$W = I_1\left[C(z) + I_3\right] = 0$$

and we can formulate the following

Theorem. The zeros of the solutions of the equation (10), given with the functions (11), can be found as a solution of $I_1 = 0, \quad I_3 = -C(z)$, where I_1 and I_3 are defined with (12).

Example 1. Let us consider the equation $\frac{\partial W}{\partial \bar{z}} = 1 \cdot W + \bar{z}, \quad A = 1, F = \bar{z}$

$$W = e^{\hat{\int} d\bar{z}}\left[C(z) + \hat{\int} \bar{z} \cdot e^{-\hat{\int} d\bar{z}} \cdot d\bar{z}\right] = e^{\bar{z}}\left[C(z) + \hat{\int} \bar{z} \cdot e^{-\bar{z}} \cdot d\bar{z}\right]$$

Applying partial integration in the integral we get

$$W = e^{\bar{z}}\left[C(z) - \bar{z}e^{-\bar{z}} - e^{-\bar{z}}\right] = C(z)e^{\bar{z}} - \bar{z} - 1$$

We get the zeroes $W = 0$, for $C(z)e^{\bar{z}} = \bar{z} + 1$

$$C(z) = \frac{\bar{z}+1}{e^{\bar{z}}}$$

This equations will not stand for every z, because the left side is analytic function and on the right side is antyanalytic function ($\overline{\frac{z+1}{e^z}}$), but it can hold for some z.

If for example $C(z) = z^2$, we have

$$z^2 = (\overline{z}+1)e^{-z}$$

$$x^2 - y^2 + 2ixy = (x+1-iy)e^{-x+iy} = (x+1-iy)e^{-x}(\cos y + i\sin y)$$

$$\Rightarrow \begin{cases} x^2-y^2=\left[(x+1)\cos y+y\sin y\right]e^{-x} \\ 2xy=\left[-y\cos y+(x+1)\sin y\right]e^{-x} \end{cases}$$

The second eqiation is fulfilled for $y = 0$, and some of the zeroes are $y = 0$, за $x^2 = (x+1)\cdot 1\cdot e^{-x}$. We can find other zeroes if we apply series for $\sin y$ and $\cos y$.

Example 2. Let us consider the same equation with different $C(z) = e^{\overline{z}}$. Then

$$e^{z}\cdot e^{\overline{z}} = \overline{z}+1$$
$$e^{2x} = (x+1)+iy.$$

So $y = 0$ and $e^{2x} = x+1$; and two zeroes are for

$x_1 = 0$ and $x_2 = \xi$: $M_1(0,0),\quad M_2\left(e^{2\xi}-\xi=1,0\right),\quad \xi<0$

Third case : Nonhomogenous linear Vekua equation with constant coefficients

That is the equation

$$\frac{\partial W}{\partial \overline{z}} = \alpha W + \beta;\quad \alpha,\beta = Const. \tag{13}$$

According to the formula (6) we have

$$W(z,\overline{z}) = e^{\alpha\int d\overline{z}}\left[\square(z)+\hat{\int}\beta e^{-\alpha\int d\overline{z}}d\overline{z}\right]=$$

$$=e^{\alpha\overline{z}}\left[\square(z)+\beta\hat{\int}e^{-\alpha\overline{z}}d\overline{z}\right]=$$

$$=\square(z)e^{\alpha\overline{z}}-\frac{\beta}{\alpha} \tag{14}$$

The zeroes $W = 0$ are in some points in which

$$\square(z)=\frac{\beta}{\alpha}e^{-\alpha\overline{z}}$$

where $\square(z)$ is an arbitrary analytic function in the role of an integral constant. If

$$\square(z) = A(x,y)+iB(x,y)$$

we have

$$A(x,y)+iB(x,y) = \frac{b_1+ib_2}{a_1+ia_2}e^{-(a_1+ia_2)(x-iy)}$$

We want to find zeroes of $A(x,y)$ and $B(x,y)$, so

$$\frac{b_1+ib_2}{a_1^2+a_2^2}(a_1-a_2i)e^{-(a_1x+a_2y)+i(-a_2x+a_1y)} =$$

$$=\frac{b_1a_1+b_2a_2+i(b_2a_1-a_2b_1)}{a_1^2+a_2^2}e^{-(a_1x+a_2y)}\cdot e^{i(-a_2x+a_1y)} =$$

$$=\frac{b_1a_1+b_2a_2+i(b_2a_1-a_2b_1)}{a_1^2+a_2^2}e^{-(a_1x+a_2y)}\left\{\cos(-a_2x+a_1y)+i\sin(-a_2x+a_1y)\right\} =$$

$$=\frac{e^{-a_1x-a_2y}}{a_1^2+a_2^2}\left\{(b_1a_1+b_2a_2)\cos(-a_2x+a_1y)-(b_2a_1-a_2b_1)\sin(-a_2x+a_1y)+\right.$$

$$\left.+i\left[(b_2a_1-a_2b_1)\cos(-a_2x+a_1y)+(b_1a_1+b_2a_2)\sin(-a_2x+a_1y)\right]\right\}$$

From here,

$$A(x,y)=\frac{e^{-a_1x-a_2y}}{a_1^2+a_2^2}\left[(b_1a_1+b_2a_2)\cos(-a_2x+a_1y)-(b_2a_1-a_2b_1)\sin(-a_2x+a_1y)\right]$$

and

$$B(x,y)=\frac{e^{-a_1x-a_2y}}{a_1^2+a_2^2}\left[(b_2a_1-a_2b_1)\cos(-a_2x+a_1y)+(b_1a_1+b_2a_2)\sin(-a_2x+a_1y)\right]$$

For the zeroes only of A: $A(x,y)=0$, we have a condition:

$$(b_1a_1+b_2a_2)\cos(-a_2x+a_1y)-(b_2a_1-a_2b_1)\sin(-a_2x+a_1y)=0$$

or $$tg(-a_2x+a_1y)=\frac{b_1a_1+b_2a_2}{b_2a_1-a_2b_1}=\lambda$$

since a_ib_i are real , so is λ ; we have that

$$-a_2x+a_1y=arctg\,\lambda=k$$

or $$a_1y=a_2x+k$$

and this represents a line for the zeroes of A .

For the zeroes only of B: $B(x,y)=0$, we have a condition:

$$(b_2a_1-a_2b_1)\cos(-a_2x+a_1y)+(b_1a_1+b_2a_2)\sin(-a_2x+a_1y)=0$$

or $$tg(-a_2x+a_1y)=-\frac{b_2a_1-a_2b_1}{b_1a_1+b_2a_2}$$

and $$-a_2x+a_1y=-arctg\,\frac{b_2a_1-a_2b_1}{b_1a_1+b_2a_2}=\nu,$$

so $$a_1y=a_2x+\nu$$

this is also a line for *B*-zeroes, that is parallel to the line for *A*-zeroes. Since they do not have intersection, there are not common A and B-zeroes, which means that the solution w of (13) does not have any zeroes.
We can state the following

Theorem. The zeros of the solutions of the equation (13), given with the functions (14), do not exist if $\beta\neq 0$. If $\beta=0$, zero of w is every zero of the analytic function $\Phi(z)$.

III. Conclusion

We can conclude that the determination of the zeros even of the most simple Vekua equation is not a trivial work. But for some cases we can say which are the zeros and/or what they depend on.

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