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Global Studies of the Solutions of some Difference Equations

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Abstract
Our aim in this paper is to investigate the boundedness, global attractively, locally asymptotically stability of the positive solution of the difference equations of form

$$x_{n+1} = \alpha x_{n-\ell} + \frac{b x_n^q}{1 + x_{n-k}^p}, \quad n = 0, 1, \ldots,$$

where $a, b, p, q \in [0, \infty)$ *and* $\ell, k \in \{0, 1\}$ *and the initial conditions* x_{-1}, x_0 *are arbitrary positive numbers.*

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I. Introduction

Recursive Sequence are recognized as description of observed development of a phenomenon, where the dominant part of estimations of a time-evolving variable are discrete. The overall developing consideration paid to the examination of different aspects, behaviors and dynamics of difference equations, for example stability, periodicity and boundedness character, is planned. See for cases [3]-[5] and [7]-[12]. Recursive sequences are also often called difference equations, which are very important in mathematical theory and application. See [10], [15] and [17]. For some more examples:

In [1] Amleh et al obtained important results for the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}.$$

In [6] Elabbasy et al. investigated the global stability character, boundedness and the periodicity of solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2}}{A x_n + B x_{n-1} + C x_{n-2}}.$$

Yang et al. [18] investigated the invariant intervals, the global attractivity of equilibrium points and the asymptotic behavior of the solutions of the recursive sequence

$$x_{n+1} = \frac{a x_{n-1} + b x_{n-2}}{c + d x_{n-1} x_{n-2}}.$$

Aloqeili [2] obtained the form of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

In [16] Yalcinkaya studied the following nonlinear difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

Our point in this paper is to explain the boundedness, the locally, and the global attractively of the positive solutions of the difference equations

$$x_{n+1} = \alpha x_{n-1} + \frac{bx_n^q}{1+x_{n-k}^p}, \quad n = 0,1,\ldots, \tag{1}$$

where $a,b,p,q \in [0,\infty)$ and $\ell, k \in \{0,1\}$ and the initial conditions x_{-1}, x_0 are arbitrary positive numbers.
Here we recall some notations and results which will be useful in our investigation.
Let J be an interval real numbers and $g: J \times J \to J$ where g is a continuously differentiable function.
Consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}) \quad \text{for all} \quad n \geq 0 \tag{2}$$

with $x_0, x_{-1} \in J$.

The equilibrium point of Eq.(2) is any point $\bar{x}$ that satisfies the condition $\bar{x} = g(\bar{x}, \bar{x})$. That is; the constant sequence $\{x_n\}$ with $x_n = \bar{x}$ for all $n \geq 0$ is a solution of the equation. The linearized equation of Eq.(2) is

$$z_{n+1} = a_1 z_n + a_2 z_{n-1},$$

where $a_1 = \dfrac{\partial g(x_n, x_{n-1})}{\partial x_n}$ and $a_2 = \dfrac{\partial g(x_n, x_{n-1})}{\partial x_{n-1}}$.

Theorem A [14]: A (Linearized Stability)
(a) If both roots of the quadratic equation

$$\lambda^2 - a_1\lambda - a_2 = 0, \tag{3}$$

lie in the open unit disk $|\lambda| < 1,$ then the equilibrium point $\bar{x}$ of Eq.(2) is locally asymptotically stable.

(b) If at least of the roots of Eq.(3) has absolute value greater that one then the equilibrium $\bar{x}$ of Eq.(2) is unstable.

Theorem B [14]: Assume that $a_1, a_2 \in R$. Then

$$|a_1| + |a_2| < 1,$$

is a sufficient condition for the locally stability of Eq.(3).

Theorem C [13]: Let $[a,b]$ be an interval of real numbers and assume that $g:[a,b] \to [a,b]$ is a continuous function satisfying the following properties:
(a) $g(x,y)$ is non-decreasing in $x \in [a,b]$ for each $y \in [a,b]$, and $g(x,y)$ is nonincreasing in $y \in [a,b]$ for each $x \in [a,b]$.
(b) If $(n,N) \in [a,b] \times [a,b]$ is a solution of the system

$$g(m,M) = m, \text{ and } g(M,m) = M.$$

then $m = M$. Then Eq.(2) has a unique equilibrium $\bar{x} \in [a,b]$ and every solution of Eq.(2) converges to $\bar{x}$.

II. Dynamics of Solutions of Eq.(1)

In this study we consider the following cases:

2.1 Case 1: $l = 1, k = 0$ and q =0

In this case Eq.(1) takes the form

$$x_{n+1} = \alpha x_{n-1} + \frac{b}{1+x_n^p}, \quad n = 0,1,\ldots \tag{4}$$

The equilibrium points of Eq.(4) are given by the relation

$$\bar{x} = a\bar{x} + \frac{b}{1+\bar{x}^p}.$$

If $a < 1$ then the unique positive equilibrium point of Eq.(4) is

$$\bar{x} + \bar{x}^{p+1} = \frac{b}{1-a}.$$

Let $f : (0,\infty)^2 \to (0,\infty)$ be a function defined by

$$f(u,v) = av + \frac{b}{1+u^p}.$$

Therefore

$$\frac{\partial f(u,v)}{\partial u} = -\frac{bpv^{p-1}}{(1+v^p)^2} \quad \text{and} \quad \frac{\partial f(u,v)}{\partial v} = a.$$

Set

$$p_1 = -\frac{p}{b}\bar{x}^{p+1}(1-a)^2 \text{ and } p_2 = a.$$

Then the linearized equation of Eq.(4) about $\bar{x}$ is

$$y_{n+1} + \frac{p}{b}\bar{x}^{p+1}(1-a)^2 y_n - ay_{n-1} = 0.$$

Theorem 1 Assume that $a < 1$ and $p > 1.$ Then the following statements are true

(i) If

$$b > \frac{p}{(p-1)^{\frac{1}{p}}}\left(\frac{1-a}{p-1}\right), \tag{5}$$

then the equilibrium point of Eq.(4) is locally asymptotically stable.

(ii) If

$$b < \frac{p}{(p-1)^{\frac{1}{p}}}\left(\frac{1-a}{p-1}\right), \tag{6}$$

then the equilibrium point of Eq.(4) is unstable, in fact is a saddle point.

(iii) If

$$b = \frac{p}{(p-1)^{\frac{1}{p}}}\left(\frac{1-a}{p-1}\right), \tag{7}$$

then the equilibrium point of Eq.(4) is a nonhyperbolic point.

Proof. (i) By Theorem A we get

$$|p_1| + |p_2| = a + \frac{bp\bar{x}^{p-1}}{(1+\bar{x}^p)^2} < 1 \quad \Leftrightarrow \quad a + \frac{p}{b}\bar{x}^{p+1}(1-a) < 1 \Leftrightarrow \bar{x} < \left(\frac{b}{p(1-a)}\right)^{\frac{1}{p+1}}.$$

Let $g(x) = x(1-a) + x^{p+1}(1-a) - b.$ Some simple calculations, using condition (5), show that

$$g\left(\left(\frac{b}{p(1-a)}\right)^{\frac{1}{p+1}}\right)=\left(\frac{b}{p(1-a)}\right)^{\frac{1}{p+1}}(1-a)+\frac{b}{p(1-a)}(1-a)-b<0$$

$$\Leftrightarrow\left(\frac{b}{p}\right)^{\frac{1}{p+1}}(1-a)^{\frac{p}{p+1}}+\frac{b}{p}<b\Leftrightarrow b\geq\frac{p}{(p-1)^{\frac{1}{p}}}\left(\frac{1-a}{p-1}\right).$$

Then since $\lim_{x\to\infty} g(x)=\infty, \overline{x}^{p+1}<\frac{b}{p(1-a)}$. (ii) The condition $p_1^2+4p_2>0$ of Theorem A is always satisfied and so $\overline{x}$ is unstable if $\overline{x}^{p+1}<\frac{b}{p(1-a)}$.

By condition (6), we have

$$g\left(\left(\frac{b}{p(1-a)}\right)^{\frac{1}{p+1}}\right)=\left(\frac{b}{p(1-a)}\right)^{\frac{1}{p+1}}(1-a)+\frac{b}{p(1-a)}(1-a)-b>0.$$

Then since $g(0)<0, \overline{x}^{p+1}<\frac{b}{p(1-a)}$. (iii) The condition $|p_1|=|1-p_2|$ is equivalent to $\overline{x}^{p+1}=\frac{b}{p(1-a)}$. Similarly by condition (7), we have $g\left(\left(\frac{b}{p(1-a)}\right)^{\frac{1}{p+1}}\right)=0$. Then $\overline{x}^{p+1}=\frac{b}{p(1-a)}$. Then the proof is completed.

Theorem 2 If $a<1$ then Eq.(4) is bounded and persists.

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(4) it follows from Eq.(4) that

$$x_{n+1}=ax_{n-1}+\frac{b}{1+x_n^p}<ax_{n-1}+b.$$

Then

$$\lim_{n\to\infty} \mathit{sup}\, x_n\leq\frac{b}{1-a}=M.$$

Then $\{x_n\}_{n=-1}^{\infty}$ is bounded from above. From Eq.(4)

$$x_{n+1}=ax_{n-1}+\frac{b}{1+x_n^p}>\frac{b}{1+x_n^p}\geq\frac{b}{1+\left(\frac{b}{1-a}\right)^p}=m.$$

Then $\{x_n\}$ is bounded from blow too. Then the result is followed. ■

Theorem 3 If a > 1 then Eq(4) has unbounded solutions.

Proof. Let $\{x_n\}$ be a solution of Eq.(4). We obtain from Eq.(4) that

$$x_{n+1} = ax_{n-1} + \frac{b}{1+x_{n-1}^p} > ax_{n-1},$$

that is

$$x_n > ax_{n-1} > a^2 x_{n-1} > \cdots > a^n x_0.$$

It follows that

$$\lim_{n\to\infty} x_n = \infty.$$

Then the proof is completed. ■

Example 1 Figure (1) shows that Eq.(4) has an unbounded solution whenever $x_{-1} = 0.8, x_0 = 3, a = 3, b = 0.5$ and $p = 3$

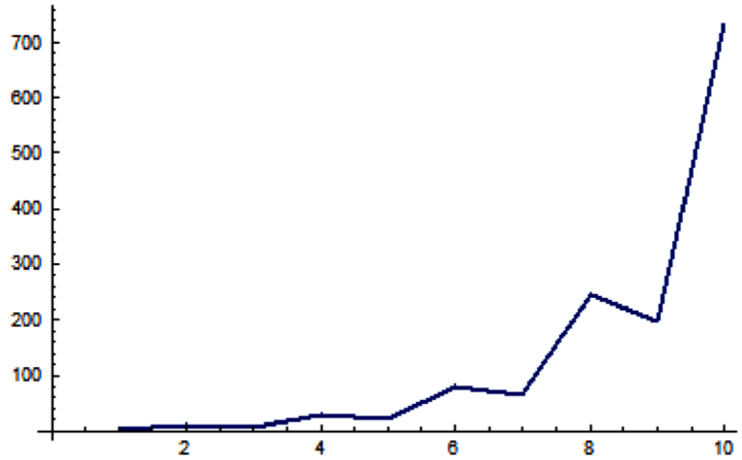

Figure (1)

Theorem 4 If a < 1 then the positive equilibrium point $\overline{x}$ is a global attractor of Eq.(4).

Proof. We can easily see that the function

$$g(u,v) = av + \frac{b}{1+u^p},$$

is increasing in v and decreasing in u. Suppose that (m, M) is a solution of the system

$$M = g(M, m), \text{ and } m = g(m, M).$$

Then from Eq.(4) we can see that

$$M = aM + \frac{b}{1+m^p}, \quad m = am + \frac{b}{1+M^p},$$

or

$$1 - a = \frac{b}{M(1+m^p)} \quad \text{and} \quad 1-a = \frac{b}{m(1+M^p)}.$$

Thus

$$m + mM^p = M + Mm^p.$$

So

$$m = M.$$

It follows by Theorem C that $\overline{x}$ is a global attractor of Eq.(4) and then the proof is complete.

Example 2 Figure (2) shows the global attractivity of the positive equilibrium point $\overline{x}$ of Eq.(4) whenever $x_{-1} = 0.8, x_0 = 3, a = 3, b = 0.5$ and $p = 3$

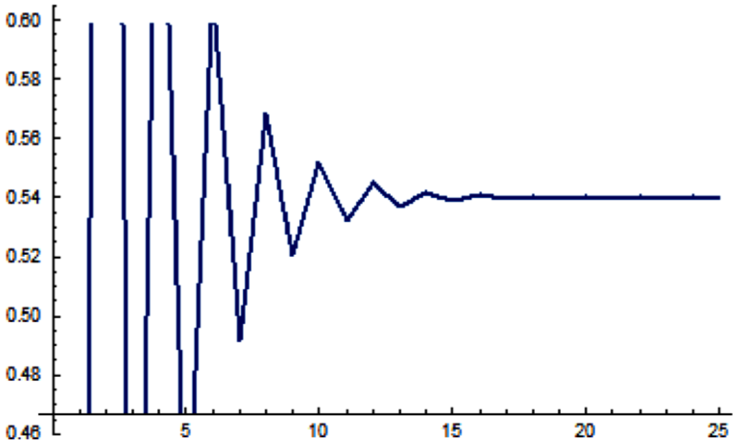

Figure (2)

Theorem 5 Let $0 < p < 1$ and $a < 1$. Then every positive solution of Eq.(4) converges to $\overline{x}$.

Proof. It was shown by Theorem 2 that every solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(4) is bounded. Thus it follows that $\lim_{n\to\infty} \mathit{inf}\, x_n = \ell,$ and $\lim \mathit{sup}\, x_n = L$.

As the sake of contradiction assume that l < L: We see from Eq.(4) that

$$a\ell + \frac{b}{1+L^p} \le \ell < L \le aL + \frac{b}{1+\ell^p},$$

which implies that

$$\frac{b}{1-a} - 1 \le \ell L^p < L\ell^p \le \frac{b}{1-a} - L,$$

i.e.

$$L - \ell \le 0 \quad (L \quad l) < 0,$$

which gives a contradiction. Hence the result follows. ■

Theorem 6 Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(4) which consists of at least two semi-cycles. Then $\{x_n\}_{n=-1}^{\infty}$ is oscillatory.

Proof. It suffices to consider the following two cases:

(I) Suppose $x_{N-1} < \overline{x} < x_N$. Then

$$x_{N+1} = ax_{N-1} + \frac{b}{1+x_N^p} < a\overline{x} + \frac{b}{1+\overline{x}^p} = \overline{x},$$

and

$$x_{N+2} = ax_N + \frac{b}{1+x_{N+1}^p} > a\overline{x} + \frac{b}{1+\overline{x}^p} = \overline{x}.$$

(II) Suppose $x_N < \overline{x} < x_{N-1}$. Then

$$x_{N+1} = ax_{N-1} + \frac{b}{1+x_N^p} > a\overline{x} + \frac{b}{1+\overline{x}^p} = \overline{x} \quad \text{and}$$

$$x_{N+2} = ax_N + \frac{b}{1+x_{N+1}^p} < a\overline{x} + \frac{b}{1+\overline{x}^p} = \overline{x}.$$

The proof is completed. ■

2.2 Case 2: $l = 0, k = 1$ and $q = 1$

In this case Eq.(1) takes the form

$$x_{n+1} = \alpha x_n + \frac{bx_n}{1+x_{n-1}^p}, \quad n = 0, 1, \ldots \tag{8}$$

The equilibrium points of Eq.(8) are given by the condition

$$\overline{x} = \alpha\overline{x} + \frac{\beta\overline{x}}{1+\overline{x}^p}$$

If $\beta + \alpha > 1$ and $\alpha < 1$ then Eq.(8) has the equilibrium points $\overline{x} = 0$ and $\overline{x} = \sqrt[p]{\frac{\beta}{1-\alpha} - 1}$. Now let $g : (0,\infty)^2 \to (0,\infty)$ be a function defined by

$$g(u,v) = \alpha u + \frac{\beta u}{1+v^p}.$$

Therefore

$$\frac{\partial g(u,v)}{\partial u} = \alpha + \frac{\beta}{1+v^p} \quad \text{and} \quad \frac{\partial g(u,v)}{\partial v} = -\frac{\beta p u v^{p-1}}{(1+v^p)^2}.$$

Then we see that

$$\frac{\partial g(\overline{x},\overline{x})}{\partial u} = \alpha + \frac{\beta}{1+\overline{x}^p} = a_1 \quad \text{and} \quad \frac{\partial g(\overline{x},\overline{x})}{\partial v} = -\frac{\beta p\, \overline{x}^p}{(1+\overline{x}^p)^2} = a_2.$$

Then the linearized equation of Eq.(8) about $\overline{x}$ is

$$z_{n+1} - \left(\alpha + \frac{\beta}{1+\overline{x}^p}\right) z_n + \left(\frac{\beta p \overline{x}^p}{(1+\overline{x}^p)^2}\right) z_{n-1} = 0.$$

Theorem 7 The following statements are true:

(i) The equilibrium point $\overline{x} = 0$ of Eq.(8) is locally asymptotically stable if $\alpha + \beta < 1$.

(ii) The equilibrium point $\overline{x} = 0$ of Eq.(8) is stable if $\alpha + \beta = 1$.

(iii) The equilibrium point $\overline{x} = 0$ of Eq.(8) is unstable if $\alpha + \beta > 1$.

Proof. Since the linearized equation of Eq.(8) about the equilibrium point $\overline{x} = 0$ can be written in the following form

$$z_{n+1} = (\alpha + \beta) z_n, \quad n = 0, 1, \ldots,$$

so the characteristic equation of Eq.(8) about $\overline{x} = 0$ is

$$\lambda^2 - (\alpha + \beta)\lambda = 0.$$

Then the proof of (i) and (ii) follows promptly from Theorem A. (iii) Let $\grave{o} > 0$ and let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(8) such that

$$|x_{-1}| + |x_0| < \delta. \tag{9}$$

It suffices to show that $|x_1| < \grave{o}$.

Now

$$0 < |x_1| = \left|\alpha x_0 + \frac{\beta x_0}{1 + x_{-1}^p}\right| < |(\alpha+\beta) x_0| = |x_0| < \delta.$$

Chose $\delta = \grave{o}$= then $|x_1| < \grave{o}$ whatever (9) holds. At that point the outcome follows by induction. ■

Theorem 8 Assume that $\alpha + \beta \le 1$. Then the zero equilibrium point of Eq.(8) is globally asymptotically stable.

Proof. We know by Theorem 7 ((i) and (ii)) that $\overline{x} = 0$ is locally asymptotically stable equilibrium point of Eq.(8) if + 1, and so it suffices to show that $\overline{x} = 0$ is global attractor of Eq.(8) as follows

$$0 \le x_{n+1} = \alpha x_n + \frac{\beta x_n}{1 + x_{n-1}^p} \le x_n,$$

Then the sequence $\{x_n\}$ is decreasing, and bounded from below by zero and since there is a unique equilibrium point $\overline{x} = 0$ in this case, then $\lim_{n\to\infty} x_n = 0$. Then the proof is completed. ■

Example 3 Figure (3) shows the global attractivity of the equilibrium point $\overline{x}$ of Eq.(8) whenever $x_{-1} = 0.87, x_0 = 0.9539, \alpha = 0.2, \beta = 0.5$ and $p = 0.442$.

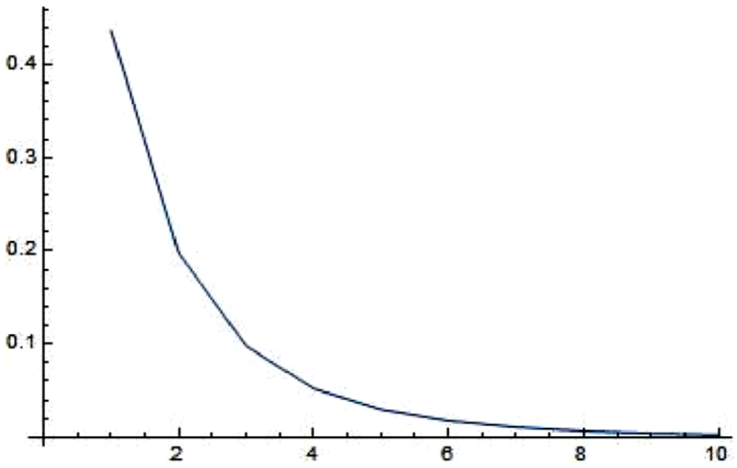

Figure (3)

Lemma 9 Assume that $\alpha + \beta \le 1$, then every solution of Eq.(8) is bounded.

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(8). It follows from Eq.(8) that

$$x_{n+1} = \alpha x_n + \frac{\beta x_n}{1 + x_{n-1}^p} \le (\alpha+\beta) x_n.$$

Then in view of the proof of Theorem 8, we have

$$x_1 \le x_0.$$

Similarly it is easy to see that

$$\cdots \le x_n \le \cdots \le x_2 \le x_1 \le x_{0.},$$

so every solution of Eq.(8) is bounded from above. ■

Lemma 10 If $\alpha > 1,$ then the Eq.(8) has unbounded solutions.

Proof. Consider $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq.(8), then it follows that

$$x_{n+1} > \alpha x_n,$$

That is

$$x_n > \alpha x_{n-1} > \cdots > \alpha^n x_0$$

$$\lim_{n\to\infty} x_n = \infty.$$

Then, the proof is completed. ■

Example 4 Figure (4) shows that Eq.(8) has an unbounded solution with the values $x_{-1} = 1.537, x_0 = 3.019, \alpha = 5, \beta = 0.9,$ and $p = 2.$

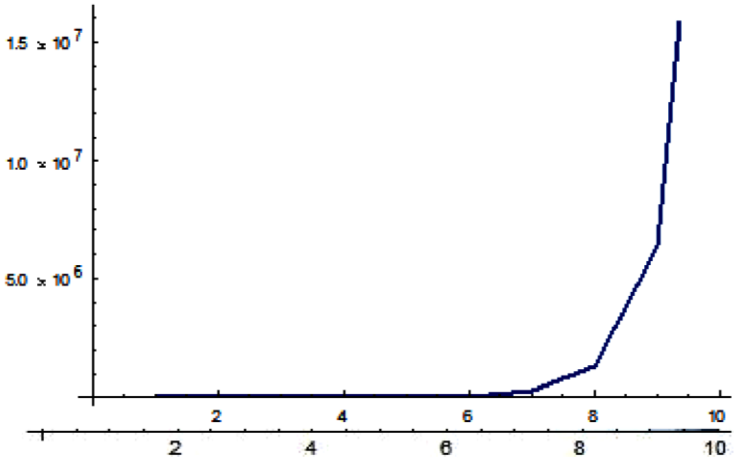

Figure (4)

Theorem 11 If $\alpha < 1$ < 1 then the positive equilibrium point x is a global attractor of Eq.(8).

Proof. We can easily see that the function

$$g(u,v) = \alpha u + \frac{\beta u}{1+v^p}$$

is increasing in u and decreasing in v. Suppose that (m, M) is a solution of the system

$$M = g(M,m), \text{ and } m = g(m,M).$$

Then from Eq.(8) we can see that

$$M = \alpha M + \frac{\beta M}{1+m^p}, \text{ and } m = \alpha m + \frac{\beta m}{1+M^p},$$

or

$$1-\alpha = \frac{\beta}{1+m^p} \text{ and } 1-\alpha = \frac{\beta}{1+M^P}.$$

We obtain from this

$$m = M.$$

It follows by Theorem C that $\overline{x}$ is a global attractor of Eq.(4) and then the proof is complete. ■

Example 5 Figure (5) shows the global attractivity of the equilibrium point $\overline{x}$ of Eq.(8) whenever $x_{-1} = 6.012, x_0 = 2.34, \alpha = 0.5, \beta = 9,$ and $p = 2.$

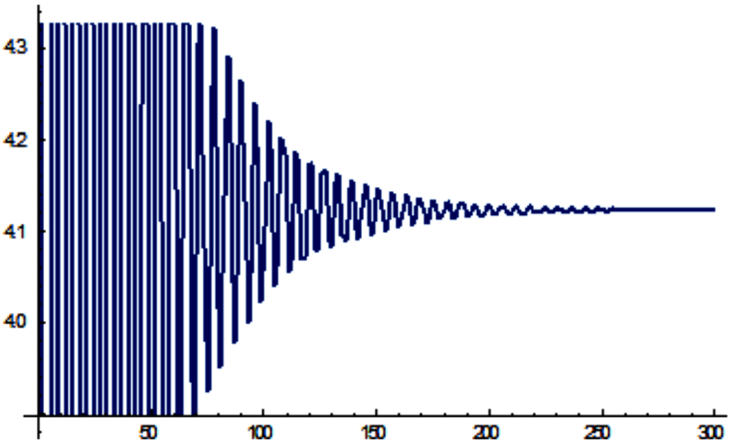

Figure (5)

Theorem 12 Let $\{x_n\}_{n=-1}^{\infty}$ be a nontrivial solution of Eq.(8). At that point the accompanying articulations are valid: (i) Every semicycle, with the exception of maybe for the first, has no less than two terms. (ii) The extreme in each semicycle occur at either the first term or the second. Besides after the main, the remaining terms in a positive semicycle are strictly decreasing and in a negative semicycle are strictly increasing.

Proof. We present the proofs for positive semicycles only. The proof for negative semicycles are similar and will be omitted. (i) Assume that for some $n \geq 0$,

$$x_{n-1} < \bar{x} \text{ and } x_n \geq \bar{x}.$$

Then

$$x_{n+1} = \alpha x_n + \frac{\beta x_n}{1 + x_{n-1}^p} > \alpha \bar{x} + \frac{\beta \bar{x}}{1 + \bar{x}^p} = \bar{x}.$$

(ii) Assume that for some $n \geq 0$, the first two terms in a positive semicycle are x_n and x_{n+1}. Then

$$x_n \geq \bar{x}, \quad x_{n+1} > \bar{x}$$

and

$$\frac{x_{n+2}}{x_{n+1}} = \frac{1}{x_{n+1}}\left[\alpha x_{n+1} + \frac{\beta x_{n+1}}{1 + x_n^p}\right] = \alpha + \frac{\beta}{1 + x_n^p} < \alpha + \frac{\beta}{1 + \bar{x}^p} = 1.:$$

Thus the proof is complete. ■

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