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Relative Coarse Homotopy

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Abstract: *The quotient coarse category Qcrs, equipped with a small amount of extra structure, is a Baues cofibration category [12], [13]. In this article we show that the pointed quotient coarse category PQcrs is also a Baues cofibration category and use the coarse cofibration category machinery to define controlled and coarse homotopy groups, compute these groups for coarse spheres, and define relative coarse homotopy. For the last we show that any two classes in the coarse category PQcrs, that are strongly coarsely homotopic relative to* A*, will be relatively coarsely homotopic.*

Keywords: *Baues cofibration Category, The quotient coarse category, relative coarse homotpy.*

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I. Introduction And Preliminaries

Coarse geometry is a model for the study of geometric objects regarding only large scale structures and where only large scale characteristics are well defined. Many notions in topology and abstract geometry have been studied over years to examine and explore their large scale versions. For example, (co)homology theory has analogues in coarse geometry, and homology groups (see [7],[8],[14]) is not away from homotopy, It is well known that homotopy groups yields homotopy theory and this has been investigated in coarse geometry long time ago and called coarse homotopy theory.

Another modern progress in studying classical homotopy theory is the use of category theoretic methods with extra structure, Baues introduced a weaker notion of cofibration category in [1], [2] as a generalization of a Quillen model category. To define coarse version of abstract homotopy groups, a coarse analogues of a notion of Baues cofibration category is defined in [13]. In this article (see [12], [13], [15], and [16]) we define controlled and coarse path components, and we define also the n-th controlled and coarse homotopy groups, and we calculate them for some important coarse spaces, such as $\mathbb{R}_+$, $\mathbb{R}$, and $\mathbb{R}^2$.

Later in this article we define the pointed quotient coarse category PQcrs and show that it is a Baues cofibration category, and we show that any classes in PQcrs that are strongly coarse homotopic relative to an initial object A are relatively coarse homotopic. Thus, these results are important in defining coarse homotopy groups in a more geometric way (see [11]) in order to proving the Whitehead theorem in algebraic topology (see for example [17] for an account).

In the large scale we can define an abstract coarse space in terms of entourages ([5], [7], [16], [18]). We recall the following definitions and propositions which found also in [12] and [13].

The following definition comes from [7].

Definition 0.1. Let X be a set. Then X is called a *unital coarse space* if it is equipped with a coarse structure, defined to be a collection ε of subsets M of $X \times X$ called entourages satisfying the following axioms:
(1): If $M \in \varepsilon$ and $M^{'} \subseteq M$, then $M^{'} \in \varepsilon$.
(2): Let $M_1, M_2 \in \varepsilon$, then $M_1 \cup M_2 \in \varepsilon$, and $M_1 M_2 \in \varepsilon$ where $M_1 M_2 = \{(x, z) \mid (x, y) \in M_1, (y, z) \in M_2$ for some $y\}$. We call $M_1 M_2$ the composite of M_1 and M_2.
(3):$\Delta_X \in \varepsilon$ where $\Delta_X = \{(x, x) : x \in X\}$.
(4):$\cup_{M \in \varepsilon} M = X \times X$.
(5): If $M \in \varepsilon, M^t = \{ (y, x) \mid (x, y) \in M \} \in \varepsilon$.
We can use (X, ε) to refer to a coarse space when we need to emphasise the collection of entourages. A subset M is called *symmetric* if $M = M^t$.
A *non-unital coarse space* is a coarse space defined as above, but we drop the axiom where Δ_X must be an entourage.

Definition 0.2. Let X and Y be (unital) coarse spaces. Then a map $f = X \to Y$ is said to be *controlled* or *coarsely uniform* if for every entourage $M \subseteq X \times X$, the image

$$f(M] = \{(f(x), f(y)) = (x, y) \in M\}$$

is an entourage- A controlled map f is called *coarse* if the inverse image of a bounded set under the map f is also bounded.
A subset $B \subseteq X$ is said to be bounded if B takes the form $M(x) = \{y: (x,y) \in M\}$ for some entourage $M \subseteq X \times X$ and point $x \in X$.

The problem of constructing a nonzero product arises in the category of proper metric spaces and proper maps (modulo closeness. Here in the non-unital case. we get many more coarse maps. Consequently, using non-unital coarse spaces, it becomes extremely easy to construct (nonzero) categorical products in the coarse category and show that the non-unital coarse category have all nonzero (projective) limits see [6]. Initially, let us focus on the unital case.

Definition 0.3. We call two coarse maps $f, g: X \to Y$ *close*, and write $f \sim_{\text{Crs}} g$, if the set $\{(f(x), g(x)) \mid x \in X\}$ is an entourage.
A coarse map $f: X \to Y$ is called a *coarse equivalence* if there is a coarse map $g: Y \to X$ such that the compositions $f \circ g$ and $g \circ f$ are close to the identity maps 1_Y and 1_X respectively.

We call two coarse spaces X and Y *coarsely equivalent* if a coarse equivalence $f: X \to Y$ exists.
Alternatively, a subset $B \subseteq X$ is said to be *bounded* if the inclusion $B \hookrightarrow X$ is close to a constant map.

Definition 0.4 . Let X be a set and e a collection of subsets of $X \times X$. *The coarse structure generated by* ε is the minimum coarse structure on X that contains ε. We write this structure $< \varepsilon >$.
Note that here we do not assume that the coarse structure generated by a collection is unital. The following definition comes from [5].
Definition 0.5. Let X be a Hausdorff space. A coarse structure on X is said to be *compatible with the topology* if every entourage is contained in an open entourage. and the closure of any bounded set is compact. We call a Hausdorff space equipped with a coarse structure compatible with the topology a *coarse topological space*.

Any coarse topological space is locally compact, and the bounded sets are precisely those which are precompact.

Example 0.6. Let (X, d) be a *proper* (i.e, closed bounded subsets are compact) metric space. Then d induces a coarse structure on X. which is called *metric structure* such that:
Let $D_r = \{(x,y) \in X \times X \mid d(x,y) < r\}$. Then $E \subseteq X \times X$ is an entourage if $E \subseteq D_r$ for some $r > 0$. Any coarse topological space is locally compact. The coarse space X equipped with the metric coarse structure is a coarse topological space if X is locally compact.
The following definition induces from [10] .

Definition 0.7 . Let R be the topological space $[0, \infty)$ equipped with the coarse structure arising from the metric. We call the space R a *generalised ray* if the following conditions hold.

- Let $M, N \subseteq R \times R$ be entourages. Then the sum
$$M + N = \{(u + x, v + y) \mid (u,v) \in M, (x,y) \in N\}$$
is an entourage.
- Let $M \subseteq R \times R$ be an entourage. Then the set
$$M^* = \{(u,v) \in R \times R \mid x \le u, v \le y, (x,y) \in M\}$$
is an entourage.
- Let $N \subseteq R \times R$ be an entourage, and $a \in R$. Then the set
$$a + N = \{(a + x, a + y) \mid (x,y) \in N\}$$
is an entourage.

For example, the space $\mathbb{R}_+$ (with the metric coarse structure) is a generalised ray.
Proposition 0.8. Let X be a coarse space, and let R be a generalised ray. Let $p, q: X \to R$ be controlled maps, then $p + q$ is a controlled map.
Proof. Let $M \subseteq X \times X$ be an entourage. Then the images $p[M], q[M]$ are entourages, Now
$$(p + q)[M] = \{((p + q)(x), (p + q)(y)): (x,y) \in M\} \subseteq p[M] + q[M]$$
which implies that $(p + q)[M]$ is an entourage. Hence $p + q$ is controlled. ■

Definition 0.9. Let X and Y be coarse spaces, equipped with collections of entourages ε_X and ε_Y respectively. Then we define *the product* of X and Y to be the Cartesian product $X \times Y$ equipped with the oarre structure generated by finite compositions, unions of entourages, and all subsets of entourages in the set

$$\{M \times N : M \in \varepsilon_X, N \in \varepsilon_Y\}$$

Unfortunately, the above product is not a product in the category-theoretic sense since the projections $p_X: X \times Y \to X$ and $p_Y: X \times Y \to Y$ are not in general coarse maps. We shall see that there is a solution to this issue in the world of non-unital coarse spaces.
Now we define two different coarse versions of disjoint union of coarse spaces. The following definition comes from [10].
Definition 0.10. Let X and Y be coarse spaces. Then we define *the disjoint union* to be the set $X \sqcup Y$ equipped with the coarse structure generated by the entourages to be subsets of unions of the form

$$M \cup N \cup (B_X \times B_Y) \cup (B_Y^{'} \times B_X^{'})$$

where $M \subseteq X \times X$ and $N \subseteq Y \times Y$ ure entourages, and $B_X, B_X^{'} \subseteq X$ and $B_Y B_Y^{'} \subseteq Y$ are bounded subsets, We denote this disjoint union by $X \sqcup Y$.
The following result is easy to check.
Proposition 0.11. Let X and Y be coarse spaces, R be a generalised ray. Let $p_X: X \to R$ and $p_Y: Y \to R$ be controlled maps, Then $X \sqcup Y$ is coarse space, and the map $p_{X \sqcup Y}: X \cup Y \to R$ defined by the formula

$$p_{X \sqcup Y}(x) = \begin{cases} p_X(x) & x \in X \\ p_Y(x) & x \in Y \end{cases}$$

is a controlled map. ■
Definition 0.12 . Let X and Y be coarse spaces, Then we define *another type of disjoint union* to be the set $X \sqcup Y$ equipped with the coarse structure generated by the entourages to be subsets of unions of the form $M \cup N$ where $M \subseteq X \times X$ and $N \subseteq Y \times Y$ are entourages. We denote this disjoint union by $X \sqcup_\infty Y$.
The space $X \sqcup_\infty Y$ is a non-unital coarse space even when X and Y are unital coarse spaces. The following is also easy to check.

Proposition 0.13. Let X and Y be coarse spaces, R be a generalised ray. Let $p_X: X \to R$ and $p_Y: Y \to R$ be controlled maps. Then the map $p_{X \sqcup \omega_\infty} Y: X \cup_\infty Y \to R$ defined by the formula

$$p_{X \cup_\infty Y}(x) = \begin{cases} p_X(x) & x \in X \\ p_Y(x) & x \in Y \end{cases}$$

is a controlled map. ■
The following definitions are prompted from [6].

Definition 0.14 . Let X, Y be coarse spaces and $f: X \to Y$ a map.

- We call f a *locally proper map* if $f|_{X'}$ is proper whenever $X^{'} \subseteq X$ is a unital coarse subspace, that is, the inverse image of a bounded set $B \subseteq Y$ under the map $f|_{X'}$ is bounded.
- We call f *a coarse map between non-umital coarsc spaces* if it is a controlled and locally proper map.

Any proper map is locally proper, but the converse in not always true. We define two maps between non-unital coarse spaces being close as follows.
Definition 0. 15. Let $f, g: X \to Y$ be two coarse maps between non-unital coarse spaces. We say that f is *close* to g if for any unital subspace $X^{'} \subseteq X$, we have $f|_{X'}$ is close to $g|_{X'}$ in sense of definition (0.3) .
We call f a coarse equivalence between non-unital coarse spaces if $f|_{X'}: X^{'} \to Y^{'}$ is a coarse equivalence in sense of definition (0.3) whenever $X^{'} \subseteq X$ and $Y^{'} \subseteq Y$ are unital coarse subspace.

Let X be a topological space. The product $X \times [0,1]$ is called a *cylinder* on X. We need to define a coarse version of the topological cylinder in order to define a coarse version of homotopy.
The following definition comes from [9].

Definition 0.16. Let X be a coarse space, R be a generalised ray, and $p; X \to R$ be some controlled map. Then we define *the p -cylinder* of X:

$$I_p X = \{(x, t) \in X \times R \mid t \leq p(x) + 1\}$$

The cylinder is a coarse space. We define the projection $p^{'}: I_p X \to R$ by the formula $p^{'}(x, t) = p(x) + t$ and we define coarse mups $i_0, i_1: X \to I_p X$ by the formula $i_0(x) = (x, 0)$ and $i_1(x) = (x, p(x) + 1)$ respectively.
Our aim in this work is to define a Banes cofibration category on the category of non-unital coarse spaces, The above definition yields ideas of homotopy and mapping cylinder which are vital to the axioms.
Definition 0.17 . Let $f_0, f_1: X \to Y$ be coarse maps between non-unital coarse spaces, *A coarse homotopy* between f_0, f_1 is a coarse map $H: I_p X \to Y$ for some controlled map $p: X \to R$ such that $f_0 = H \circ i_0$ and $f_1 = H \circ i_1$ respectively, We say the maps $f_0, f_1: X \to Y$ are *coarsely homotopic* between non-unital coarse spaces if $f_0|_{X'}$ is coarsely homotopic to $f_1|_{X'}$ whenever $X^{'} \subseteq X$ is a unital coarse subspace. *A* coarse map $f: X \to Y$ is termed a *coarse homotopy equivalence* if there is a coarse map $g: Y \to X$ such that the compositions $g \circ f$ and $f \circ g$ are

coarsely homotopic to the identities 1_X and 1_Y respectively. In this case, we call X and Y are *coarsely homotopic equivalent*, and denoted by $X \approx_{\text{Crs}} Y$.

Example 0.18 , Let X and Y be coarse spaces, and let $p: X \to R$ be a controlled map. Consider two close coarse maps $f_0, f_1: X \to Y$. Then we can define a coarse homotopy $H: I_p X \to Y$ between the maps f_0 and f_1 by the formula

$$H(x,t) = \begin{cases} f_0(x) & \text{t} < 1 \\ f_1(x) & \text{t} \geq 1 \end{cases}$$

Similarly we can define controlled homotopy between f_0 and f_1, when f_0 and f_1 are close controlled maps.
Thus, close coarse maps are also coarse homotopic. In particular, any coarse equivalence is a coarse homotopy equivalence which is also a controlled homotopy equivalence since coarse equivalence and controlled equivalence are the same.
The following definition is from [10].

Definition 0.19 . Let X be a subspace of the unit sphere S^{n-1}. Then we define *the open cone* on X to be the metric space

$$CX = \{\lambda x: \lambda \in \mathbb{R}^+, x \in X\} \subseteq \mathbb{R}^n$$

The open cone CX is a coarse space. The coarse structure is defined by the Euclidean metric on $\mathbb{R}^n$.
The cone of S^{n-1} is the Euclidean space $\mathbb{R}^n$, and the n-cell D^n can be viewed as the upper hemisphere in the cone of S^n, so its cone is $\mathbb{R}^n \times \mathbb{R}_+$. The following definition comes from [8] and [4] .

Definition 0.20. Let R be a generalized ray, $n \in \mathbb{N}$. Write

$$S_R^{n-1} = (R \sqcup R)^n, \qquad D_R^n = (R \sqcup R)^n \times R$$

We call S_n^{n-1} *a coarse R -sphere of dimension* $n-1$, D_R^n *a coarse R -cell of dimension n*, and the coarse R-sphere $\{(x,0) \in D_R^n : x \in S_R^{n-1}\}$ is called *the boundary of the coarse R -cell* D_R^n, i.e. $\partial D_R^n = S_R^{n-1} \times \{0\}$.

1-CONTROLLED AND COARSE PATH COMPONENTS

Definition 1.1. Let $f_0, f_1: X \to Y$ be controlled maps. A *controlled homotopy* between f_0, f_1 is a controlled map $H: I_p X \to Y$ for some controlled map $p: X \to R$ such that $f_0 = H \circ i_0$ and $f_1 = H \circ i_1$ respectively.
A controlled map $f: X \to Y$ is termed *controlled homotopy equivalence* if there is a controlled map $g: Y \to X$ such that the compositions $g \circ f$ and $f \circ g$ are controlled homotopic to the identities 1_X and 1_Y respectively. In this case, we call X and Y are *controlledly homotopic equivalent*, and denoted by $X \approx_{\text{Crd}} Y$.

Definition 1.2. Let $f: X \to Y$ be a controlled map. *The controlled equivalence class of a controlled map f under the equivalence relation of controlled homotopy* is denoted

$$[f]^{Crd} = \{ \text{ controlled map } g: X \to Y \mid g \simeq_{\text{Crd}} f\}$$

and called *the controlled homotopy class of f, The family of all such controlled homotopy classes* is denoted by $[X,Y]^{Crd}$.

Definition 1.3. Let $f: X \to Y$ be a coarse map. *The coarse equivalence class of a coarse map f under the equivalence relation of coarse homotopy* is denoted

$$[f]^{\text{Crs}} = \{ \text{ coarse map } g: X \to Y \mid g \simeq_{Crs} f\}$$

and called *the coarse homotopy class of f. The family of all such coarse homotopy classes* is denoted by $[X,Y]^{Crs}$.
We define $\pi_0^{\text{Crd}}(X)$ to be *the set of all controlled homotopy classes* $[f]$ *of controlled maps* $f: \mathbb{R} \to X$, and we define $\pi_0^{Crs}(X)$ to be *the set of all coarse homotopy classes* $[f]$ *of coarse maps* $f: \mathbb{R} \to X$, These sets for different choices of $\mathbb{R}$ are related by natural bijections.

Proposition 1.4. $\pi_0^{Crs}(\mathbb{R}_+)$ has one element which is the identity on $\mathbb{R}_+$.

Proof. Let $f: \mathbb{R}_+ \to \mathbb{R}_+$ be a coarse map, we need to show that f is coarsely homotopic to a homotopy class of the identity. Define a map $H: I_f \mathbb{R}_+ \to \mathbb{R}_+$ by

$$H(s,t) = \begin{cases} s+t & t \leq f(s) \\ s+f(s) & t \geq f(s) \end{cases}$$

Then H is a coarse homotopy between $id_{R_+}, id_{R_+} + f$ by lemma (2.2.7) and corollary (2.2.8)(1)) in [12].
Similarly we define a map $H': I_{id_{\mathbb{R}_+}} \mathbb{R}_+ \to \mathbb{R}$ by

$$H'(s,t) = \begin{cases} t+f(s) & t \leq s \\ s+f(s) & t \geq s \end{cases}$$

Then H' is a coarse bomotopy between f, $id_{\mathbb{R}_+} + f$ again by lemma ((2.2.7)(2)) and corollary ((2.2.8)(2)) in [12]. By theorem (2.2.5) in [12] , we have $id_{\mathbb{R}_+}$ is coarsely homotopic to f. This shows that $\pi_0^{Crs}(\mathbb{R}_+) = \{1\}$. ■

Proposition 1.5. (1): Lef $f: \mathbb{R}_+ \to \mathbb{R}$ he a controlled map, then f is close to some Lipschitz controlled map.
(2): Let $f: \mathbb{R}_+ \to \mathbb{R}$ be a coarse map, then f is close to some Lipschitz coarse map.

Proof. We prove (2), and (1) is identical. Since $\mathbb{R}_+, \mathbb{R}$ are geodesic, then by proposition (1.4) in [3] , the map f is asymptotically Lipschitz, and so the map $f|_{z_+}$ is coarse and asymptotically Lipschits, that is, we have two constants $A, B \geq 0$ such that

$$|f|_{z_+}(x) - f|_{z_+}(y)| \leq A|x - y\ | + B, \qquad x, y \in Z_+$$

If $x \neq y$, $|x - y| \geq 1$ so $B \leq B|x - y|$, then

$$|f|_{z_+}(x) - f|_{z_+}(y)| \leq A|x - y| + B|x - y| = (A + B)|x - y\ |$$

which implies that $f|_{z_+}$ is a Lipschitz map. Now we extend the map $f|_{z_+}$ piecewise linearly to the map $g: \mathbb{R}_+ \to \mathbb{R}$ as follows

$$g(k + t) = tf|_{z_+}(k + 1) + (1 - t)f|_{z_+}(k), k \in \mathbb{Z}_+, \quad t \in [0,1]$$

Then the map g is a coarse, Lipschitz map, and close to the map f. ∎

Proposition 1.6. $\pi_0^{Crs}(\mathbb{R})$ has two elements.

Proof. Let $f: \mathbb{R}_+ \to \mathbb{R}$ be a coarse map, then f is close to the coarse map g defined in proposition 1.5. This implies that $g^{-1}[0]$ is bounded, so $g^{-1}[0] \subseteq [0, a]$ for some $a > 0$. Now since g is continuous, and by the intermediate value theorem either:

$$g(x) > 0 \text{ for all } x > a, \text{ or } g(x) < 0 \text{ for all } x > a$$

Then $g|_{(a,\infty)}$ is never zero, and since g is continuous which means that $g|_{(a,\infty)}$ is always positive or always negative.

Now we need to show that g is coarsely homotopic to $r: \mathbb{R}_+ \to \mathbb{R}$ defined by $r(x) = x$ or g is coarsely homotopic to $s: \mathbb{R}_+ \to \mathbb{R}$ defined by $s(x) = -x$.

Without loss of generality, let g be always positive. In other words $g(x) > 0$ for all $x > 0$.

First, we show that $r + g$ is a coarse map which is so by definition of generalised ray, and it can be shown also as follows; Let $R > 0$ such that $|x - y| < R$, then there is $S > 0$ such that

$$\begin{aligned} |(r + g)(x) - (r + g)(y)| &= |r(x) + g(x) - r(y) - g(x)| \\ &\leq |x - y| + |g(x) - g(y)| \\ &< R + S \end{aligned}$$

Now let $B \subseteq \mathbb{R}_+$ be a bounded set, then we can choose $a > 0$ such that $B \subseteq [0, a]$. Hence

$$(r + g)^{-1}(B) = \{x \in \mathbb{R}: r + g(x) \leq a\} \subseteq \{x \in \mathbb{R}: g(x) \leq a\} = g^{-1}[0, a]$$

So the inverse image $(r + g)^{-1}(B)$ is a bounded set. Therefore $r + g$ is a coarse map. Now define a map $H: I_p\mathbb{R}_+ \to \mathbb{R}$ by

$$H(s,t) = \begin{cases} t + r(s) & t \leq p(s) \\ r(s) + g(s) & t \geq p(s) \end{cases}$$

where $p: \mathbb{R}_+ \to \mathbb{R}_+$ is the identity map. Then H is a coarse homotopy between r, $r + g$ by lemma (2.2.7) and corollary ((2.2.8)(1)) in [12] . Similarly we define a map $H': I_p\mathbb{R}_+ \to \mathbb{R}$ by

$$H'(s,t) = \begin{cases} t + g(s) & t \leq p(s) \\ r(s) + g(s) & t \geq p(s) \end{cases}$$

Then H' is a coarse homotopy between $g, r + g$ again by lemma (2.2.7) and corollary ((2.2.8)(1)) in [12].

By theorem (2.2.5) in [12] , we have r is coarsely homotopic to g, and hence we have r is coarsely homotopic to f.

Now let $g(x) < 0$ for all $x > 0$. Similarly we show that g is coarsely homotopic to s such that $s(x) = -x$ where $x \in \mathbb{R}_+$. But g is close to f, and so they are coarsely homotopic. Therefore f is coarsely homotopic to r or s, and hence $\pi_0^{Crs}(\mathbb{R})$ has only two elements. ∎

Proposition 1.7. $\pi_0^{Crs}(\mathbb{R}^2)$ has one element.

To prove this proposition we need the following propositions.

Proposition 1.8. Let $m, n \in \mathbb{N}$, and let $f: \mathbb{R}^m \to \mathbb{R}^n$ be a coarse map. Then the map f is close to some coarse Lipschitz map.

Proof. This is a higher dimension case of proposition 1.5. ∎

Proposition 1.9. Let $f: \mathbb{R}_+ \to \mathbb{R}^2$ be a coarse map. Then the map f is coarsely homotopic to the map $i: \mathbb{R}_+ \to \mathbb{R}^2$ defined by $i(s) = (s, 0)$.

Proof. By proposition (1.8), we can assume without loss of generality that f is Lipschitz. In polar coordinates, we can write f as follows:

$$f(s) = (r(s), \theta(s))$$

where the map $\theta: \mathbb{R}_+ \to \mathbb{R}$ is bounded. In polar coordinates the map i is defined by the formula $i(s) = (s, 0)$. It is not hard to prove that i is coarsely homotopic to the map $j: \mathbb{R}_+ \to \mathbb{R}^2$ defined by $j(s) = (r(s), 0)$.

Define a map $H: I_p\mathbb{R}_+ \to \mathbb{R}^2$ by writing

$$H(s,t)=\begin{cases}\left(r(s),\dfrac{t}{r(s)+1}\right) & \text{t} \leq (r(s)+1)\theta(s)\\ (r(s),\theta(s)) & \text{t} \geq (r(s)+1)\theta(s)\end{cases}$$

where p is the identity on $\mathbb{R}_+$. The map $s \mapsto (r(s)+1)\theta(s)$ is a controlled map, so H is a coarse homotopy between f and j and since coarse homotopy is transitive, so f is coarsely homotopic to i. ■

Proving all the above propositions proves proposition (1.7). ■

Note that the above propositions presented in this section are special cases of a main result found in [10].

Example 1.10. Let B be a bounded coarse space. Then there are no coarse maps $\mathbb{R}_+ \to B$. Hence $\pi_0^{Crs}(B)$ can not be defined.

Theorem 1.11. For any coarse space $X, \pi_0^{\text{Crd}}(X)$ is always the one point set.

Proof. Let $f: \mathbb{R}_+ \to X$ be a controlled map (not necessarily coarse). Let $p: \mathbb{R}_+ \to \mathbb{R}_+$ be the map $p(x)=x$ for all $x \in \mathbb{R}_+$. Define a map $H: I_p\mathbb{R}_+ \to X$ by

$$H(x,t)=\begin{cases}f(t) & t \leq x\\ f(x) & t \geq x\end{cases}$$

for all $x,t \in \mathbb{R}_+, 0 \leq t \leq p\{x\}+1$. Then H is a controlled map since f is so, and $H(x,0)=f(0)$ for all $x \in \mathbb{R}_+$ which clearly shows that f is controlledly homotopic to a constant map. By proposition 1.1 .11 in [12] then any constant maps are close, and so by example (0.18) they are controlledly homotopic. ■

The above theorem tells us that the set of controlled path components of any space X has one element.

The following proposition is induced from proposition (4.9) in [10] , and it resembles its classical analogue.

Proposition 1.12. Let $f: X \to Y$ be coarse map. Then there is a functorially induced homomorphism $f_*: \pi_0^{\text{Crs}}(X) \to \pi_0^{\text{Crs}}(Y)$ defined by $f_0[h]=[f \circ h]$ where $h: R \to X$ is a coarse map. Furthermore, if $f,g=X \to Y$ are coarsely homotopic maps. Then the functorial induced maps $f_*, g_*: \pi_0^{\text{Crs}}(X) \to \pi_0^{\text{Crs}}(Y)$ are equal.

Proof. Suppose that $f,g: X \to Y$ are coarsely homotopic maps, then there is a coarse map $H: I_pX \to Y$ such that $H(x,0)=f(x)$ and $H(x,p(x)+1)=g(x)$ for some controlled map $p: X \to R$. Define maps $f_*, g_*: \pi_0^{\text{Crs}}(X) \to \pi_0^{\text{Crs}}(Y)$ by $f_*[h]=|f \circ h|$ and $g_*[h]=[g \circ h]$ where $h: R \to X$ is a coarse map, then $f \circ h$ and $g \circ h$ are coarse maps. Define a map $F: I_{p\circ h}R \to Y$ by $F(s,t)=H(h(s),t)$ then F is a coarse map and

$$F(s,0)=f(h(s)), \text{ and } F(s,p(h(s))+1)=g(h(s))$$

Hence $f \circ h$ is coarsely homotopic to $g \circ h$ as required. ■

Corollary 1.13. Let $f: X \to Y$ be a coarse homotopy equivalence, then

$$\left|\pi_0^{\text{Crs}}(X)\right|=\left|\pi_0^{\text{Crs}}(Y)\right|$$

Proof. It is straightforward from the above proposition. ■

2. CONTROLLED AND COARAE HOMOTOPY GROUPS

Here we define coarse homotopy groups, and in order to do that we need a notion of basepoint. The purpose of this article is to develop some notions of homotopy theory in the coarse category. These homotopies have to end eventually, but the end time will be allowed to depend on the given point in the coarse space (and to go to infinity as one goes to infinity). This will be measured by coarse maps $p: X \to R$, which we call *basepoint projection*, and which will be a part of the structure for us.

Definition 2.1. Let X be a coarse space in the category of coarse maps, R a generalised ray. A *besepoint* for X is a coarse map $i_X: R \to X$ such that $p_X \circ i_X = id_R$ where $p_X: X \to R$ is a controlled map. A coarse space equipped with a basepoint is termed *pointed coarse space*.

If Y is another coarse space with basepoint in the category Crs, then a coarse map $f: X \to Y$ is termed *pointed coarse map* if $f \circ i_X = i_Y$

We term the category of pointed coarse spaces and pointed coarse maps *the category of pointed coarse* maps. It has an initial object, namely the space R, and we denote this category by PCrs.

Similarly we define *the category of pointed controlled maps*, and we denote it by PCrd.

Example 2.2. Let B be a bounded coarse space, Then there are no coarse maps $i_B: R \to B$, so B has no coarse basepoint.

Definition 2.3. Let X and Y be coarse pointed spaces. Then we write $[X,Y]_R^{Crs}$ to denote *the set of coarse homotopy classes of pointed coarse maps from X to Y relative to R*. That is, all coarse homotopies are pointed. The set is equipped with a base element that is defined to be *the relative coarse homotopy class* of the map

$$X \xrightarrow{px} R \xrightarrow{iY} Y$$

And similarly $[X,Y]_A^{\text{Crd}}$ is *the set of controlled homotopy classes of pointed controlled maps from X to Y relative to R*.

The following definition is directly inspired by the classical definition of homotopy groups, See [10].

Definition 2.4. Let X be a coarse pointed space. Let $n > 0$. Then we define *the* $n - th$ *coarse homotopy group with respect to* $R \sqcup R$ to be the set of coarse homotopy classes of pointed coarse maps

$$\pi_n^{\mathrm{PCrs}}(X, R) = [(R \sqcup R)^{n+1}, X]_A^{\mathrm{Crs}}$$

where $(R \cup R)^{n+1}$ the cone of the n-sphere S^n (see [4], [13]).

And for controlled case, we have the following definition as the evident candidate of the above definition.

Definition 2.5. Let X be a coarse pointed space. Let $n > 0$. Then we define *the* $n - th$ *controlled homotopy group with respect to* $R \sqcup R$ to be the set of controlled homotopy classes of pointed controlled mapes

$$\pi_n^{\mathrm{PCrd}}(X, R) = [(R \sqcup R)^{n+1}, X]_A^{\mathrm{Crd}}$$

Example 2.6. Let B be a bounded coarse space. There are no coarse maps $(R \sqcup R]^n \to B$ so $\pi_n^{\mathrm{PCrd}}(B)$ is not defined for any $n > 0$.

Proposition 2.7. Let $n \geq 1$. Then the set $\pi_n^{\mathrm{PCrd}}(X, R)$ is a group. For $n \geq 2$ then the set $\pi_n^{\mathrm{PCrd}}(X, R)$ is an abelain group.

Proof. Straightforward by the statement of proposition (4.8) in [10] . ■

Similarly, we prove the following proposition.

Proposition 2.8. Let $n \geq 1$. Then the set $\pi_n^{\mathrm{PCrs}}(X, R)$ is a group. For $n \geq 2$ the set $\pi_n^{\mathrm{PCrs}}(X, R)$ is an abelain group. ■

The following result is proved in [10] .

Theorem 2.9. The coarse homotopy groups $\pi_k^{\mathrm{PCrs}}(\mathbb{R}^{n+1})$ is isomorphic to the basic homotopy groups $\pi_k(S^n)$. ■

Example 2.10. $\pi_1^{\mathrm{PCrs}}(\mathbb{R})$ is isomorphic to $\pi_1(S^0)$, but $\pi_1(S^0) = \{0\}$ so $\pi_1^{\mathrm{PCrs}}(\mathbb{R})$ is isomorphic to $\mathbb{Z}$.

Proposition 2.11. Let $f; X \to Y$ be a coarse homotopy equivalence map. Then the functorial induced map $f_*: \pi_n^{\mathrm{PCrs}}(X) \to \pi_n^{\mathrm{PCrs}}(Y)$ is a bijection when $n = 0$ and isomorphism when $n > 0$.

Proof. A direct consequence to proposition (1.12). ■

Theorem 2.12. For any coarse space $X, \pi_n^{\mathrm{PCrd}}(X)$, the set of homotopy controlled classes of pointed controlled maps has one element.

Proof. Similar argument to theorem (1.11). ■

The above theorem tells us that the controlled category is trivial from a homotopy point of view and this puts an end to this idea since there is no evident way to have interesting homotopy groups based on controlled maps.

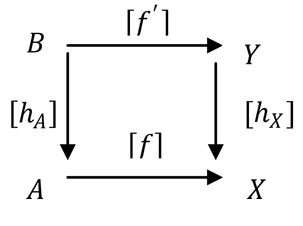
3. RELATIVE COARSE HOMOTOPY

Cofibration categories carry an abstract notion of relative homotopy. There is a more intuitive version of relative homotopy in the quotient coarse category. In this section we define and compare these two notions.

The following definition generalized from [1].

Recall that $Qcrs$ the quotient coarse cofibration category defined and proved in [12],[13] to be The category of non-unital coarse spaces and closeness equivalence classes of coarse maps (between non-unital coarse spaces), and we denote this category by $Qcrs$. Denote such classes by $[f]: X \to Y$ where f is a representative coarse map. A coarse map $f: X \to Y$ is a coarse equivalence in the category of coarse maps if and only if the closeness equivalence class is an isomorphism in the category $Qcrs$.

Definition 3.1. Let $Qcrs$ be the quotient coarse cofibration category. Then we define $Pair(Qcrs)$ to be the category in which objects are morphisms $[h_X]: Y \to X$ in Qcrs, the morphisms are the pairs $([f], [f']): h_A \to h_X$ such that the diagram



commutes in $Qcrs$.

The morphism $([f], [f'])$ is a coarse homotopy equivalence class if $[f], [f']$ are coarse homotopy equivalence classes, and $([f], [f'])$ is a coarse cofibration class if $[f']$ and $([f], [h_X]): A \vee_B Y \to X$ are coarse cofibration classes in $Qcrs$.

We call $(f], [f'])$ a *push out* if the diagram is a pushout diagram with $[h_A \mid: B \mapsto A$ a coarse cofibration class.

The proof of the following theorem is found in lemma ((1.5) , chapter (II), [1]).

Theorem 3.2. The category *Pair(Qcrs)* with coarse cofibration classes and coarse homotopy equivalence classes as in the previous definition is a Baues cofibration category.

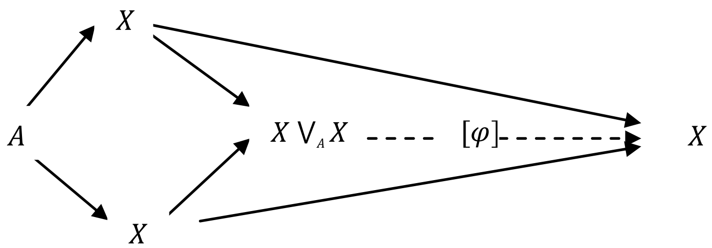
An object $[h_A]: B \to A$ is *fibrant* in $Pair(Qcrs)$ if and only if B and A are fibrant in $Qcrs$. ■

The following definition comes from [1] , and [2] .

Definition 3.3. A *based object* in a cofibration category C is a *cofibrant object* X (that is, $*\rightarrow X$ is a cofibration) with a map $p: X \rightarrow *$ from X to the initial object $*$ termed the *trivial map*. This defines the trivial map $i_U \circ p: X \rightarrow * \rightarrow U$ for all objects U in C representing $i_U \circ p \in [X, U]$ the set of maps from X to U relative to $*$.

A map $f: A \rightarrow B$ between based objects is *based* if $pf = p$.

Definition 3.4. We term the category of non-unital pointed coarse spaces and closeness equivalence classes of pointed coarse maps *the pointed quotient coarse category*. It has an initial object, namely the space R, and we denote this category by $PQcrs$.
In this category for later requirement, we need to know that our basepoint inclusion in a space we consider is a coarse cofibration, that is, all objects are cofibrant. For this point to be true for spaces of interest, we need to check at least we have the following result.
Lemma 3.5. The inclusion $i: R \hookrightarrow R^n$ is a coarse cofibration.
Proof. The inclusion i is a coarse homotopy equivalence by example (3.9) in [10]. Now by lemma 9(2.2.13)(2)) and a similar argument to that used in proposition (3.8.2) in [12] shows that i is a coarse cofibration.
Proposition 3.6. The category $PQcrs$ is a Baues cofibration category. The weak equivalences are coarse homotopy equivalence classes relative to R, and cofibrations are pointed coarse cofibration classes.
Proof. By definition (1.4) of chapter III in [2], the category $PQcrs$ is a subcategory of the category $Pair(Qcrs)$. Objects are the non-unital pointed coarse spaces, and the maps are the pointed coarse classes. Weak equivalences and cofibrations in the category $Qcrs$ yield the structure of Baues cofibration category for the category $PQcrs$. ■
So we have the category $PQcrs$ is a Baues cofibration category which has weak equivalences to be coarse homotopy equivalence classes relative to R in $Qcrs$ and cofibrations are defined to be the pointed coarse cofibration classes. The cofibrant objects X in $PQcrs$ are the coarse cofibration classes $R \hookrightarrow X$.
Definition 3.7. Let $[i]: A \rightarrow X$ be a coarse cofibration class. *The folding class* $[\varphi]: X \vee_A X \rightarrow X$ defined in $Qcrs$ by the commutative diagram

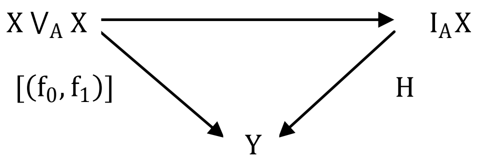


where the maps in the left square all are coarse cofibration classes.
In our pointed quotient coarse cofibration category, the folding class $[\varphi]$ is defined by writing $[\varphi \circ \theta] = [id]$, where $\theta: X \sqcup_\infty X \rightarrow X \vee_A X$ is the *coequalizer map* as defined in 4.4 in [13] .
In general, since the folding map $[\varphi]$ is a morphism in a cofibration category C, it can be written as a composite:

$$X \vee_A X \xrightarrow{[i']} I_A X \xrightarrow{[r]} X$$

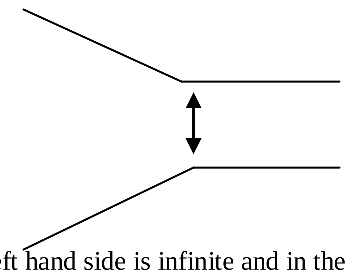
where $[i']$ is a coarse cofibration class, $[r]$ a coarse homotopy equivalence class. The space $I_A X$ is called a *relative* cylinder for the pair (X, A).

Definition 3.8. Let $[f_0], [f_1]: X \rightarrow Y$ be morphisms in the quotient coarse cofibration category *Qcrs*. Suppose we have a coarse cofibration class $[i]: A \rightarrow X$ such that $[f_0 \circ i] = [f_1 \circ i]$. Then we say that the maps $[f_0], [f_1]$ are *strongly coarse homotopic relative to A on the relative cylinder $I_A X$* if there is a commutative diagram

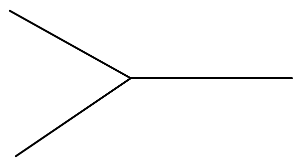


such that $[H \circ i_0] = [f_0]$, and $[H \circ i_1] = [f_1]$
By proposition (2.2), chapter (II) in [1], the notion of strong coarse homotopy is independent of the choice of relative cylinders. The following definition found in [13].

Definition 3.9. Let $[f]: X \to Y$ be the closeness equivalence class of a coarse map, and $p: X \to R$ be a controlled map. Then we define *the mapping cylinder* of $[f], C_f$ to be the push out $I_p X \vee_X Y$ which is defined to be $\mathrm{Coe}q([[\tilde{f}], [\tilde{\imath}])$ where $\tilde{f} : X \to I_p \mathrm{X} \sqcup_\infty Y$ and $\tilde{\imath}: X \to I_p \mathrm{X} \sqcup_\infty Y$ are coarse maps.
Thus for the quotient coarse category, by definition 3.9 and our work in [13] we have the mapping cylinder $I_p (X \vee_A X) \vee_X X$ (defined by the coequalizer) where $p: X \vee_A X \to R$ is some controlled map.
By definition of mapping cylinder we can choose this mapping cylinder to be our relative cylinder. We consider the disjoint union $R \sqcup R$ to be the line $(-\infty, \infty)$ equipped with the metric coarse structure. Initially, we can not view the above space $I_A X$ explicitly in abstract general picture, but the following example will give a picture of what the space look like.
The space $I_R(R \sqcup R)$. First the space $(R \sqcup R) \vee_R (R \sqcup R)$ can be viewed as follows:



where the distance in the left hand side is infinite and in the hand right side is finite.
Since this space is pointed, the easy way of showing a picture of the space $I_R(R \sqcup R)$ is to define a coarse equivalence between the space $(R \sqcup R) \vee_R (R \sqcup R)$ equipped with the coequalizer coarse structure and *the glued coarse space* (we use the notation "*glued*" to not be confused with the quotient space defined in the previous section) $((R \sqcup R) \sqcup_\infty (R \sqcup R))/R$, with the later space pictured as:



If we consider the disjoint union $R \sqcup R$ to be the line $(-\infty, \infty) = \mathbb{R}_+ \cup \mathbb{R}$ - equipped with the metric coarse structure, where $\mathbb{R}_+ = \{x : x \geq 0\}$ and $\mathbb{R}_+ = \{x : x \leq 0\}$ and the ray R to be the line $\mathbb{R}_+$. So the glued space has two apart different copies of $\mathbb{R}_-$ and one copy of $\mathbb{R}_+$ while the other space still have two apart different copies of $\mathbb{R}$ - and two copies of $\mathbb{R}_+$ within finite distance as explained in the above pictures.
Lemma 3.10. There is a coarse equivalence between the spaces $(R \sqcup R) \vee_R (R \sqcup R)$ and the coarse space $((R \cup R) \sqcup_\infty (R \sqcup R))/R$.
Proof. : Define a map $f: ((R \sqcup R) \cup_\infty (R \cup R))/R \to (R \sqcup R) \vee_R (R \cup R)$ by writing

$$f(x_1) = x_1 \text{ and } f(x_2) = x_2$$

If x_1 and x_2 are in different copies of $\mathbb{R}_-$, and

$$f(x) = x$$

where $x \in \mathbb{R}_+$. That is, the map f defines the inclusion for any $\in \mathbb{R}_+$. It is clear that this map is a coarse map.
Now define another map $g: (R \sqcup R) \vee_R (R \sqcup R) \to ((R \sqcup R) \sqcup_\infty (R \sqcup R))/R$ by writing

$$g(x_1) = x_1 \text{ and } g(x_2) = x_2$$

if x_1 and x_2 are in different copies of $\mathbb{R}_-$, and

$$g(x_1) = g(x_2) = x_1$$

If x_1 and x_2 are from different coples of $\mathbb{R}_+$. This is also a coarse map which clearly sends entourages to entourages and the inverse image of a bounded set under the map g restricted to any unital coarse subspace of $(R \sqcup R) \vee_R (R \sqcup R)$ is a bounded set.
The composite $g \circ f = id_{((R \sqcup R) \sqcup_\infty (R \sqcup R))/R}$ and the composite $f \circ g$ is close to the identity $id_{(R \sqcup R) \vee_R (R \sqcup R)}$ as follows. $f \circ g(x_1) = x_1$ and $f \circ g(x_2) = x_2$ if x_1 and x_2 are in different copies of $\mathbb{R}_-$, so $f \circ g = id_{(R \sqcup R) \vee_R (R \sqcup R)}$ in this case. $f \circ g(x_1) = f(x_2) = x_2$ where x_1 and x_2 are in different copies of $\mathbb{R}_+$.
By definition of the coequalizer coarse structure, the two copies of $\mathbb{R}_+$, which are apart, are within finite distance as in the picture. This implies that $d(f \circ g(x_1), x_1) = d(x_2, x_1) < c$ for some $c > 0$, so the composite is close to the identity. Hence the above spaces are coarsely equivalent. The above implies that the space $I_R(R \sqcup R)$ is coarsely equivalent to the space $(I_R(R \sqcup R))_{Glue}$ (We use "Glue" for the glued coarse structure) which can be viewed as follows:

$\{(x,t) \in (R \sqcup R)^2 : -|x|-1 \le t \le |x|+1\}/\sim$ such that $(s,t) \sim (s,-t)$ for all $s \in R, -|x|-1 \le t \le |x|+1$ which is equivalent to the following picture:

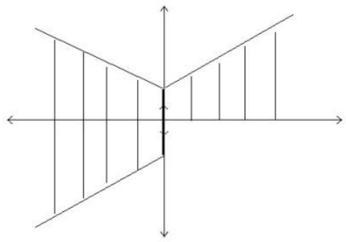


Figure 1. $(I_R(R \sqcup R))_{Glue}$

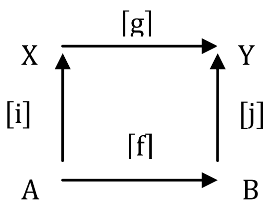
Definition 3.11. Suppose we have a closeness equivalence class $[i] = A \to X$. A *relative coarse hemotopy* is a coarse homotopy $F: I_pX \to Y$ such that the map $t \to F(x,t)$ is constant if $x = g(a)$ for any $g \in [i]$ and some point $a \in A$.
If $F: I_pX \to Y$ is a relative coarse homotopy, the closeness equivalence classes $[f_0 |: X \to Y$ and $[f_1]: X \to Y$ are said to be *coarsely homotopic relative to A* if representative maps f_0 and f_1 are defined by the formulae

$$f_0(x) = F(x,0), f_1(x) = F(x, p(x)+1)$$

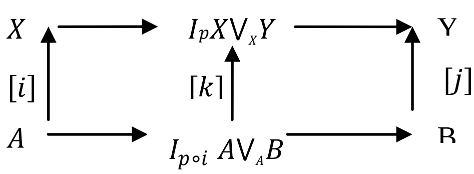
respectively.

Lemma 3.12. The notion of relative coarse homotopy between closeness equivalence classes of coarse maps is an equivalence relation.
Proof. : By the same method used in proof of theorem 3.2 in [13].

Lemma 3.13. Let X be a non-unital warse space, and $p: X \to R$ be some controlled map. Let $[i]: A \hookrightarrow X$ be a coarse cofibration class. Then the induced cless $[i_*]: I_{p\circ i}A \hookrightarrow I_pX$ defined by the formula $i_*(a,t) = (i(a),t)$ is a coarse cofibration class.
Proof. : It is enough to show that the representative map i_* is a coarse cofibration. Let $q: I_pX \to R$ be the controlled map defined by the formula $q(x,t) = p(x) + t$. By lemma (3.5) in [13], it suffices to show that the inclusion class $[j_*]: I_{q\circ i_*}(I_{p\circ i}A) \cup (I_pX \times \{0\}) \hookrightarrow I_q(I_pX)$ has a retraction, that is, there exists a coarse homotopy $[r_*]: I_q(I_pX) \to I_{q\circ i_*}(I_{p\circ i}A) \cup (I_pX \times \{0\})$ such that $r_* \circ j_* = id_{I_p \circ i A}$
Since $[i]: A \to X$ is a coarse cofibration class, then using lemma (3.5) in [13], there is a coarse class $[r]: I_pX \to (I_{p\circ i}A) \cup (X \times \{0\})$ such that $r \circ j = id_{(I_{p\circ i}A) \cup (X \times \{0\})}$.
We define the class $[r_*]$ by writing $r_*(x,t) = (r(x),t)$, then r_* is a representative coarse map, and $r_* \circ j_*(x,t) = r_*(j(x),t) = (r(j(x)),t) = (x,t)$ for all $x \in I_{q\circ i_*}(I_{p\circ i}A) \cup (I_pX \times \{0\})$ as required.
Lemma 3.14. Suppose that we have a commutative diagram

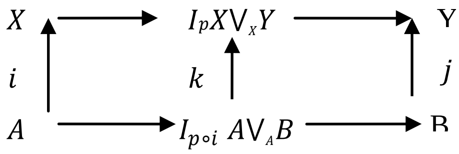


such that the inclusions $[i], [j]$ are coarse cofibration classes. Then we have a canonical coarse cofibration class $[k]: I_{p\circ i}A \vee_A B \to I_pX \vee_X Y$ such that the following diagram



commutes.

Proof. : First we have the diagram



by the factorization axiom in the quotient coarse cofibration category. Similar to the argument in the last lemma we need to find a retraction, that is; to show that there exists a coarse homotopy class

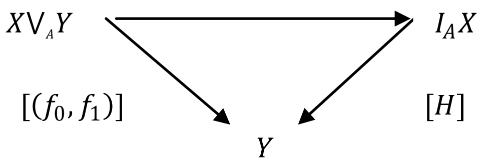
$$[h_*]: I_{q_1}\big(I_pX \vee_X Y\big) \to I_{q_1\circ i}\big(I_{p\circ i}A \vee_A B\big) \cup \Big(\big(I_pX \cup_X Y\big) \times \{0\}\Big)$$

Let $q_1: I_pX \vee_X Y \to R$ is some controlled map. By the above lemma since $[j]$ is a coarse cofibration class, we have the induced class $[i_*]: I_{p\circ i}A \hookrightarrow I_pX$ a coarse cofibration class. So the induced classes
$[j_*^1]: \big(I_{q_1\circ f'\circ j}B\big) \cup (Y \times \{0\}) \to I_{q_1\circ f'}Y$ and $[j_*^2]: I_{q_1\circ i'\,\circ i_*}\big(I_{p\circ i}A\big) \cup \big(I_pX \times \{0\}\big) \to I_{q_1\circ\, i'}\big(I_pX\big)$
have retractions
$[r_*^1]: I_{q_1\circ f'}Y \to \big(I_{q_1\circ f'\circ j}B\big) \cup (Y \times \{0\})$ and $[r_*^2]: I_{q_1\circ\, i'}\big(I_pX\big) \to I_{q_1\circ i'\,\circ i_*}\big(I_{p\circ i}A\big) \cup \big(I_pX \times \{0\}\big)$
where $f': I_pX \to I_pX \vee_X Y$ and $i': Y \to I_pX \vee_X Y$ are coarse maps defined by
$f'(x,t) = \theta(x,t)$ for any $x \in X$, and $i'(y) = \theta(y)$ for $y \in Y$ (the map θ is the coequalizer map).
Define the class $[h_*]$ by the formula:

$$h_*(\theta(y),t) = r_*^1(\theta(y),t), y \in Y$$
$$h_*(\theta(x,s),t) = r_*^2(\theta(x,s),t), (x,s) \in I_pX$$

Then$[h_*]$is the required retraction, and we are done.■

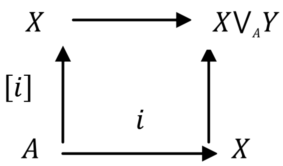
Theorem 3.15. Let $[i]: A \hookrightarrow X$ be a coarse cofibration class, and suppose that we have coarse classes $[f_0],[f_1]: X \to Y$ such that $[f_0 \circ i] = [i'] \; i]$.
Suppose that the classes $[f_0],[f_1]$ are strongly coorse homotopic relative to A. then $[f_0],[f_1]$ are also relatively coarse homotopic.
Proof. Suppose we have a commutative diagram of the form



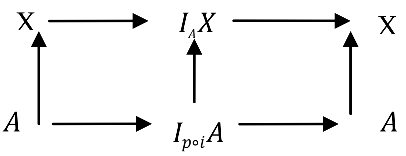
Let I_AX' be the quotient space $I_AX/\sim$, where the equivalence relation $\sim$ is defined by writing

$$(i(a),s) \sim (i(a),t) \text{ whenever } -p(i(a)) - 1 \leq s,t \leq p((a)) + 1$$

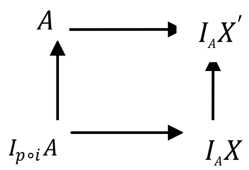
We need to prove that the spaces I_AX and I_AX' are coarsely homotopy equivalent.
First, note that by the following push out diagram



The obvious class $X \to X \vee_A X$ is a coarse cofibration class. Hence the composite class $A \to X \to X \vee_A X$ is also a coarse cofibration class, so by lemma 3.14 and 3.13 we have a commutative diagram



where the vertical arrows are all coarse cofibration class. Now the space $I_A X'$ is obtained by a push out diagram



The class $I_{\text{poi}}\, A \to A$ is certainly a coarse homotopy equivalence class, so by proposition (3.22) in [13] the class $I_A X \to I_A X'$ is also a coarse homotopy equivalence class, and we are done.

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