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Extremal Topology as an Ideal Extension

Yousef Al-Gadid [(1)] Mabruk Ali Sola [(2)]
Department of Mathematics, Tripoli University, Faculty Of Science, Tripoli, Libya

Abstract: *The aim of this paper is to find a formula for a topology J on X such that any extremal topology τ on X is an ideal extension to J for some ideal I on X. Characterizations related to such ideals are also discussed.*
Keyword: *Extremal topology, filter, ultrafilter, ideal, ideal extension*

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I. Preliminaries

If X is a non-empty set, a non-empty collection $\mathcal{F}$ of subsets of X is called a filter on X if (i) $\emptyset \notin \mathcal{F}$, (ii) if $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$, (iii) if $\mathrm{F} \in \mathcal{F}$ and $\mathrm{G} \subset \mathrm{X}$ with $\mathrm{F} \subset \mathrm{G}$ then $\mathrm{G} \in \mathcal{F}$. A filter $\mathcal{F}$ on X is said to be free filter provided $\bigcap_{\mathrm{F}\in\mathcal{F}} \mathrm{F} = \emptyset$ otherwise it is called a fixed filter. A filter $\mathcal{F}$ is called an ultrafilter if it is a maximal filter; that is if $\mathcal{G}$ is a filter containing $\mathcal{F}$, then $\mathcal{F} = \mathcal{G}$. A filter $\mathcal{F}$ is an ultrafilter on X if and only if for any $E \subseteq X$ either $E \in \mathcal{F}$ or $X \setminus E \in \mathcal{F}$ and an ultrafilter $\mathcal{F}$ is fixed ultrafilter if and only if there exists $\mathrm{b} \in \mathrm{X}$ such that $\bigcap_{\mathrm{F}\in\mathcal{F}} \mathrm{F} = \{\mathrm{b}\}$ [1]. If K is any set, $P(K)$ denotes the power set of K.

In [2] extremal topology was defined, and it was proved that for any $\mathrm{a}, \mathrm{b} \in \mathrm{X}$, $\mathrm{a} \neq \mathrm{b}$, $\tau_{\{\mathrm{a},\mathrm{b}\}} = \mathrm{P}(\mathrm{X}\setminus\{\mathrm{a}\}) \cup \{\{\mathrm{a}\} \cup \mathrm{A}: \mathrm{A} \subset \mathrm{P}(\mathrm{X}\setminus\{\mathrm{a}\}), \mathrm{b} \in \mathrm{A}\}$ is an extremal topology and if X is finite then every extremal topology on X has to be of the form $\tau_{\{\mathrm{a},\mathrm{b}\}}$ for some $a, b \in X$, $a \neq b$.

Theorem 1-2 and Theorem2-1 of [2] were generalized in[3, Theorem2] which states: A topology τ on X is extremal if and only if there exists $a \in X$ such that $\tau = \mathrm{P}(\mathrm{X}\setminus\{\mathrm{a}\}) \cup \{\{\mathrm{a}\} \cup \mathrm{F}: \mathrm{F} \in \mathcal{F}\}$ for some ultrafilter $\mathcal{F}$ on $X\setminus\{a\}$. An ideal I on a nonempty set X is a nonempty collection of subsets of X which satisfies (1) $A \in I$ and $B \subset A$ implies $B \in I$ (2) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given a topological space (X, τ) with an ideal I on X. For any $A \subset X$, let $A(I, \tau) = \{x \in X: A \cap U \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau: x \in U\}$. The mapping $(.)^*: P(X) \to P(X)$ defined by $A^* = A \cup A(I, \tau)$[4] is a closure operator defined on the power set $P(X)$ and thus defines a topology $\tau^*(I, \tau)$ on X finer than τ such that $U \in \tau^*(I, \tau)$ if and only if $(X\setminus U)^* = X\setminus U$. The topology $\tau^*(I, \tau)$ is called an ideal extension to τ over the ideal I. When there is no chance for confusion, we will simply write τ^* for $\tau^*(I, \tau)$. Clearly, every topology is an ideal extension to itself simply by taking $I = \{\emptyset\}$.

II. The main results

The following Proposition holds for any set X with $|X| > 1$.

Proposition 2.1. Let $\tau_{\{a,b\}}$ be an extremal topology on X for some $a, b \in X$. Taking the topology $J = P(X\setminus\{a\}) \cup \{X\}$ and the ideal $I = P(X\setminus\{b\})$ on X makes $\tau_{\{a,b\}}$ an ideal extension to J over I i.e., $J^* = \tau_{\{a,b\}}$.

Proof. Because $J \subset J^*$, it is enough to show that $\{\{a\} \cup A: A \in P(X\setminus\{a\}), b \in A\} \subset J^*$ and $\{a\} \notin J^*$ (J^* is not discrete).
Now, $((\{a\} \cup A)^c)^* = (\{a\} \cup A)^c \cup \{x \in X: (\{a\} \cup A)^c \cap U \notin I$ for every $U \in J(x)\}$. If $x \notin (\{a\} \cup A)^c \Rightarrow x \in \{a\} \cup A$ and we have two cases:
Case 1. $x = a$, then $(\{a\} \cup A)^c \cap X = (\{a\} \cup A)^c \in I$;
Case 2. $x \neq a$, then $(\{a\} \cup A)^c \cap \{x\} = \{\ \} \in I$.
Thus, $((\{a\} \cup A)^c)^* = (\{a\} \cup A)^c$ and this implies that $\{\{a\} \cup A: A \in P(X\setminus\{a\}), b \in A\} \subset J^*$. However, $(X\setminus\{a\})^* = X\setminus\{a\} \cup \{x \in X: X\setminus\{a\} \cap U$ for every $U \in J(x)\}$, X is the only openset in J contating a and $X\setminus\{a\} \cap X = X\setminus\{a\} \notin I$.
Thus, $(X\setminus\{a\})^* = X$ and this implies $\{a\} \notin J^*$.

Example 2.1. Let τ be any extremal topology on the set $X = \{a, b, c\}$. Without loss of generality [2, Theorem2-1], assume $\tau = \tau_{\{a,b\}} = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}\}$. Then by Proposition2.1, τ is an ideal extension to the topology $J = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$ where the ideal $I = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. In other words, $J^* = \tau$.

Remark. One can easily verify that τ is also an ideal extension to the same topology J and the ideal $I = \{\emptyset, \{c\}\}$.

Throughout the next X is an infinite set unless otherwise explicitly stated.

The following Proposition characterizes such ideals for extremal topology induced by fixed ultrafilter.
Proposition 2.2. If $\tau = P(X\backslash\{a\}) \cup \{\{a\} \cup F: F \in \mathcal{F}\}$ is an ideal extension to the topology $J = P(X\backslash\{a\}) \cup \{X\}$ for some ideal I on X ($J^* = \tau$) where $\mathcal{F}$ is a fixed ultrafilter on $X\backslash\{a\}$ and $a \in X$, then $I \subset P(X\backslash\{b\})$ where $\{b\} = \bigcap_{F\in\mathcal{F}} F$.

Proof. By [3, Corollary 3] and Proposition 2.1, such an ideal I exists. Also, we have $\{a, b\} = \{a\} \cup \{b\} \in \tau = J^*$.
Now, suppose not i.e., there exists $b \in K \in I \Rightarrow \{b\} \in I.(\{a, b\}^c)^* = \{a, b\}^c \cup \{x \in X: \{a, b\}^c \cap U \notin I$ for every $U \in J(x)\}$. Take x = a, then the only open set in J containing a is X and $\{a, b\}^c \cap X = \{a, b\}^c$, so we have two cases:
Case1. $\{a, b\}^c \notin I \Rightarrow (\{a, b\}^c)^* \neq \{a, b\}^c \Rightarrow \{a, b\} \notin J^*$, which is a contradiction.
Case2. $\{a, b\}^c \in I \Rightarrow X\backslash\{a\} = \{b\} \cup \{a, b\}^c \in I$ and since X is the only open set in J containing a with $X\backslash\{a\} \cap X = X\backslash\{a\} \in I$, and this implies that $(X\backslash\{a\})^* = X\backslash\{a\} \Rightarrow \{a\} \in J^* = \tau$, which contradicts being τ extremal. Therefore, $I \subset P(X\backslash\{b\})$.

Lemma2.1. If $\mathcal{F}$ is a filter on $X\backslash\{a\}$ and $a \in X$, then $I_a(\mathcal{F}) = \{X\backslash(F \cup \{a\}) = F`: F \in \mathcal{F}\}$ is an ideal on $X$ (F`:= the complement of F w.r.t. $X\backslash\{a\}$).

Proof. If $A \subset F`(F \in \mathcal{F}) \Rightarrow F \cup \{a\} = X\backslash F` \subset X\backslash A$
$\Rightarrow F \subset X\backslash(A \cup \{a\}), (a \notin F)$
$\Rightarrow A` = X\backslash(A \cup \{a\}) \in \mathcal{F} \Rightarrow A \in I_a(\mathcal{F})$.
If F` and G` are in $I_a(\mathcal{F})(F, G \in \mathcal{F}) \Rightarrow (F` \cup G`) = (F \cap G)` \in I(F \cap G \in \mathcal{F})$. Therefore, $I_a(\mathcal{F})$ is an ideal on X.

Proposition2.3. Let τ be any extremal topology on X, $\tau = P(X\backslash\{a\}) \cup \{\{a\} \cup F: F \in \mathcal{F}\}$ where $\mathcal{F}$ is an ultrafilter on $X\backslash\{a\}$ and $a \in X$. Then τ is an ideal extension to the topology $J = P(X\backslash\{a\}) \cup \{X\}$ over the ideal $I_a(\mathcal{F}) = \{X\backslash(\{a\} \cup F): F \in \mathcal{F}\}$ on X ($J^* = \tau$).

Proof. Because $J \subset J^*$, it is enough to show that $\{a\} \cup F \in J^*$ for every $F \in \mathcal{F}$ and J^* is not discrete. $(X\backslash(\{a\} \cup F))^* = X\backslash(\{a\} \cup F) \cup \{x \in X: X\backslash(\{a\} \cup F) \cap U \notin I$ for every $U \in J(x)\}$.
If $x \notin X\backslash(\{a\} \cup F) \Rightarrow x \in \{a\} \cup F$ and we have two cases:
Case1. $x = a$, then $a \in X \in J$ with $X\backslash(\{a\} \cup F) \cap X = X\backslash(\{a\} \cup F) \in I$;
Case2. $x \in F$, then $\{x\} \in J$ with $X\backslash(\{a\} \cup F) \cap \{x\} = \{\ \ \} \in I$.
Thus, $(X\backslash(\{a\} \cup F))^* = X\backslash(\{a\} \cup F) \Rightarrow \{a\} \cup F \in J^*$.
Also $(X\backslash\{a\})^* = X\backslash\{a\} \cup \{x \in X: X\backslash\{a\} \cap U \notin I$ for every $U \in J(x)\}$. Now, X is the only open set in J containing a and $X\backslash\{a\} \cap X = X\backslash\{a\} \notin I \Rightarrow (X\backslash\{a\})^* = X \Rightarrow \{a\} \notin J^*$.

The following ideal characterization holds for any extremal topology.
Proposition2.4. If an extremal topology τ on X (τ as in Proposition 2.3) is an ideal extension to the topology $J = P(X\backslash\{a\}) \cup \{X\}$ for some ideal K on X ($J^* = \tau$), then $I_a(\mathcal{F}) \subset K$.

Proof. Recall that $(X\backslash(\{a\} \cup F))^* = X\backslash(\{a\} \cup F) \cup \{x \in X: X\backslash(\{a\} \cup F) \cap U \notin K$ for every $U \in J(x)\}$ and X is the only open set in J containing a. Therefore if $X\backslash(\{a\} \cup F) \notin K$ for some $F \in \mathcal{F}$, then $(X\backslash(\{a\} \cup F))^* \neq X\backslash(\{a\} \cup F) \Rightarrow \{a\} \cup F \notin J^* = \tau$. A contradiction.

The following is consequence of Proposition 2.2 and Proposition 2.4.
Corollary2.1. If an extremal topology τ on X (τ as in Proposition 2.3) is an ideal extension to the topology $J = P(X\backslash\{a\}) \cup \{X\}$ for some ideal K on X, then $I_a(\mathcal{F}) \subset K \subset P(X\backslash\{b\})$ where $\bigcap_{F\in\mathcal{F}} F = \{b\}$.

References

[1] John L. Kelley, General Topology, Springer- Verlag 1955.
[2] Kherie M. Mera and Mabruk A. Sola, Extremal Topology, DamascusUniversity Journal for basic science, vol.21, No1, 2005.
[3] Sola, M. A. and Najm M. S, On Extremal Topology, The Libyan Journal of Sicience,19B,39-42, 2016.
[4] Vaidyanathaswamy, R., Set Topology, Chelsea Publishing Company, 1946.