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Double Slowly Oscillating Sequences and its Relation to Uniform Continuity of two Dimensional Real-Valued Functions

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Abstract: *In this paper, we study the concept of functions preserving slowly oscillating double sequences and obtain some inclusion relations between double oscillating sequences, uniform continuity of two dimensional real-valued functions, sequential continuity and a newly introduced type of continuity of factorable double functions defined on a double subset $A \times A$ of $\boldsymbol{R}^2$ into $\boldsymbol{R}$. A double sequence $x = \{x_{k,l}\}$ of points in $\boldsymbol{R}$ is slowly oscillating if for any given $\varepsilon > 0$, there exist $\alpha = \alpha(\varepsilon) > 0, \delta = \delta(\varepsilon) > 0$, and $N = N(\varepsilon)$ such that $|x_{k,l} -$ xs,t<ε whenever k,l≥Nε andk≤s≤1+αk,l≤t≤1+δl. Suppose that I×I is any two dimensional bounded interval. Then a two dimensional factorable real-valued function is uniformly continuous on $I \times I$ if and only if it is defined on $I \times I$ and preserves factorable double slowly oscillating sequences from $I \times I$. Extensions and variations of the above theorem was also presented.*

Keywords: *Multiple sequences and series, Matrix methods, Continuity and related questions*

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I. Introduction

The concept of convergence of real double sequences was introduced by Pringsheim (1900).

Four years later, Hardy (1904) introduced the notion of regular convergence for double sequences in the sense that double sequence has a limit in Pringsheim's sense and has one sided limits. A considerable number of papers which appeared in recent years study double sequences from various points of view. Some results in the investigation are generalizations of known results concerning simple sequences to certain classes of double sequences, while other results reflect a specific nature of the Pringsheim convergence (e.g., the fact that a double sequence may converge without being bounded). First usage of the slowly oscillating concept of real single sequences goes back to beginning of twentieth century, Hardy (1907) and Landou (1910) while the slowly oscillating concept of real double sequences seems to be first studied in Knopp (1939).

The aim of this paper is to review functions preserving slowly oscillating double sequences and obtain some inclusion relations between double oscillating sequences, uniform continuity of two dimensional real-valued functions and a newly introduced type of continuity of factorable double functions.

II. Preliminaries

Throughout this paper a factorable double sequence will mean a double sequence $x = \{x_{m,n}\}$ of real numbers which can be written in the form that $x_{m,n} = x_m^{m,n}.x_n^{m,n}$ where $x_m^{m,n}$ and $x_n^{m,n}$ are real numbers for each $m, n \in N$; a factorable double function f will mean a real valued function f defined on a double subset $E \times E$ of $\boldsymbol{R}^2$ such that there are $f^{x_1,x_2}(x_1)$, and e $f^{x_1,x_2}(x_2)$ satisfying $f(x_1, x_2)$ = e $f^{x_1,x_2}(x_1).\ f^{x_1,x_2}(x_2)$ for all $(x_1, x_2) \in E \times E$; and a double sequence $(f_{m,n})$ of two dimensional factorable real-valued functions from a double interval $I \times I$ of $\boldsymbol{R}^2$ will be called uniformly P – convergent to a function f, if for each $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $m, n > N$ implies that $|f_{m,n}(x) - f(x)| < \varepsilon$ for all $x \in I \times I$. A (single) sequence $\boldsymbol{x} = (x_k)$ is said to be λ-statistically convergent to a number L if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \varepsilon\}| = 0,$$

where (λ_n) is a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, and $I_n = [n - \lambda_n + 1, n]$ for each $n \in \boldsymbol{N}$.

Definition 2.1 (Pringsheim [1900]): A double sequence $x = \{x_{k,l}\}$ is Cauchy provided that, given an $\varepsilon > 0$ there exists an $N \in \boldsymbol{N}$ such that $|x_{k,l} - x_{s,t}| < \varepsilon$ whenever $k, l, s, t > N$.

Definition 2.2 (Pringsheim [1900]): A double sequence $x = \{x_{k,l}\}$ has a Pringsheim limit L (denoted by P-lim $x = L$) provided that, given an $\varepsilon > 0$ there exists an $N \in \boldsymbol{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. Such an x is described more briefly as "P-convergent".

If for every $M > 0$ there are $n_1, n_2 \in \boldsymbol{N}$ such that $|x_{m,n}| > M$ whenever $m > n_1, n > n_2,$ then $x = \{x_{m,n}\}$ is said to be definitely divergent. This is denoted by P-lim$|x| = \infty$.
A double sequence $x = \{x_{m,n}\}$ is bounded if there is an $M > 0$ such that $|x_{m,n}| < M$ for all $m, n \in N$. Notice that a P-convergent double sequence need not be bounded.

Definition 2.3 (Patterson [2000]): A double sequence $\boldsymbol{y}$ is a double subsequence of $\boldsymbol{x}$ provided that there exist increasing index sequences $\{n_j\}$ and $\{k_j\}$ such that, if $\{x_j\} = \{x_{n_j,k_j}\}$, then y is formed by

$$\begin{matrix} x_1 & x_2 & x_5 & x_{10} \\ x_4 & x_3 & x_6 & - \\ x_9 & x_8 & x_7 & - \\ - & - & - & - \end{matrix}$$

III. Main Results

Definition 3.1 (M´oricz, F. [1994]): A double sequence $x = \{x_{k,l}\}$ of points in $\boldsymbol{R}$ is called slowly oscillating if for any given $\varepsilon > 0$, there exist $\alpha = \alpha(\varepsilon)$, $\delta = \delta(\varepsilon) > 0$, and $N = N(\varepsilon)$ such that $|x_{k,l} - x_{s,t}| < \varepsilon$, if $k, l \geq N(\varepsilon)$ and $k \leq s \leq (1+\alpha)k, l \leq t \leq (1+\delta)l$.

Any Cauchy double sequence is slowly oscillating, so any P-convergent double sequence is. The converse is easily seen to be false as in the single dimensional case as the following example shows.

Example 3.1 Write $m_n = \log n$ for each positive integer n. Then the double sequence defined by

$$\begin{matrix} m_1 & m_2 & m_3 & m_4 & \cdots \\ m_2 & m_2 & m_3 & m_4 & \cdots \\ m_3 & m_3 & m_3 & m_4 & \cdots \\ m_4 & m_4 & m_4 & m_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

is not P-convergent nor Cauchy, however it is a slowly oscillating double sequence.

Theorem 3.1 If a factorable double function f defined on a double subset $A \times A$ of $\boldsymbol{R}^2$ is uniformly continuous, then it preserves factorable slowly oscillating double sequences from
$A \times A$.

Proof. Suppose that f is uniformly continuous, and let

$$\begin{matrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\ x_{3,1} & x_{3,2} & x_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

be any slowly oscillating factorable double sequence. To prove that $\{(x_{n,m})\}$ is slowly oscillating, take any $\varepsilon > 0$. Uniform continuity of f implies that there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$ for $x, y \in \boldsymbol{A} \times \boldsymbol{A}$ where the absolute value in the latter in $\boldsymbol{R}^2$. Since $\{x_{n,m}\}$ is slowly oscillating for this δ, there exist $\alpha_1 = \alpha_1(\delta) > 0, \delta_1 = \delta_1(\delta) > 0$ and $N = N(\delta)$ such that t $|x_{k,l} - x_{s,t}| < \delta$, if $k, l \geq N(\delta)$ and $k \leq s \leq (1+\alpha_1)k, l \leq t \leq (1+\delta_1)l$. Hence $|x_{k,l} - x_{s,t}| < \delta$, if $k, l \geq N(\delta)$ and $k \leq s \leq (1+\alpha_1)k, l \leq t \leq (1+$ $\delta 1l$. It follows from this that {fxn,m} is slowly oscillating. This completes the proof of the theorem. ∎

Theorem 3.2 If a factorable double function f defined on a double subset $A \times A$ of $\boldsymbol{R}^2$ preserves factorable slowly oscillating double sequences from $A \times A$, then it preserves factorable P-convergent double sequences from $A \times A$.

Proof. Suppose that f preserves factorable slowly oscillating double sequences from $A \times A$. Let

$$\begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

be any P-convergent factorable double sequence with P-limit L. Then the sequence

$$\begin{matrix} a_{1,1}L & a_{1,2}L & a_{1,3}L & \cdots & & & \\ L & L & L & L & L & L & \cdots \\ a_{2,1}L & a_{2,2}L & a_{2,3}L & \cdots & & & \\ L & L & L & L & L & L & \cdots \\ a_{3,1}L & a_{3,2}L & a_{3,3}L & \cdots & & & \\ L & L & L & L & L & L & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

is also P-convergent with P-limit L. Since any P-convergent double sequence is slowly oscillating, this sequence is slowly oscillating. So the transformed sequence of the sequence is slowly oscillating. Thus it follows that

$$\begin{matrix} f(a_{1,1})\,f(L) & f(a_{1,2})\,f(L) & f(a_{1,3})\,f(L) & \ldots & & \\ f(L) & f(L) & f(L) & f(L) & f(L) & f(L)\ \ldots \\ f(a_{2,1})\,f(L) & f(a_{2,2})\,f(L) & f(a_{2,3})\,f(L) & \ldots & & \\ f(L) & f(L) & f(L) & f(L) & f(L) & f(L)\ \ldots \\ f(a_{3,1})\,f(L) & f(a_{3,2})\,f(L) & f(a_{3,3})\,f(L) & \ldots & & \\ f(L) & f(L) & f(L) & f(L) & f(L) & f(L)\ \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix}$$

is a slowly oscillating double sequence. Hence

$$\begin{matrix} f(a_{1,1}) - f(L) & f(a_{1,2}) - f(L) & f(a_{1,3}) - f(L) & \ldots \\ f(a_{2,1}) - f(L) & f(a_{2,2}) - f(L) & f(a_{2,3}) - f(L) & \ldots \\ f(a_{3,1}) - f(L) & f(a_{3,2}) - f(L) & f(a_{3,3}) - f(L) & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

a P-convergent factorable double sequence with P-limit 0 This implies that the transformed double sequence

$$\begin{matrix} f(a_{1,1}) & f(a_{1,2}) & f(a_{1,3}) & \ldots \\ f(a_{2,1}) & f(a_{2,2}) & f(a_{2,3}) & \ldots \\ f(a_{3,1}) & f(a_{3,2}) & f(a_{3,3}) & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

is P-convergent with P-limit $f(L)$. This completes the proof of the theorem. ■

Corollary 3.1 If a factorable double function f defined on a double subset $A \times A$ of $\boldsymbol{R}^2$ preserves factorable slowly oscillating double sequences from $A \times A$, then it preserves $\curlywedge$-statistically convergent (single) sequences from $A \times A$.

Proof. The proof follows from the regularity and subsequentiality of $\curlywedge$-statistically sequential method so is omitted (see [4]). ■

Theorem 3.3 Suppose that $A \times A$ is a bounded subset of $\boldsymbol{R}^2$. A two dimensional factorable real-valued function is uniformly continuous on $A \times A$ if and only if it preserves factorable slowly oscillating double sequences from $A \times A$.

Proof. It immediately follows from Theorem 3.1 that two dimensional uniformly continuous functions preserve slowly oscillating sequences. Conversely, suppose that f defined on $A \times A$ is not uniformly continuous. Then there exists an $\varepsilon > 0$ such that for any $\delta > 0$ there exist $(a,b), (\bar{a}, \bar{b}) \in A \times A$ with $\sqrt{(a-\bar{a})^2 + (b-b)^2} < \delta$ but $|f(a,b) - f(\bar{a},b)| \geq \varepsilon$, $|f(a,b) - f(a,\bar{b})| \geq \varepsilon$, and $|f(a,b) - f(\bar{a},\bar{b})| \geq \varepsilon$, respectively. Thus for each positive integer n we can choose $(a_n, b_n), (\bar{a}_n, \bar{b}_n) \in A \times A$ with $\sqrt{(a_n - \bar{a}_n)^2 + (b_n - b_n)^2} < \frac{1}{n}$ but $|f(a_n, b_n) - f(\bar{a}_n, b)| \geq \varepsilon$, $|f(a_n, b_n) - f(a_n, \bar{b}_n)| \geq \varepsilon$, and $|f(a_n, b_n) - f(\bar{a}_n, \bar{b}_n)| \geq \varepsilon$. Then since $A \times A$ is bounded there exists a slowly oscillating double subsequence of the double sequence $\{a_n, b_n\}$ by a simple extension of Bolzano-Weierstrass theorem, $\{a_{n_k}, b_{n_k}\}$ say. Thus the corresponding double sequence$\{\bar{a}_{n_k}, \bar{b}_{n_k}\}$ has a slowly oscillating double subsequence, say$\{\bar{a}_{n_{k_m}}, \bar{b}_{n_{k_m}}\}$. It is easy to see that $\{\bar{a}_{n_{k_m}}, \bar{b}_{n_{k_m}}\}$ is a slowly oscillating sequence. Since f preserves slowly oscillating double sequences by the hypothesis, $\{f(a_{n_{k_m}}, b_{n_{k_m}})\}$ and $\{f(\bar{a}_{n_{k_m}}, \bar{b}_{n_{k_m}})\}$ and are slowly oscillating. This is impossible. This contradiction completes the proof of the theorem. ■

It is well known that uniform limit of a sequence of continuous functions is continuous. This is also true for two dimensional factorable real-valued functions that preserve slowly oscillating double sequences, i.e. uniform limit of a sequence of two dimensional factorable real-valued functions preserving slowly oscillating double sequences from $A \times A$ of $\boldsymbol{R^2}$also preserves slowly oscillating double sequences from $A \times A$.

Theorem 3.4 If (f_n) is a sequence of two dimensional factorable real-valued functions preserving slowly oscillating double sequences from a double interval $I \times I$ of $\boldsymbol{R^2}$ and (f_n) is uniformly convergent to a function f, then f preserves slowly oscillating double sequences from$I \times I$.

Proof. Let (x_{nk}) be a slowly oscillating double sequence and $\varepsilon > 0$. Then there exists a positive integer N such that $|f_n(a,b) - f(\bar{a}, \bar{b})| < \frac{\varepsilon}{3}$ for all $(a,b), (\bar{a}, \bar{b}) \in I \times I$ whenever $n \geq N$. As $f_N$preserves slowly oscillating double sequences from$I \times I$, there exist a $\delta > 0$ and a positive integer$N_1 = N_1(\varepsilon)$, greater than$N$, such that

$$|f_N(x_{k,l}) - f_N(x_{s,t})| < \frac{\varepsilon}{3},$$

for $n \geq N_1$ and $k \leq s \leq (1+\delta)k, l \leq t \leq (1+\delta)l$. Now for $n \geq N_1$ and $k \leq s \leq (1+\delta)k, l \leq t \leq (1+\delta)l$. Thus for $n \geq N_1$ and $k \leq s \leq (1+\delta)k, l \leq t \leq (1+\delta)l$ we have

$$|f(x_{k,l}) - f(x_{s,t})| \leq |f(x_{k,l}) - f_N(x_{k,l})| + |f_N(x_{k,l}) - f_N(x_{s,t})|$$

$$+ |f_N(x_{s,t}) - f(x_{s,t})| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

This completes the proof of the theorem. ■

Theorem 3.5 If $(f_{m,n})$ is a double sequence of two dimensional factorable real-valued functions preserving slowly oscillating double sequences from a double interval $I \times I$ of $\boldsymbol{R^2}$ and $(f_{m,n})$ is uniformly P-convergent to a function f, then f preserves slowly oscillating double sequences from $I \times I$.

The proof is similar to the last theorem and as of such it is omitted.

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