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q- Generalization and Merging of Hyperbolic Functions with Circular Functions

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Abstract- *An unification of the circular and hyperbolic functions is proposed. The q-analysis is performed to obtain q-extension of more general function. The general function produces circular and hyperbolic functions as different states of the unified function. q- De Moivre formula and q- Euler formula for unified function are proposed.*

Keywords: *q-extension, unification of trigonometric functions, q- De Moivre formula, q- Euler formula for unified function*
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I. Introduction:

The problem of unification of basics laws or theories or definitions always remains of immense importance and keen interest. The mathematician, physicists and scientists devote their triumphs to formulate the ideas in least possible conceptual dimensions. The conciseness is the soul of any theoretical framework. In this direction we devote our study to unify circular functions with hyperbolic functions. The q-generalization yields the most elegant form. Gauss hyper geometric functions proposed by Carl Frederic Gauss[1] plays an important role to incorporate many functions as various states of it.

II. q-Generalization of hyper geometric function

$$ {}_2F_1\begin{pmatrix} a & , & b & ; & x \\ & c & & & \end{pmatrix} = 1 + \frac{a \quad b}{1 \quad c}x + \frac{a \quad (a+1) \quad b \quad (b+1)}{1 \quad 2 \quad c \quad (c+1)}x^2 + ... \qquad \text{… (1),} $$

where |x|< 1 and c > 0.

Introducing Pochhammer symbols,

$$ (a)_j = a(a+1)(a+2)...(a+j-1), \qquad \text{…(2),} $$

the hyper geometric series takes the form

$$ {}_2F_1\begin{pmatrix} a & , & b & ; & x \\ & c & & & \end{pmatrix} = \frac{(a)_0 \quad (b)_0}{(1)_0 \quad (c)_0}x^0 + \frac{(a)_1 \quad (b)_1}{(1)_1 \quad (c)_1}x^1 + \frac{(a)_2 \quad (b)_2}{(1)_2 \quad (c)_2}x^2 + ... \qquad \text{…(3).} $$

Eduard Heine[2] made the q-generalization of it, in the following manner

$$ {}_2\phi_1\begin{pmatrix} a & , & b & ; & x \\ & c & & & \end{pmatrix} = 1 + \frac{[a]_q \quad [b]_q}{[1]_q \quad [c]_q}x^1 + \frac{[a]_q[a+1]_q \quad [b]_q[b+1]_q}{[1]_q[2]_q \quad [c]_q[c+1]_q}x^2 + ... \qquad \text{…(4),} $$

with the fulfillment that

$$ \lim_{q\to 1} {}_2\phi_1 = {}_2F_1 \qquad \text{... (5).} $$

It is worthwhile to mention that

$$ \lim_{q\to 1}\frac{1-q^n}{1-q} = \lim_{q\to 1}\left\{1+q+q^2+...+q^{n-1}\right\} = n \qquad \text{…(6)} $$

Hence Heine introduced

$$ [n]_q = \frac{1-q^n}{1-q} \quad \text{and} \quad [n]_q! = \prod_{j=1}^{n}[j]_q \qquad \text{…(7)} $$

III. Tsallis ‘ q-analysis [3]

Constantino Tsallis utilized these ideas of q-analysis in statistics and contributed a great deal of knowledge in the field. Following Tsallis we consider the following differential equation

$$\frac{dy}{dx}-y^{q}=0,\quad y(0)=1 \qquad \text{…(8a)}$$

Method of separation of variables immediately yields

$$y=\left[1+(1-q)x\right]^{\left(\frac{1}{1-q}\right)} \qquad \text{…(9a)}$$

Direct observation of Eq. (8a) suggests that for q=1, we have the solution y = exp (x).
Hence we write

$$y=e_q^{x}=\left[1+(1-q)x\right]^{\left(\frac{1}{1-q}\right)} \qquad \text{…(9b).}$$

Eq. (8a) can be written as

$$\frac{dx}{dy}-y^{-q}=0,\quad y(0)=1 \qquad \text{…(8b).}$$

For q=1 , Eq. (8b) has the solution x=ln(y). Therefore, for $q\neq 1$, using Eq. (9b) we may write

$$x=\ln_q y=\frac{y^{1-q}-1}{1-q} \qquad \text{…(10).}$$

It can be simply verified that $\ln_q x$ and e_q^x are inverse to each other. Moreover, we see that such extension doesn’t preserve linearity.

$$\ln_q(uv)=\frac{(uv)^{1-q}-1}{1-q}=\frac{\left\{(u)^{1-q}-1\right\}\left\{(v)^{1-q}-1\right\}+\left\{(u)^{1-q}-1\right\}+\left\{(v)^{1-q}-1\right\}}{1-q}$$

$$=\frac{u^{1-q}-1}{1-q}+\frac{v^{1-q}-1}{1-q}+(1-q).\frac{u^{1-q}-1}{1-q}.\frac{v^{1-q}-1}{1-q}=\ln_q(u)+\ln_q(v)+(1-q).\ln_q(u).\ln_q(v)$$

…(11)

This non linear relation becomes linear only when q=1.

IV. q-exponential as solution of nonlinear oscillator equation[4]-[6]

Eq. (8a) with solution as Eq. (9b) suggests that:

$$\frac{d}{dx}e_q^{x}=\left(e_q^{x}\right)^{q} \qquad \text{…(12)}$$

Eq. (8a) with substitution z = y ^{b} suggests that:

$$\left(e_q^{x}\right)^{a}=e^{ax}_{1-\left(\frac{1-q}{a}\right)} \qquad \text{…(13)}$$

Combining Eq. (12) and Eq. (13), we obtain an elegant chain of solutions:

$$\frac{d}{dx}\left[e_q^{ax}\right]=a.e_{q'}^{q.ax},\qquad q'=2-q^{-1}.$$

$$\frac{d^2}{dx^2}\left[e_q^{ax}\right]=a^2q.e_{q''}^{q'.(q.ax)},\qquad q''=2-q'^{-1}=2-\left(2-q^{-1}\right)^{-1}.$$

$$\frac{d^3}{dx^3}\left[e_q^{ax}\right]=a^3q^2q'.e_{q'''}^{q''q'qax},\qquad q'''=2-q''^{-1}=2-\left(2-\left(2-q^{-1}\right)^{-1}\right)^{-1}.$$

…(14)

Using Eq. (14) we can obtain non linear oscillator differential equation

$$\frac{d^2}{dx^2}\left[e_q^{ikx}\right]^{\nu}+\gamma^2\left[e_q^{ikx}\right]^{\mu}=0,\quad q=\frac{\mu-\nu}{2}+1,\quad k^2=\frac{2\gamma^2}{\nu(\mu+\nu)} \qquad \text{…(15)}$$

V. Maclaurin 's series for q-exponential

Eq.(8a) on successive differentiation yields

$$\frac{d^n y}{dx^n} = (1.q-0)(2.q-1)(3.q-2)...\{((n-1).q-(n-2))\}\, y^{[nq-(n-1)]}$$

$$= Q_{n-1}.y^{[nq-(n-1)]}$$

...(16)

Where $Q_n = (1.q-0)(2.q-1)(3.q-2)...[nq-(n-1)]$ ...(17)

Using Eq. (8a) and Eq.(16) we obtain

$$e_q^x = 1 + \sum_{n=1}^{\infty} Q_{n-1}.\frac{x^n}{n!} \quad ...(18)$$

This expansion on replacing x by ix can be separated into real and imaginary parts. It leads to Euler formula for q-generalization

$$e_q^{ix} = \cos_q x + i\sin_q x \quad ...(19)$$

Similarly Eq. (18) can be expressed for hyperbolic entities

$$e_q^x = \cosh_q x + \sinh_q x \quad ...(20)$$

VI. Unification of circular and hyperbolic functions

Let us introduce an e[…] symbol as follows

$$e\begin{bmatrix} r \\ s \end{bmatrix}; \quad x\Bigg] = \frac{e^{x(-1)^s} + re^{-x(-1)^s}}{2r^s} \quad ...(21)$$

To ensure analyticity we impose that r≠0.
This newly introduced function gives

$$e\begin{bmatrix} \pm 1 & ; & x \\ 2 & & \end{bmatrix} = \begin{matrix}\cosh \\ \sinh\end{matrix}\, x \quad ...(22)$$

$$e\begin{bmatrix} \pm 1 & ; & x \\ \frac{1}{2} & & \end{bmatrix} = \begin{matrix}\cos \\ \sin\end{matrix}\, x \quad ...(23)$$

Motivating with this ansatz we propose q-extension of our newly defined function

$$e_q\begin{bmatrix} r & ; & x \\ s & & \end{bmatrix} = \frac{e_q^{\,x(-1)^s} + re_q^{\,-x(-1)^s}}{2r^s} \quad ...(24)$$

We obtain the analogous trigonometric functions as follows

$$e_q\begin{bmatrix} \pm 1 & ; & x \\ 2 & & \end{bmatrix} = \begin{matrix}\cosh_q \\ \sinh_q\end{matrix}\, x \quad ...(25)$$

$$e_q\begin{bmatrix} \pm 1 & ; & x \\ \frac{1}{2} & & \end{bmatrix} = \begin{matrix}\cos_q \\ \sin_q\end{matrix}\, x \quad ...(26)$$

Euler formulas can be expressed as

$$e_q^{ix} = e_q\begin{bmatrix} 1 & ; & x \\ ½ & & \end{bmatrix} + ie_q\begin{bmatrix} -1 & ; & x \\ ½ & & \end{bmatrix} \quad ...(27)$$

$$e_q^{x} = e_q\begin{bmatrix} 1 & ; & x \\ 2 & & \end{bmatrix} + e_q\begin{bmatrix} -1 & ; & x \\ 2 & & \end{bmatrix} \quad ...(28)$$

De Moivre Theorem reads

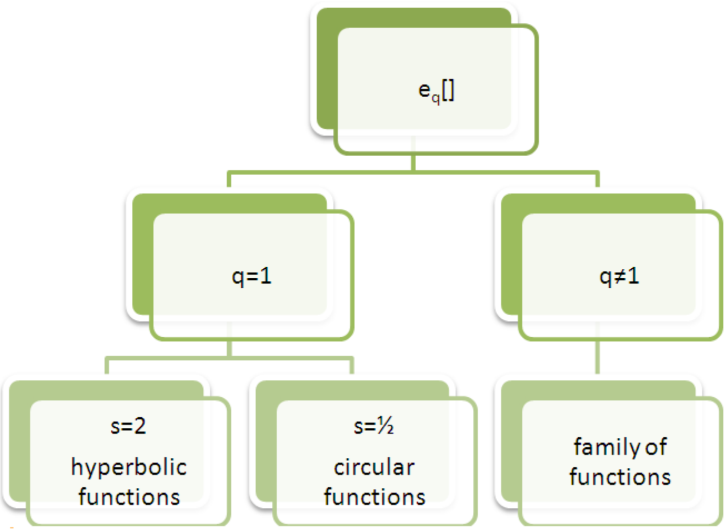
$$\left\{ e_q\left[\begin{matrix}1\\2\end{matrix};\quad x\right] + e_q\left[\begin{matrix}-1\\2\end{matrix};\quad x\right]\right\}^n = e_{q'}\left[\begin{matrix}1\\2\end{matrix};\quad nx\right] + e_{q'}\left[\begin{matrix}-1\\2\end{matrix};\quad nx\right] \mid\ 1.(1-q) = n.(1-q') \quad \ldots(29)$$

$$\left\{ e_q\left[\begin{matrix}1\\\frac{1}{2}\end{matrix};\quad x\right] + ie_q\left[\begin{matrix}-1\\\frac{1}{2}\end{matrix};\quad x\right]\right\}^n = e_{q'}\left[\begin{matrix}1\\\frac{1}{2}\end{matrix};\quad nx\right] + ie_{q'}\left[\begin{matrix}-1\\\frac{1}{2}\end{matrix};\quad nx\right] \mid\ 1.(1-q) = n.(1-q') \quad \ldots(30)$$

i.e., $$\left\{ \sum_{r=\pm1} r^s.e_q\left[\begin{matrix}r\\s\end{matrix};\quad x\right]\right\}^n = \sum_{r=\pm1} r^s.e_{q'}\left[\begin{matrix}r\\s\end{matrix};\quad nx\right] \mid\ 1.(1-q) = n.(1-q') \quad \ldots(31)$$

VII. Discussion And Conclusion:

We can regard different stares of newly introduced q-extended function as circular and hyperbolic functions in the limit q→1, as depicted in the below flowchart



It is to be noted that the generalized function need not to satisfy the nonlinear harmonic oscillator equation unlike e_q^z. Moreover, nonlinearity causes amplitude of oscillator varying unlike constant in case of q=1. For complex argument of the function, real and imaginary part get intermingled for q≠1. Euler 's formulas are straight forward due to Maclaurin's expansion but de Moivre 's formulas get modified due to change of q-value as a result of exponent as obtained in Eq, (13). Hopefully , the proposed unification and q-extension might play a role of immense use in theoretical physics and associated mathematics.

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