IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 16, Issue 6 Ser. III (Nov. – Dec. 2020), PP 48-54
www.iosrjournals.org

On the Qualitative Study of some Difference Equations

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Abstract
In this paper, we study some qualitative properties of the solutions for the following difference equation

$$y_{n+1} = \frac{\alpha + \alpha_0 y_n^r + \alpha_1 y_{n-1}^r + \cdots + \alpha_k y_{n-k}^r}{\beta + \beta_0 y_n^r + \beta_1 y_{n-1}^r + \cdots + \beta_k y_{n-k}^r}, \quad n \geq 0, \qquad (*)$$

where $r, \alpha, \alpha_0, \alpha_1, \ldots, \alpha_k, \beta, \beta_0, \beta_1, \ldots, \beta_k \in (0, \infty)$ *and k isanonnegativeinteger number. We find the equilibrium points for Eq.(*) and then classify these points in terms of local stability or not. We investigate the boundedness and the global stability of the considered equation. Also we study the existence of periodic solutions ofEq.(*).*

Keywords: *boundedness, local stability, periodicity, global stability.*
MathematicsSubjectClassification:39A10

Date of Submission: 01-12-2020 Date of Acceptance: 15-12-2020

I. Introduction

Inthispaperwestudytheboundednessandtheglobalattractvityofthesolutionsof the differenceequation

$$y_{n+1} = \frac{\alpha + \alpha_0 y_n^r + \alpha_1 y_{n-1}^r + \cdots + \alpha_k y_{n-k}^r}{\beta + \beta_0 y_n^r + \beta_1 y_{n-1}^r + \cdots + \beta_k y_{n-k}^r}, \quad n \geq 0, \qquad (1)$$

where $r, \alpha, \alpha_0, \alpha_1, \ldots, \alpha_k, \beta, \beta_0, \beta_1, \ldots, \beta_k \in (0, \infty)$ and kisanonnegativeintegernumber. Also we investigate the periodicity character of the solutions ofEq.(1).

The study of the properties of the solutions for the difference equations such as periodicity, global stability and boundedness has been discussed by many authors. See, for examples the following papers and the references therein:

Cinar [1] studied the properties of the positive solution for the equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \qquad n = 0,1, \ldots .$$

Yang *et al* [17] investigated the qualitative behavior of the recursive sequence

$$x_{n+1} = \frac{a x_{n-1} + b x_{n-2}}{c + d x_{n-1} x_{n-2}}, \qquad n = 0,1, \ldots .$$

Li *etal*[13]studiedtheglobalasymptoticofthefollowingnonlineardifferenceequation

$$x_{n+1} = \frac{a + x_{n-1} + x_{n-2} + x_{n-3} + x_{n-1}x_{n-2}x_{n-3}}{a + x_{n-1}x_{n-2} + x_{n-1}x_{n-3} + x_{n-2}x_{n-3} + 1}, \quad n = 0,1, \ldots ,$$

with $a \geq 0$.

Kulenovic and Ladas [10] presented a summary of a recent work and a large of openproblemsandconjecturesonthethirdorderrationalrecursivesequenceofthe form

$$x_{n+1}=\frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}+Dx_{n-2}},\qquad n=0,1,\ldots$$

In [15],Xianyi and Deming proved that the positive equilibrium of the difference equations

$$x_{n+1}=\frac{x_nx_{n-1}+x_{n-2}+a}{x_n+x_{n-1}x_{n-2}+a},\quad \text{and}\quad x_{n+1}=\frac{x_{n-1}+x_nx_{n-2}+a}{x_nx_{n-1}+x_{n-2}+a},\quad n=0,1,\ldots,$$

withpositive initialvaluesx_{-2}, x_{-1}, $x_0$and a nonnegativeparametera,isgloballyasymptotically stable.
Simsek *et al* [14] obtained the solutions of the following difference equation

$$x_{n+1}=\frac{x_{n-3}}{1+x_{n-1}},\qquad n=0,1,\ldots.$$

In [5], Yalçınkaya et al. investigated the dynamics of the difference equation

$$x_{n+1}=\frac{ax_{n-k}}{b+cx_n^px_{n-1}},\quad n=0,1,\ldots.$$

For more related results see [2-5], [7-8] and[11-12].
Inthefollowingwepresentsomedefinitionsandsomeknownresultsthatwillbe useful in the investigation ofEq.(1).
Now consider the difference equation

$$x_{n+1}=f(x_n,x_{n-1},\ldots,x_{n-k}),\qquad n=0,1,\ldots,\qquad (2)$$

with$x_{-k}, x_{-k+1},\ldots, x_0\in I$, for some interval$I\subset R$ andf: $I^{k+1}\to R$ be a continuous function.

Defintion 1: Eq.(2) is saidto be permanent if there exist numbers m and M with $0<m<M<\infty$ such that for any initial conditions $x_{-k}, x_{-k+1},\ldots, x_0\in(0,\infty)$ there exists sa positive integer N which depends on these initial conditions such that

$$m<x_n<M\qquad \text{for all}\quad n\geq N.$$

Defintion 2:

(i) The equilibrium point $\bar{x}$ of Eq.(2) is locally stable if for every$\varepsilon>0$ there exists $\delta>0$ such that for all $x_{-k}, x_{-k+1},\ldots,$ $x_0\in I$ with $|x_{-k}-\bar{x}|+|x_{-k+1}-\bar{x}|+\ldots+|x_0-\bar{x}|<\delta$ we have $|x_n-\bar{x}|<\varepsilon$ for all $n\geq -k$.
(ii) The equilibrium point $\bar{x}$ of Eq.(2) is globally asymptotically stable if x is locally stable and there exists $\lambda>0$,such that for all $x_{-k}, x_{-k+1},\ldots, x_0\in I$ with$|x_{-k}-\bar{x}|+|x_{-k+1}-\bar{x}|+\ldots+|x_0-\bar{x}|<\lambda$,we have$\lim_{n\to\infty}x_n=\bar{x}$.
(iii) The equilibrium point $\bar{x}$ of Eq.(2) is global attractor if for all $x_{-k}, x_{-k+1},\ldots, x_0\in I$, we have$\lim_{n\to\infty}x_n=\bar{x}$.
(iv) The equilibrium point $\bar{x}$ of Eq.(2) is globally asymptotically stable if $\bar{x}$ is locally stable, and $\bar{x}$ is also a global attractor of Eq.(2).
(v) The equilibrium point $\bar{x}$ of Eq.(2) is unstable if $\bar{x}$ is not locally stable.
Observe that, the linearized equation of Eq.(2) about the equilibrium point $\bar{x}$ is

$$y=p_0y_n+p_1y_{n-1}+\cdots+p_ky_{n-k},(3)$$

where

$$p_0=\frac{\partial f(\bar{x},\bar{x},\ldots,\bar{x})}{\partial x_n},p_1=\frac{\partial f(\bar{x},\bar{x},\ldots,\bar{x})}{\partial x_{n-1}},\ldots,p_k=\frac{\partial f(\bar{x},\bar{x},\ldots,\bar{x})}{\partial x_{n-k}},$$

and the characteristic equation of Eq.(3) is

$$\lambda^{k+1}-\sum_{i=0}^{k}p_i\lambda^{k-i}=0.$$

Theorem A [9]: Assume that $p_1,p_2,\ldots,p_k\in R$. Then the condition;

$$\sum_{i=0}^{k}|p_i|<1$$

is a sufficient condition for the locally stability of Eq.(3).

Theorem B [6]: Let J be some interval of real numbers, $f\in C[J^{v+1},J]$ and let $\{x_n\}_{n=-v}^{\infty}$ be a bounded solution of the difference equation

$$x_{n+1}=f(x_n,x_{n-1},\ldots,x_{n-v}),\qquad n=0,1,\ldots,\qquad (4)$$

with

$$I=\lim_{n\to\infty}inf\,x_n,\qquad S=\lim_{n\to\infty}sup\,x_n,\qquad I,S\in J.$$

Then there exist two solutions $\{I_n\}_{n=-\infty}^{\infty}$and $\{S_n\}_{n=-\infty}^{\infty}$ of Eq.(4) with

$$I_0=I,\quad S_0=S,\quad I_nS_n\in[I,s]\text{for all } n\in Z.$$

and such that for everyN $\in Z$, $I_N$and$S_N$are limit points of $\{x_n\}_{n=-v}^{\infty}$.
Furthermore, for every $m\leq -\upsilon$,there exist two subsequences $\{x_{r_n}\}$and $\{x_{l_n}\}$of the solution $\{x_n\}_{n=-v}^{\infty}$ such that the following are true

$$\lim_{n\to\infty}x_{r_n}=I_N \text{ and } \lim_{n\to\infty}x_{l_n}=S_N\text{for every}\quad N\geq m.$$

The solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ are called **full limiting sequences** of Eq.(4).

II. Boundedness for the solutions of Eq.(1)

In this section we study the boundedness of the solutions for Eq.(1).

Theorem1: Every solution of Eq.(1) is bounded and persists.

Proof Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1) and assume that $\alpha^* = \min\{\alpha,\alpha_0,\alpha_1,\dots,\alpha_k\}$, $\alpha^{**} = \max\{\alpha,\alpha_0,\alpha_1,\dots,\alpha_k\}$, $\beta^* = \min\{\beta,\beta_0,\beta_1,\dots,\beta_k\}$ and $\beta^{**} = \max\{\beta,\beta_0,\beta_1,\dots,\beta_k\}$. It follows from Eq.(1) that

$$y_{n+1} = \frac{\alpha + \alpha_0 y_n^r + \alpha_1 y_{n-1}^r + \cdots + \alpha_k y_{n-k}^r}{\beta + \beta_0 y_n^r + \beta_1 y_{n-1}^r + \cdots + \beta_k y_{n-k}^r}$$
$$\leq \frac{\max\{\alpha,\alpha_0,\alpha_1,\dots,\alpha_k\}\,(1 + y_n^r + y_{n-1}^r + \cdots + y_{n-k}^r)}{\min\{\beta,\beta_0,\beta_1,\dots,\beta_k\}\,(1 + y_n^r + y_{n-1}^r + \cdots + y_{n-k}^r)}$$
$$= \frac{\max\{\alpha,\alpha_0,\alpha_1,\dots,\alpha_k\}}{\min\{\beta,\beta_0,\beta_1,\dots,\beta_k\}} = \frac{\alpha^{**}}{\beta^*}.$$

Similarly it easy to see that

$$y_n \geq \frac{\min\{\alpha,\alpha_0,\alpha_1,\dots,\alpha_k\}}{\max\{\beta,\beta_0,\beta_1,\dots,\beta_k\}} = \frac{\alpha^*}{\beta^{**}}.$$

Thus we get

$$0 < \gamma := \frac{\alpha^*}{\beta^{**}} \leq y_n \leq \frac{\alpha^{**}}{\beta^*} := \delta < \infty, \quad \text{for all} \quad n \geq 1.$$

Therefore every solution of Eq.(1) is bounded and persists. Hence the result holds.

Theorem 2: Every solution of Eq.(1) is bounded and persists.

Proof Let $\{y_n\}_{n=-k}^{\infty}$ be a positive solution of Eq.(1). Then it follows that

$$y_{n+1} = \frac{\alpha}{\beta + \beta_0 y_n^r + \cdots + \beta_k y_{n-k}^r} + \frac{\alpha_0 y_n^r}{\beta + \beta_0 y_n^r + \cdots + \beta_k y_{n-k}^r} + \cdots$$
$$\dots + \frac{\alpha_k y_{n-k}^r}{\beta + \beta_0 y_n^r + \beta_1 y_{n-1}^r + \cdots + \beta_k y_{n-k}^r}$$
$$\leq \frac{\alpha}{\beta} + \frac{\alpha_0}{\beta_0} + \frac{\alpha_1}{\beta_1} + \cdots + \frac{\alpha_k}{\beta_k} = \frac{\alpha}{\beta} + \sum_{i=0}^{k} \frac{\alpha_i}{\beta_i} := D.$$

Then $\{y_n\}_{n=-k}^{\infty}$ is bounded from above by D, that is $y_n \leq D$ for all $n \geq 1$.

Now we can obtain the lower bound of $\{y_n\}_{n=-k}^{\infty}$ by two ways;

(I) By the change of variables $y_n = \frac{1}{z_n}$ for all $n \geq 1$, Eq.(1) can be rewritten in the form

$$z_{n+1} = \frac{\beta z_n^r z_{n-1}^r \dots z_{n-k}^r + \beta_0 z_{n-1}^r z_{n-2}^r \dots z_{n-k}^r + \cdots + \beta_k z_n^r z_{n-1}^r \dots z_{n-k+1}^r}{\alpha z_n^r z_{n-1}^r \dots z_{n-k}^r + \alpha_0 z_{n-1}^r z_{n-2}^r \dots z_{n-k}^r + \cdots + \alpha_k z_n^r z_{n-1}^r \dots z_{n-k+1}^r}$$
$$\leq \frac{\beta}{\alpha} + \frac{\beta_0}{\alpha_0} + \frac{\beta_1}{\alpha_1} + \cdots + \frac{\beta_k}{\alpha_k} = \frac{\beta}{\alpha} + \sum_{i=0}^{k} \frac{\beta_i}{\alpha_i} := d^*.$$

That is $y_n \geq \frac{1}{d^*}$ for all $n \geq 1$ and therefore

$$d := \frac{1}{d^*} = \frac{1}{\frac{\beta}{\alpha} + \sum_{i=0}^{k} \frac{\beta_i}{\alpha_i}} \leq y_n \leq \frac{\alpha}{\beta} + \sum_{i=0}^{k} \frac{\alpha_i}{\beta_i} = D, \quad \text{for all } n \geq 1,$$

and this completes the proof.

(II) Since $y_n \leq D$ for all $n \geq 1$, we get from Eq.(1) that

$$y_{n+1} \geq \frac{\alpha}{\beta + \beta_0 y_n^r + \beta_1 y_{n-1}^r + \cdots + \beta_k y_{n-k}^r} \geq \frac{\alpha}{\beta + D^r \sum_{i=0}^{k} \beta_i} := d^{**}.$$

Then we see again that

$$d^{**} = \frac{\alpha}{\beta + D^r \sum_{i=0}^{k} \beta_i} \leq y_n \leq \frac{\alpha}{\beta} + \sum_{i=0}^{k} \frac{\alpha_i}{\beta_i} = D, \quad \text{for all } n \geq 1,$$

thus the proof is so completed.

III. Stability Analysis

In this section we investigate the global asymptotic stability of Eq.(1). Observe that the equilibrium points of Eq.(1) are given by $\bar{y} = \frac{\alpha + A(\bar{y})^r}{\beta + B(\bar{y})^r}$, where $A = \sum_{i=0}^{k} \alpha_i$ and $B = \sum_{i=0}^{k} \beta_i$.

Lemma 1: Eq.(1) has a unique positive equilibrium point if one of the following is true:

(i) $A\beta \le B\alpha$,

(ii) $r < 1$,

(iii) $rA\beta < A\beta + \alpha[B(r+1) + \beta(\bar{y})^r] + AB(\bar{y})^r$, or

(iv) $rA\beta < B(r\alpha + 2\beta\bar{y} + B(\bar{y})^{r+1}) + \beta^2(\bar{y})^{1-r}$.

Proof Define the function $f(x) = \frac{\alpha + Ax^r}{\beta + Bx^r} - x,\ x \in R$. Therefore

$$\grave{f}(x) = \frac{rx^{r-1}(A\beta - B\alpha)}{(\beta + Bx^r)^2} - 1,\ \ f(0) = \frac{\alpha}{\beta} > 0,\ \ \lim_{n\to\infty} f(x) = -\infty \text{ and } \lim_{n\to-\infty} f(x) = \infty.$$

Thus in particular $\grave{f}(\bar{y}) = \frac{r(\bar{y})^{r-1}(A\beta - B\alpha)}{(\beta + B(\bar{y})^r)^2} - 1$. Now we discuss the following cases:

(I) If (i) holds we obtain that $f(\bar{y}) < 0$ for all $\bar{y} \in R^+$ and then Eq.(1) has a unique positive equilibrium point $\bar{y}$ satisfies the relation $\bar{y} = \frac{\alpha + A(\bar{y})^r}{\beta + B(\bar{y})^r}$, and this completes the proof of (I).

(II) Note that

$$\begin{aligned} \grave{f}(\bar{y}) < 0 &\Leftrightarrow r(\bar{y})^{r-1}(A\beta - B\alpha) < [\beta + B(\bar{y})^r]^2 \\ &\Leftrightarrow r(\bar{y})^r(A\beta - B\alpha) < [\alpha + A(\bar{y})^r][\beta + B(\bar{y})^r] \\ &\Leftrightarrow rA\beta(\bar{y})^r - rB\alpha(\bar{y})^r < \alpha\beta + \alpha B(\bar{y})^r(r+1) + AB(\bar{y})^{2r} \\ &\Leftrightarrow A\beta(\bar{y})^r(r-1) < \alpha\beta + \alpha B(\bar{y})^r(r+1) + AB(\bar{y})^{2r}, \end{aligned}$$

which is true by (ii), then the result follows.

(III) Again, we see from case (iii) that

$$\begin{aligned} \grave{f}(\bar{y}) < 0 &\Leftrightarrow A\beta(r-1) < B\alpha(r+1) + \alpha\beta(\bar{y})^{-r} + AB(\bar{y})^r \\ &\Leftrightarrow rA\beta < A\beta + \alpha[B(r+1) + \beta(\bar{y})^{-r}] + AB(\bar{y})^r. \end{aligned}$$

Therefore again Eq.(1) has a unique positive equilibrium point $\bar{y}$.

(IV) The proof of the case wherever (iv) holds is similar to the previous cases and will be omitted. Thus the proof of the theorem is so completed.

Define the function F by

$$F(y_n, y_{n-1}, \dots, y_{n-k}) = \frac{\alpha + \alpha_0 y_n^r + \alpha_1 y_{n-1}^r + \cdots + \alpha_k y_{n-k}^r}{\beta + \beta_0 y_n^r + \beta_1 y_{n-1}^r + \cdots + \beta_k y_{n-k}^r}. \qquad (5)$$

Then

$$\frac{\partial F}{\partial y_{n-i}} = \frac{r y_{n-i}^{r-1}[\alpha_i\beta - \alpha\beta_i) + \sum_{j=0, j\ne i}^{k}(\alpha_i\beta_j - \alpha_j\beta_i)\, y_{n-i}^r]}{(\beta + \beta_0 y_n^r + \beta_1 y_{n-1}^r + \cdots + \beta_k y_{n-k}^r)^2}, \qquad i = 0,1,\dots .$$

Thus the linearized equation of Eq.(5) about the equilibrium point $\bar{y}$ of Eq.(5) is the linear difference equation

$$w_{n+1} - \frac{r(\bar{y})^{r-1}}{[\beta + B(\bar{y})^r]^2}\sum_{i=0}^{k}[(\alpha_i\beta - \alpha\beta_i) + (\bar{y})^r\sum_{j=0}^{k}(\alpha_i\beta_j - \alpha_j\beta_i)]\, w_{n-i} = 0,$$

whose characteristic equation is

$$\emptyset^{k+1} - \frac{r(\bar{y})^{r-1}}{[\beta + B(\bar{y})^r]^2}\sum_{i=0}^{k}[(\alpha_i\beta - \alpha\beta_i) + (\bar{y})^r\sum_{j=0}^{k}(\alpha_i\beta_j - \alpha_j\beta_i)]\, \emptyset^{k-i} = 0.$$

Then it follows by Theorem A that the equilibrium point $\bar{y}$ of Eq.(1) is locally as-ymptotically stable if

$$\sum_{i=0}^{k} r\bar{y}^r\left|(\alpha_i\beta - \alpha\beta_i) + \bar{y}^r\sum_{j=0}^{k}(\alpha_i\beta_j - \alpha_j\beta_i)\right| < (\alpha + A\bar{y}^r)(\beta + B\bar{y}^r).$$

Remark 1: For any partial order of the quotients $\frac{\alpha}{\beta}, \frac{\alpha_0}{\beta_0}, \frac{\alpha_1}{\beta_1}, \dots, \frac{\alpha_k}{\beta_k}$, the function $F(y_n, y_{n-1}, \dots, y_{n-k})$ defined by relation (5) has the monotonicity character in some of its arguments.

Therom 3: Every solution of Eq.(1) is globally asymptotically stable if one of the following holds

(i) $A > 2G$ and $B > 2L$

(ii) $A < 2G$, $B < 2L$ and

$$\beta + L(\gamma^r + \bar{y}^r) > (2G - A)\delta^{r-1} + \left(2L - B + \frac{2G - A}{\bar{y}}\right)\sum_{j=0}^{r-1}\bar{y}^{2r-i}\delta^i. \qquad (6)$$

(iii) $A < 2G$, $B > 2L$ and

$$\beta + L(\gamma^r + \bar{y}^r) + (B - 2L)\sum_{i=0}^{r-1}\gamma^{r-i}\bar{y}^i > (2G - A)\left(\gamma^{r-1} + \sum_{i=0}^{r-1}\delta^i\bar{y}^{r-i}\right). \quad (7)$$

(iv) $A > 2G$, $B < 2L$ and

$$\beta + L(\gamma^r + \bar{y}^r) + (A - 2G)\left(\delta^{r-1} + \sum_{i=0}^{r-1}\gamma^{r-i}\bar{y}^i\right) > (2L - B)\sum_{i=0}^{r-1}\delta^i\bar{y}^{r-i}. \quad (8)$$

Proof: Assume that $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.(1). Observe that it was proven in **Theorem 4:** that every solution of Eq.(1) is bounded and therefore it follows by Theorem B (Method of Full Limiting Sequences [6])that there exist solutions $\{I_n\}_{n=-\infty}^{\infty}$) and $\{S_n\}_{n=-\infty}^{\infty}$ of Eq.(1) with

$$\gamma \le I = I_0 = \lim_{n\to\infty}\inf y_n \le \lim_{n\to\infty}\sup y_n = S_0 = S \le \delta,$$

where

$$I_n, S_n \in [I, S], n = \cdots, -1, 0, 1, \ldots .$$

Now since $S \ge 1$, it suffices to show that $I \ge S$, Now we obtain from Eq.(1) that

$$I = I_0 = \frac{\alpha + \alpha_0 I_{-1}^r + \alpha_1 I_{-2}^r + \cdots + \alpha_k I_{-k}^r}{\beta + \beta_0 I_{-1}^r + \beta_1 I_{-2}^r + \cdots + \beta_k I_{-k}^r} \ge \frac{\alpha + GI^r + HS^r}{\beta + KI^r + LS^r},$$

where $H = A - G$ and $K = B - L$. Then we obtain

$$\alpha + GI^r + HS^r \le \beta I + KI^{r+1} + LIS^r,$$

or equivalently

$$\alpha \le \beta I + KI^{r+1} + LIS^r - GI^r - HS^r,$$

Similarly it is easy to see that

$$\alpha \ge \beta S + KS^{r+1} + LSI^r - GS^r - HI^r.$$

Therefore it follows from Eqs.(7) and (8) that

$$\beta S + KS^{r+1} + LSI^r - GS^r - HI^r \le \beta I + KI^{r+1} + LIS^r - GI^r - HS^r$$

or

$$\beta(I - S) + K(I^{r+1} - S^{r+1}) - LIS(I^{r-1} - S^{r-1}) - (G - H)(I^r - S^r) \ge 0,$$

or equivalently

$$\beta(I - S) - K(I^r + I^{r-1}S + \cdots + S^{r-1}I + S^r) - LIS(I^{r-2} + I^{r-3}S + I^{r-4}S^2 + \cdots$$
$$\ldots + IS^{r-3} + S^{r-2}) - (G - H)[(I^{r-1} + I^{r-2}S + \cdots + IS^{r-2} + S^{r-1})] \ge 0,$$

or

$$(I - S)[\beta + K(I^r + S^r) + (H - G)S^{r-1} + \left(K - L - \frac{G - H}{S}\right)(I^{r-1}S + \cdots + IS^{r-1})] \ge 0.$$

Note that $H = A - G$ and $K = B - L$, then

$$(I - S)\left[\beta + K(I^r + S^r) + (A - 2G)S^{r-1} + \left(B - 2L + \frac{A - 2G}{S}\right)\sum_{i=0}^{r-1} I^{r-i}S^i\right] \ge 0. \quad (9)$$

Note that if (i) is true then we obtain that

$$\beta + K(I^r + S^r) + (A - 2G)S^{r-1} + \left(B - 2L + \frac{A - 2G}{S}\right)\sum_{i=0}^{r-1} I^{r-i}S^i > 0.$$

Thus it follows from (9) that $I \ge S$ and the this completes the proof (i).
Proof of (ii): Note that $\gamma \le I \le \bar{y} \le S \le \delta$, therefore we see that

$$\beta + L(\gamma^r + \bar{y}^r) < \beta + L(I^r + S^r), \quad (10)$$

and

$$(A - 2G)S^{r-1} + \left(B - 2L + \frac{A - 2G}{S}\right)\sum_{i=0}^{r-1} I^{r-i}S^i < (2G - A)\delta^{r-1}$$
$$\left(B - 2L + \frac{A - 2G}{S}\right)\sum_{i=0}^{r-1}\bar{y}^{r-i}\delta^i. \quad (11)$$

Then we get from (6), (10) and (11) that

$$\beta + L(I^r + S^r) + (A - 2G)S^{r-1} + \left(B - 2L + \frac{A - 2G}{S}\right)\sum_{i=0}^{r-1} I^{r-i}S^i > 0.$$

Thus it follows again from (9) that $I \ge S$ and this completes the proof of (ii).
The proofs of cases (iii) and (iv) are similar to the proofs of the previous cases and will be omitted. The proof of the theorem is so completed.

IV. Existence of Periodic Solutions of Eq.(1)

In this section we investigate the existence of periodic solutions of prime period two of Eq.(1). In fact to achieve the existence of periodic solutions of Eq.(1) we need some very complicated computations so we here consider the case whenever *r = 1* the cases when *r > 1* and *r < 1* are similar. Let

$$D = \sum_{\substack{i=0 \\ i-odd}} \alpha_i\,, \quad E = D = \sum_{\substack{j=0 \\ j-even}} \alpha_j\,, \quad F = \sum_{\substack{i=0 \\ i-odd}} \beta_i \text{ and } R = \sum_{\substack{j=0 \\ j-ven}} \beta_j.$$

Theorem 5: Assume that $r = 1, D > E + \beta$ and $R > F$ then Eq.(1) has periodic solutions of prime period two if and only if

$$(R - F)(D - E - \beta)^2 > 4F[\alpha F + E(D - E - \beta)]. \qquad (12)$$

Proof First suppose that there exists a periodic solution $\{\dots, \phi, \psi, \phi, \psi, \phi, \psi, \dots\}$ of Eq.(1), where ϕ and ψ are distinct positive real numbers. Then it follows from Eq.(1) that ϕ, ψ satisfy the following

$$\phi = \frac{\alpha + D\phi^r + E\psi^r}{\beta + F\phi^r + R\psi^r} \text{ and } \quad \psi = \frac{\alpha + D\psi^r + E\phi^r}{\beta + F\psi^r + R\phi^r},$$

which are equivalent to

$$\phi\beta + F\phi^{r+1} + R\phi\psi^r = \alpha + D\phi^r + E\psi^r, \qquad (13)$$

and

$$\psi\beta + F\psi^{r+1} + R\psi\phi^r = \alpha + D\psi^r + E\phi^r. \qquad (14)$$

Subtracting (14) from (13) gives

$$\beta(\phi - \psi) + F(\phi^{r+1} - \psi^{r+1}) + R\phi\psi(\psi^{r-1} - \phi^{r-1}) = (D - E)(\phi^r - \psi^r).$$

Wherever *r = 1* we see that

$$\beta(\phi - \psi) + \beta(\phi - \psi)(\phi + \psi) = (D - E)\beta(\phi - \psi).$$

Since $\phi \neq \psi$, we have

$$\phi + \psi = \frac{D - E - \beta}{F}. \qquad (15)$$

By adding (13) and (14) we obtain

$$\beta(\phi + \psi) + F(\phi^2 + \psi^2) + 2R\phi\psi = (D + E)(\phi + \psi), \qquad (16)$$

and therefore

$$\phi\psi = \frac{\alpha F + E(D - E - \beta)}{F(R - F)}.$$

Thus ϕ and ψ are the roots of the following quadratic equation

$$u^2 - \frac{D - E - \beta}{F}u + \frac{\alpha F + E(D - E - \beta)}{F(R - F)} = 0. \qquad (17)$$

Again since $\phi \neq \psi$, we obtain

$$\left(\frac{D - E - \beta}{F}\right)^2 > \frac{4[\alpha F + E(D - E - \beta)]}{F(R - F)},$$

which implies that $(R - F)(D - E - \beta)^2 > 4[\alpha F + E(D - E - \beta)]$. Thus (12) holds.

Second suppose that the condition (12) is true. We will show that Eq.(1) has positive prime period two solutions.

Now assume that *k* is odd (the case wherever *k* is even is similar and will be left to the reader). Choose

$$y_{-k} = \dots = y_{-3} = y_{-1} = \phi = \frac{\frac{D-E-\beta}{F} + \sqrt{\left(\frac{D-E-\beta}{F}\right)^2 - \frac{4[\alpha F + E(D-E-\beta)]}{F(R-F)}}}{2},$$

and

$$y_{-k+1} = \dots = y_{-2} = y_0 = \psi = \frac{\frac{D-E-\beta}{F} - \sqrt{\left(\frac{D-E-\beta}{F}\right)^2 - \frac{4[\alpha F + E(D-E-\beta)]}{F(R-F)}}}{2}.$$

It is easy by direct substitution in Eq.(1) to prove that $y_1 = y_{-1} = \phi$ and $y_2 = y_0 = \psi$ and then it follows by Mathematical Induction that

$$y_{2n+1} = \phi \qquad \text{and} \qquad y_{2n} = \psi \qquad \text{for all} \quad n \geq 1.$$

Thus Eq.(1) has the positive prime period two solution

$$\{\dots, \phi, \psi, \phi, \psi, \phi, \psi, \dots\},$$

where ϕ and ψ are the distinct roots of the quadratic equation (17) and so the proof is completed.

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