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Periodic solutions of an impulsive functional differential equation with delay and a parameter

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Abstract: *This paper is concerned with a second order impulsive functional differential equation with delay and a parameter. By using Krasnoselskii's fixed point theorem, sufficient conditions for the existence of positive periodic solutions to the impulsive delay differential equation are obtained.*
Keywords: *Periodic solution; Delay differential equation; Fixed point theorem; Impulse.*

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I. Introduction

In recent years, impulsive and periodic boundary value problems have been studied extensively in the literature, see [1-9]. In [2,4,5,10], periodic boundary value problems were studied extensively. Jiang [4] has applied Krasnoselskii's fixed point theorem to establish the existence of positive solution to problem

$$\begin{cases} -x'' + a^2 x = f(t, x), & t \in [0, 2\pi] \\ x(0) = x(2\pi), \; x'(0) = x'(2\pi) \end{cases} \tag{1.1}$$

he proved that there exists at least one positive solution. Zhang and Wang [10] studied (1.1) for singularity. They gave the existence of multiple positive solutions via the Krasnoselskii's fixed point theorem.
On the other hand, impulsive differential equations were studied extensively. In [6,8,9], authors used the method of lower and upper solutions with monotone iterative technique to study impulsive differential equations. In [1,7], authors used the Krasnoselskii's fixed point theorem in a cone to impulsive differential equations and obtained the existence of positive solutions.
Motivated by the above works, in this paper, we shall deal with the existence of positive periodic solutions of a second order impulsive delay functional differential equation with periodic coefficients

$$\begin{cases} x''(t) + a(t)x'(t) + b(t)x(t) = \lambda c(t) f(t, x(t), x(t-\tau(t))), & t \neq t_j, \\ \Delta x\big|_{t=t_j} = I(x(t_j)), \; -\Delta x'\big|_{t=t_j} = J_j(x(t_j)), & j \in z^+, \end{cases} \tag{1.2}$$

where

(A1) $a, b : R \to R^+$, $c, \tau : R \to R$ are all continuous T-periodic functions, and $\int_0^T a(s)ds > 0$, $\int_0^T b(s)ds > 0$, $\tau'(t) \neq 1$, for all $t \in [0, T]$;

(A2) $f : R^3 \to R$ is continuous for any $(t, x, y) \in R^3$ and is T-periodic in t for all $(x, y) \in R^2$.

(A3) There exist positive constants L and E such that

$$\left| f(t, x, y) - f(t, z, w) \right| \leq L\left|x - z\right| + E\left|y - w\right|.$$

(A4) $I_k \in C(R^+, R)$, $J_k \in C(R^+, R^+)$ with a constant m such that $-\frac{1}{m} J_k(x) < I_k(x) < \frac{1}{m} J_k(x)$, and $\Delta x\big|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $-\Delta x'\big|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$, where $x(t_j^+)$ and $x(t_j^-)$ represent the right and the left limit of $x(t_j)$, there exist an integer $p > 0$ such that $t_{j+p} = t_j + T$, $I_{j+p} = I_j$, $j \in z^+$.

For convenience, we first introduce the related definition and the fixed point theorem applied in the paper.

Definition 1.1 Let X be a Banach space and K be a closed nonempty sunset of X, K is a cone if

(1) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta \geq 0$;

(2) $u, -u \in K$ imply $u = 0$.

Theorem 1.1 (Krasnoselskii [11]) Let X be a Banach space, and let $K \subset X$ be a cone in X. Assume that Ω_1, Ω_2 are open bounded subsets of X with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, and let

$\phi : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$

be a completely continuous operator such that either

(1) $\|\phi y\| \le \|y\|, \forall y \in K \cap \partial\Omega_1$ and $\|\phi y\| \ge \|y\|, \forall y \in K \cap \partial\Omega_2$; or

(2) $\|\phi y\| \ge \|y\|, \forall y \in K \cap \partial\Omega_1$ and $\|\phi y\| \le \|y\|, \forall y \in K \cap \partial\Omega_2$.

Then ϕ has a fixed point in $K \cap (\overline{\Omega_2} \setminus K \cap \partial\Omega_1)$.

In this paper we always assume that

(H1) $f(t,\xi,\eta) \ge 0$ for all $(t,\xi,\eta) \in R \times BC(R,R_+) \times R_+$.

II. Preliminaries

In order to define the solution of (1.2) we consider the following Banach spaces:

$$PC(R,R) = \{x : R \to R : x|_{(t_j,t_{j+1})} \in C(t_j,t_{j+1}), x(t_j^-) = x(t_j), \exists x(t_j^+), j \in z^+\}$$

is a Banach space with the norm $\|x\|_{PC} = \sup_{t\in[0,T]} \sum_{j=1}^{n} |x_j(t)|$.

$$PC^1(R,R) = \{x : R \to R : x|_{(t_k,t_{k+1})}, x'|_{(t_k,t_{k+1})} \in C(t_k,t_{k+1}), x(t_k^-) = x(t_k), x'(t_k^-) = x'(t_k), \exists x(t_k^+), x(t_k), j \in z^+\}$$

is also a Banach space with the norm $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$.

Lemma 2.1. ([12]) Suppose that (A1, A4) holds and

$$\frac{R_1[\exp(\int_0^T a(u)du) - 1]}{Q_1 T} \ge 1, \tag{2.1}$$

$$R_1 = \max_{t\in[0,T]} \left| \int_t^{t+T} \frac{\exp(\int_t^s a(u)du)}{\exp(\int_0^T a(u)du) - 1} b(s)ds \right|, Q_1 = \left(1 + \exp(\int_0^T a(u)du)\right)^2 R_1^2,$$

there exist continuous T-periodic functions p and q such that $q(t) > 0$, $\int_0^T p(u)du > 0$, and

$p(t) + q(t) = a(t), q'(t) + p(t)q(t) = b(t)$ for all $t \in R$.

Therefore

$p(t) + q(t) = a(t), q'(t) + p(t)q(t) = b(t), t \in R$.

Lemma 2.2. ([13]) Suppose the conditions of Lemma 2.1 hold and $\varphi(t) \in X$. The equation

$$x''(t) + a(t)x'(t) + b(t)x(t) = \varphi(t) \tag{2.2}$$

has a T-periodic solution. Moreover, the periodic solutions can be expressed by

$$x(t) = \int_t^{t+T} G(t,s)\varphi(s)ds, \tag{2.3}$$

where

$$G(t,s) = \frac{\int_t^s \exp[\int_t^u q(v)dv + \int_u^s p(v)dv]du + \int_s^{t+T} \exp[\int_t^u q(v)dv + \int_u^{s+T} p(v)dv]du}{[\exp(\int_0^T p(u)du) - 1][\exp(\int_0^T q(u)du) - 1]}.$$

Therefore, the equation $x''(t) + a(t)x'(t) + b(t)x(t) = \lambda c(t) f(t, x(t), x(t-\tau(t)))$ has a T-periodic solution, it can be expressed by

$$x(t) = \int_t^{t+T} G(t,s)\lambda c(s) f(s, x(s), x(s-\tau(s)))ds$$

and by (H1), we have

$$G(t,s)\lambda c(s)f(s,x(s),x(s-\tau(s)))\geq 0,(t,s)\in R^2.$$

The following lemma is fundamental to our discussion. Since the method is similar to that in the literature [14], we omit the proof.

Lemma 2.3. $x\in X$ is a solution of (1.2) if and only if $x\in X$ is a solution of the equation

$$x(t)=\int_t^{t+T}G(t,s)\lambda C(s)f(s,x(s),x(s-\tau(s)))ds+\sum_{j:t_j\in[t,t+T]}G(t,t_j)J_j(x(t_j))$$
$$+\sum_{j:t_j\in[t,t+T]}\left.\frac{\partial G(t,s)}{\partial s}\right|_{s=t_j}I_j(x(t_j)). \tag{2.4}$$

Corollary 2.1. Green's function $G(t,s)$ satisfies the following properties:

$$G(t,t+T)=G(t,t),\ \ G(t+T,s+T)=G(t,s),$$

$$\frac{\partial}{\partial s}G(t,s)=p(s)G(t,s)-\frac{\exp\int_t^s q(v)dv}{\exp\int_0^T q(v)dv-1},$$

$$\frac{\partial}{\partial t}G(t,s)=-q(s)G(t,s)+\frac{\exp\int_t^s p(v)dv}{\exp\int_0^T p(v)dv-1}.$$

Lemma 2.4. ([13]) Let $A=\int_0^T a(u)du$, $B=T^2\exp(\frac{1}{T}\int_0^T \ln b(u)du)$. If $A^2\geq 4B$, (2.5)

then

$$\min\left\{\int_0^T p(u)du,\int_0^T q(u)du\right\}\geq\frac{1}{2}(A-\sqrt{A^2-4B}):=l,$$

$$\max\left\{\int_0^T p(u)du,\int_0^T q(u)du\right\}\leq\frac{1}{2}(A+\sqrt{A^2-4B}):=m.$$

Therefore the function $G(t,s)$ satisfies

$$0<N_1=:\frac{T}{(e^m-1)^2}\leq G(t,s)\leq\frac{T\exp(\int_0^T a(u)du)}{(e^l-1)^2}:=M_1,s\in[t,t+T],$$

$$1\geq\frac{G(t,s)}{M_1}\geq\frac{N_1}{M_1}=\sigma.$$

Now, before presenting our main results, we give the following assumptions.

(H2) $f(t,\phi(t),\phi(t-\tau(t)))$ is a continuous function of t for each $\phi\in BC(R,R^+)$.

(H3) For any $L>0$ and $\varepsilon>0$, there exists $\delta>0$, such that

$\{\phi,\psi\in BC,\|\phi\|\leq L,\|\psi\|\leq L,\|\phi-\psi\|<\delta,0\leq s\leq T\}$

imply $\left|f(s,\phi(s),\phi(s-\tau(s)))-f(s,\psi(s),\psi(s-\tau(s)))\right|<\varepsilon$.

III. Main Results

For every positive solution of (1.2), one has

$$\|x\|=\sup_{t\in[0,T]}\{|x(t)|,x\in X\}.$$

Let K is a cone in X, which is defined as

$$K=\left\{x\in X:x(t)\geq\sigma\|x\|,t\in[0,T]\right\}.$$

Now we define a mapping $T:K\to K$,

$$(Tx)(t) = \int_t^{t+T} G(t,s)\lambda C(s) f(s, x(s), x(s-\tau(s)))ds + \sum_{j:t_j\in[t,t+T]} G(t,t_j)J_j(x(t_j))$$

$$+ \sum_{j:t_j\in[t,t+T]} \left.\frac{\partial G(t,s)}{\partial s}\right|_{s=t_j} I_j(x(t_j)),$$

then we have

$$(Tx)(t) = \int_t^{t+T} G(t,s)\lambda C(s) f(s, x(s), x(s-\tau(s)))ds + \sum_{j:t_j\in[t,t+T]} G(t,t_j)J_j(x(t_j))$$

$$+ \sum_{j:t_j\in[t,t+T]} \left(p(t_j)G(t,t_j) - \frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv - 1} \right) I_j(x(t_j))$$

$$= \int_t^{t+T} G(t,s)\lambda C(s) f(s, x(s), x(s-\tau(s)))ds + \sum_{j:t_j\in[t,t+T]} G(t,t_j)J_j(x(t_j))$$

$$+ \sum_{j:t_j\in[t,t+T]} G(t,t_j)\, p(t_j) I_j(x(t_j)) - \sum_{j:t_j\in[t,t+T]} \left(\frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv - 1} \right) I_j(x(t_j)).$$

Lemma 3.1. $T: K \to K$ is well-defined.

Proof. For each $x \in K$, by (H2) we have $(Tx)(t)$ is continuous and

$$(Tx)(t+T) = \int_{t+T}^{t+2T} G(t,s)\lambda C(s) f(s, x(s), x(s-\tau(s)))ds + \sum_{j:t_j\in[t,t+T]} G(t+T, t_j+T)J_j(x(t_j+T))$$

$$+ \sum_{j:t_j\in[t,t+T]} \left(p(t_j+T)G(t+T, t_j+T) - \frac{\exp\int_{t+T}^{t_j+T} q(v)dv}{\exp\int_0^T q(v)dv - 1} \right) I_j(x(t_j+T))$$

$$= \int_t^{t+T} G(t+T, v+T)\lambda C(v+T) f(v+T, x(v+T), x(v+T-\tau(v+T))dv$$

$$+ \sum_{j:t_j\in[t,t+T]} G(t,t_j)J_j(x(t_j)) + \sum_{j:t_j\in[t,t+T]} \left(p(t_j)G(t,t_j) - \frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv - 1} \right) I_j(x(t_j))$$

$$= \int_t^{t+T} G(t,v)\lambda C(v) f(v, x(v), x(v-\tau(v))dv + \sum_{j:t_j\in[t,t+T]} G(t,t_j)I_j(x(t_j))$$

$$+ \sum_{j:t_j\in[t,t+T]} \left(p(t_j)G(t,t_j) - \frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv - 1} \right) I_j(x(t_j))$$

$$= (Tx)(t).$$

Thus, $Tx \in PC(J,R)$, since

$$N_1 \le G(t,s) \le M_1, \quad s \in [t, t+T],$$

and
$$\left.\frac{\partial G(t,s)}{\partial s}\right|_{s=t_j} = p(t_j)G(t,t_j) - \frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv - 1}, \quad t_j \in [t, t+T],$$

$$N_2 \le \left.\frac{\partial G(t,s)}{\partial s}\right|_{s=t_j} \le M_2, \quad t_j \in [t, t+T].$$

We define $M = \max\{M_1, M_2\}, N = \min\{N_1, N_2\}$.

Hence, for $x \in K$, we have

$$\|Tx\| \le M\left(\int_0^T \left|\lambda c(s) f(s, x(s), x(s-\tau(s)))\right| ds + \sum_{j: t_j \in [t,t+T]} J_j(x(t_j)) + \sum_{j: t_j \in [t,t+T]} I_j(x(t_j))\right), \qquad (3.1)$$

and

$$(Tx)(t) \ge N\left(\int_0^T \left|\lambda c(s) f(s, x(s), x(s-\tau(s)))\right| ds + \sum_{j: t_j \in [t,t+T]} J_j(x(t_j)) + \sum_{j: t_j \in [t,t+T]} I_j(x(t_j))\right)$$

$$= \frac{N}{M} M\left(\int_0^T \left|\lambda c(s) f(s, x(s), x(s-\tau(s)))\right| ds + \sum_{j: t_j \in [t,t+T]} J_j(x(t_j)) + \sum_{j: t_j \in [t,t+T]} I_j(x(t_j))\right)$$

$$\ge \sigma \|Tx\|.$$

Therefore, $Tx \in K$. This completes the proof.

Lemma 3.2. $T: K \to K$ is completely continuous.

Proof. We first show that T is continuous .

By (H3), for any $L > 0$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that $\{\phi, \psi \in BC, \|\phi\| \le L, \|\psi\| \le L, \|\phi - \psi\| \le \delta\}$ imply

$$\sup_{0 \le s \le T} \left| f(s, \phi(s), \phi(s-\tau(s))) - f(s, \psi(s), \psi(s-\tau(s)))\right| < \frac{\varepsilon}{2\lambda MTC},$$

where $C = \max_{0 \le t \le T} |c(t)|$.

Since $J_j, I_j \in C(R, R)$, we have $\left|J_j(\phi) - J_j(\psi)\right| < \dfrac{\varepsilon}{4Mp}$, $\left|I_j(\phi) - I_j(\psi)\right| < \dfrac{\varepsilon}{4Mp}$.

If $x, y \in K$ with $\|x\| \le L, \|y\| \le L, \|x - y\| \le \delta$, then

$$\left|(Tx)(t) - (Ty)(t)\right| \le \int_t^{t+T} |G(t,s)| \left|\lambda c(s) f(s, x(s), x(s-\tau(s))) - \lambda c(s) f(s, y(s), y(s-\tau(s)))\right| ds$$

$$+ \sum_{j: t_j \in [t,t+T]} \left|G(t,t_j)\right| \left|J_j(x(t_j)) - J_j(y(t_j))\right| + \sum_{j: t_j \in [t,t+T]} \left|\left.\frac{\partial G(t,s)}{\partial s}\right|_{s=t_j}\right| \left|I_j(x(t_j)) - I_j(y(t_j))\right|$$

$$\le \lambda MC \int_0^T |G(t,s)| \left| f(s, x(s), x(s-\tau(s))) - f(s, y(s), y(s-\tau(s)))\right| ds$$

$$+ M\sum_{j=1}^{P} \left|J_j(x(t_j)) - J_j(y(t_j))\right| + M\sum_{j=1}^{P} \left|I_j(x(t_j)) - I_j(y(t_j))\right|$$

$$< M\lambda TC \frac{\varepsilon}{2M\lambda TC} + 2Mp\frac{\varepsilon}{4Mp} = \varepsilon$$

for all $t \in [0, T]$, this yields $\|Tx - Ty\| < \varepsilon$, thus T is continuous.

Next we show that T maps any bounded sets in K into relatively compact sets. Now we first prove that f maps bounded sets into bounded sets. Indeed, let $\varepsilon = 1$, by (H3), for any $\mu > 0$, there exists $\delta > 0$ such that $\{x, y \in BC, \|x\| \le \mu, \|y\| \le \mu, \|x - y\| \le \delta, 0 \le s \le T\}$ imply

$$\left| f(s, x(s), x(s-\tau(s))) - f(s, y(s), y(s-\tau(s)))\right| < 1.$$

Choose a positive integer N such that $\frac{\mu}{N} < \delta$. Let $x \in BC$ and define

$x^k(t) = \frac{x(t)k}{N}, k = 0, 1, 2 \cdots, N$.

If $\|x\| < \mu$, then

$$\|x^k - x^{k-1}\| = \sup_{t\in R}\left|\frac{x(t)k}{N} - \frac{x(t)(k-1)}{N}\right| \le \|x\|\frac{1}{N} \le \frac{\mu}{N} < \delta .$$

Thus,

$$\left|f(s, x^k(s), x^k(s-\tau(s)) - f(s, x^{k-1}(s), x^{k-1}(s-\tau(s))\right| < 1$$

for all $s \in [0, T]$, this yields

$$\begin{aligned} &\left|f(s, x(s), x(s-\tau(s))\right|_0 = \left|f(s, x^N(s), x^N(s-\tau(s))\right| \\ &\le \sum_{k=1}^{N}\left|f(s, x^k(s), x^k(s-\tau(s)) - f(s, x^{k-1}(s), x^{k-1}(s-\tau(s))\right| + \left|f(s,0,0)\right| \qquad (3.2) \\ &< N + \|f\| =: W , \end{aligned}$$

and

$$\left|J_j(x(t_j))\right| = \left|J_j(x^N(t_j))\right| \le \sum_{k=1}^{N}\left|J_j(x^N(t_j)) - J_j(x^{N-1}(t_j))\right| + \left|J_j(0)\right| \le N + \left|J_j(0)\right| := U_1,$$

$$\left|I_j(x(t_j))\right| = \left|I_j(x^N(t_j))\right| \le \sum_{k=1}^{N}\left|I_j(x^N(t_j)) - I_j(x^{N-1}(t_j))\right| + \left|I_j(0)\right| \le N + \left|I_j(0)\right| := U_2,$$

we define $U = \max\{U_1, U_2\}$.

It follows from (3.1) that for $t \in [0, T]$,

$$\begin{aligned} &\|Tx\| = \sup_{t\in R}\left|(Tx)(t)\right| \\ &\le M\lambda C\int_0^T\left|f(s, x(s), x(s-\tau(s))\right|ds + M\left(\sum_{j:t_j\in[t,t+T]}\left|I_j(x(t_j))\right| + \sum_{j:t_j\in[t,t+T]}\left|J_j(x(t_j))\right|\right) \\ &\le M\lambda CTW + 2MpU . \end{aligned}$$

Finally, for $t \in R$, we have

$$\begin{aligned} (Tx)'(t) = &\int_t^{t+T}\left[-q(s)G(t,s) + \frac{\exp\int_t^s p(v)dv}{\exp\int_0^T p(v)dv - 1}\right]\lambda c(s) f(s, x(s), x(s-\tau(s))ds \\ &+ \sum_{j=1}^{p}\left(-q(s)G(t,s) + \frac{\exp\int_t^s p(v)dv}{\exp\int_0^T p(v)dv - 1}\right)J_j(x(t_j)) \\ &+ \sum_{j=1}^{p}\left(p(t_j)\left(-q(t_j)G(t,t_j) + \frac{\exp\int_t^{t_j} p(v)dv}{\exp\int_0^T p(v)dv - 1}\right) + \frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv - 1}q(t)\right)I_j(x(t_j)). \end{aligned}$$

(3.3)

Combine (3.1)-(3.3) and Corollary 2.1, we obtain

$$\left|\frac{d}{dt}(Tx)(t)\right| = \sup_{t\in R}\left|(T_j x)'(t)\right|$$

$$\le \int_t^{t+T}\left|\lambda c(s) f(s, x(s), x(s-\tau(s)))\right|\left|-q(s)G(t,s)+\frac{\exp\int_t^s p(v)dv}{\exp\int_0^T p(v)dv-1}\right|ds$$

$$+\sum_{j=1}^{p}\left|-q(s)G(t,s)+\frac{\exp\int_t^s p(v)dv}{\exp\int_0^T p(v)dv-1}\right|\left|J_j(x(t_j))\right|$$

$$+\sum_{j=1}^{p}\left(\left|-q(t_j)p(t_j)G(t,t_j)\right|+\left|\frac{\exp\int_t^{t_j} p(v)dv}{\exp\int_0^T p(v)dv-1}p(t_j)\right|+\left|\frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv-1}q(t)\right|\right)\left|I_j(x(t_j))\right|$$

$$\le\left(\lambda C\int_t^{t+T}\left|f(s,x(s),x(s-\tau(s)))\right|+\sum_{j=1}^{p}\left|J_j(x(t_j))\right|+\sum_{j=1}^{p}\left|I_j(x(t_j))\right|\left|p(t_j)\right|\right)ds\left((M\left\|Q\right\|+\frac{e^m}{e^l-1})\right)$$

$$+\sum_{j=1}^{p}\left|\frac{\exp\int_t^{t_j} q(v)dv}{\exp\int_0^T q(v)dv-1}q(t)\right|\left|I_j(x(t_j))\right|$$

$$\le \lambda C(TW+U+PU)(M\left\|Q\right\|+\frac{e^m}{e^l-1})+\frac{e^m}{e^l-1}\left\|Q\right\|U,$$

where $\left\|Q\right\| = \max_{0\le t\le T}\left|q(t)\right|$, $\left\|P\right\| = \max_{0\le t\le T}\left|p(t)\right|$.

Hence $\{Tx : x\in K, \left\|x\right\|\le \mu\}$ is a family of uniformly bounded and equicontinuous functions on $[0,T]$. By a theorem of Ascoli-Arzela, the function T is completely continuous.

Theorem 3.1. Suppose that (H1)-(H3), (2.1) and (2.5) and that there are positive constants R_1 and R_2 with $R_1 < R_2$ such that

$$\sup_{\left\|\phi\right\|=R_1,\phi\in K}\int_0^T\left|f(s,\phi(s),\phi(s-\tau(s)))\right| = P_1, \tag{3.4}$$

$\sup_{\left\|\phi\right\|=R_1,\phi\in K}\left|I_j(\phi(t_j))\right| = I_1$,

and

$$\inf_{\left\|\phi\right\|=R_2,\phi\in K}\int_0^T\left|f(s,\phi(s),\phi(s-\tau(s)))\right| = P_2, \tag{3.5}$$

$\inf_{\left\|\phi\right\|=R_2,\phi\in K}\left|I_j(\phi(t_j))\right| = I_2$,

for each λ satisfy

$$\frac{R_2}{MCP_2} < \lambda < \frac{R_1}{MCP_1}. \tag{3.6}$$

Then (1.2) has a positive T -periodic solution x with $R_1\le\left\|x\right\|\le R_2$.

Proof. Let $x\in K$ and $\left\|x\right\| = R_1$. By (3.4) and (3.6), we have

$$\left|(Tx)(t)\right| \leq M \int_t^{t+T} \left|\lambda c(s) f(s, x(s), x(s-\tau(s))\right| ds + M \sum_{j:t_j \in [t,t+T]} \left|I_j(x(t_j))\right|$$

$$\leq \lambda M C \int_t^{t+T} \left|f(s, x(s), x(s-\tau(s))\right| ds + M \sum_{j:t_j \in [t,t+T]} \left|I_j(x(t_j))\right|$$

$$< \frac{R_1}{MCP_1} MCP_1 + MpI_1 = R_1$$

for all $t \in [0, T]$. This implies that $\|Tx\| \leq \|x\|$ for $x \in K \cap \partial\Omega_1, \Omega_1 = \{x \in X, \|x\| < R_1\}$.

If $x \in K$ and $\|x\| = R_2$. By (3.5) and (3.6), we have

$$\left|(Tx)(t)\right| \geq N \int_t^{t+T} \left|\lambda C(s) f(s, x(s), x(s-\tau(s))\right| ds$$

$$\geq \lambda N C \int_t^{t+T} \left|f(s, x(s), x(s-\tau(s))\right| ds$$

$$> \frac{R_2}{NCP_2} NC \int_t^{t+T} \left|f(s, x(s), x(s-\tau(s))\right| ds \geq R_2$$

for all $t \in [0, T]$. Thus, $\|Tx\| \geq \|x\|$ for $x \in K \cap \partial\Omega_2, \Omega_2 = \{x \in X, \|x\| < R_2\}$.

By Krasnoselskii's fixed point theorem, T has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$. It is easy to say that (1.2) has a positive T-periodic solution x with $R_1 \leq \|x\| \leq R_2$. This completes the proof.

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