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Periodic solutions for a class of second order higher-dimensional functional differential equations

Lili Wang
School of Mathematics and Statistics, Anyang Normal University, Anyang, Henan 455000, China

Abstract:
By using Krasnoselskii's fixed point theorem in cones to study the existence of periodic solutions for a higher-dimensional of second order nonlinear functional differential equations of the form

$$x''(t)+A(t)x'(t)+B(t)x(t)=\lambda C(t)f(t,x(t),x(t-\tau(t)),t\in R$$

where

$A(t)=diag[a_1(t),a_2(t),\cdots,a_n(t)],\ \ B(t)=diag[b_1(t),b_2(t),\cdots,b_n(t)],\ \ C(t)=diag[c_1(t),\cdots,c_n(t)],$ $a_j,b_j,c_j:R\to R^+$, $\tau:R\to R$ *are all continuous* T *-periodic functions,* $\lambda>0$, $f:R\times R^n\times R^n\to R^n$ *is continuous and* T *-periodic function.*

Keywords: *Periodic solution; Delay differential equations; Fixed point theorem; Parameter*

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I. Introduction

In the past two decades, nonlinear second-order differential equations have developed very rapidly owing to their many applications in almost all the branches of science. For example, Liu and Ge [1] investigated the following nonlinear Duffing equation with delay and variable coefficients:

$$x''(t)+p(t)x'(t)+q(t)x(t)=\lambda h(t)f(t,x(t-\tau(t))+r(t).$$

The existence and nonexistence of positive periodic solutions are obtained with suitable conditions imposed on f by using a fixed point theorem in cones.

However, there are few results on the existence of periodic solutions for higher-dimensional of high order functional differential equations. Motivated by the works of [1-8], in this paper, we shall use Krasnoselskii's fixed point theorem in cones to study the existence of periodic solutions for a higher-dimensional of second order nonlinear functional differential equations with periodic coefficients

$$x''(t)+A(t)x'(t)+B(t)x(t)=\lambda C(t)f(t,x(t),x(t-\tau(t)),t\in R, \tag{1}$$

where

(A1) $A(t)=diag[a_1(t),a_2(t),\cdots,a_n(t)],B(t)=diag[b_1(t),b_2(t),\cdots,b_n(t)],C(t)=diag[c_1(t),\cdots,c_n(t)]$;

(A2) $a_j,b_j,c_j:R\to R^+,\ \ \tau:R\to R$ are all continuous T -periodic functions, and $\int_0^T a_j(s)ds>0,$ $\int_0^T b_j(s)ds>0,\quad j=1,2,\cdots,n$;

(A3) f is a function defined on $R\times BC\times R^n$, satisfying $f(t+T,x(t+T),y)=f(t,x(t),y)$ for all $t\in R,\ x\in BC, y\in R^n$, where BC denotes the Banach space of bounded continuous functions $\eta:R\to R^n$ with the norm $\|\eta\|=\sup_{\theta\in R}\sum_{j=1}^n|\eta_j(\theta|)$ where $\eta=(\eta_1,\eta_2,\cdots\eta_n)^T$. In the sequel, we denote $f=(f_1,f_2,\cdots f_n)^T$.

Let $R=(-\infty,+\infty),R_+=(0,+\infty),R_+{}^n=\{(x_1,x_2,\cdots x_n)^T\in R^n:x_j>0,j=1,2,\cdots n\}$. We say that x is positive whenever $x\in R_+{}^n$. For every $x=\{(x_1,x_2,\cdots x_n)^T\in R^n$, the norm of x is defined as

$\left|x_0\right|=\sum_{j=1}^{n}\left|x_j\right|$. $BC(X\to Y)$ denotes the set of bounded continuous function $\phi: X\to Y$.

For convenience, we first introduce the related definition and the fixed point theorem applied in the paper.

Definition 1.1 Let X be a Banach space and K be a closed nonempty sunset of X, K is a cone if

(1) $\alpha u+\beta v\in K$ for all $u,v\in K$ and all $\alpha,\beta\geq 0$;

(2) $u,-u\in K$ imply $u=0$.

Theorem 1.1 (Krasnoselkii [9]) Let X be a Banach space, and let $K\subset X$ be a cone in X. Assume that Ω_1,Ω_2 are open bounded subsets of X with $0\in\Omega_1,\overline{\Omega_1}\subset\Omega_2$, and let

$$\phi: K\bigcap(\overline{\Omega_2}\setminus\Omega_1)\to K$$

be a completely continuous operator such that either

(1) $\|\phi y\|\leq\|y\|,\forall y\in K\bigcap\partial\Omega_1$ and $\|\phi y\|\geq\|y\|,\forall y\in K\bigcap\partial\Omega_2$; or

(2) $\|\phi y\|\geq\|y\|,\forall y\in K\bigcap\partial\Omega_1$ and $\|\phi y\|\leq\|y\|,\forall y\in K\bigcap\partial\Omega_2$.

Then ϕ has a fixed point in $K\bigcap(\overline{\Omega_2}\setminus K\bigcap\partial\Omega_1)$.

In this paper we always assume that

(H1) $f_j(t,\xi,\eta)\geq 0$ for all $(t,\xi,\eta)\in R\times BC(R,R_+{}^n)\times R_+{}^n$, $j=1,2,\cdots n$.

II. Some preparation

Let T be a positive constant. We define two sets

$$X=\{x: C(R,R^n),x(t+T)=x(t),t\in R\}$$

endow with the usual linear structure as well as the norm

$$\|x\|=\sup_{t\in R}\sum_{j=1}^{n}\left|x_j(t)\right|,\ \ |x|_0=\sum_{j=1}^{n}\left|x_j(t)\right|,$$

and

$$K=\left\{x\in X,x_j(t)\geq\sigma\left\|x_j\right\|,t\in[0,T],x=(x_1,x_2,\cdots x_n)^T\right\}.$$

Obviously, X is a Banach space and K is a cone.

Similar to the proof in [1], we can get:

Lemma 2.1. Suppose that (A1, A2) holds and

$$\frac{R_{1j}[\exp(\int_0^T a_j(u)du)-1]}{Q_{1j}T}\geq 1, \tag{2}$$

$$R_{1j}=\max_{t\in[0,T]}\left|\int_t^{t+T}\frac{\exp(\int_t^s a_j(u)du)}{\exp(\int_0^T a_j(u)du)-1}b_j(s)ds\right|,Q_{1j}=\left(1+\exp(\int_0^T a_j(u)du)\right)^2 R_{1j}{}^2.$$

Then there exist continuous T-periodic functions p_j and q_j such that $q_j(t)>0$, $\int_0^T p_j(u)du>0$, and

$$p_j(t)+q_j(t)=a_j(t),q_j'(t)+p_j(t)q_j(t)=b_j(t)\ \text{ for all } t\in R,\ j=1,2,\cdots n.$$

Therefore

$$p(t)+q(t)=A(t),q'(t)+p(t)q(t)=B(t),t\in R,$$

where $p=diag[p_1,p_2,\cdots,p_n],q=diag[q_1,q_2,\cdots,q_n]$.

Similar to the proof in [3], we can get the following lemmas.

Lemma 2.2. Suppose the conditions of Lemma 2.1 hold and $\varphi(t)\in X$. Then the equation

$$x''(t)+A(t)x'(t)+B(t)x(t)=\varphi(t) \tag{3}$$

has a T-periodic solution. Moreover, the periodic solutions can be expressed by

$$x(t)=\int_t^{t+T}G(t,s)\varphi(s)ds, \tag{4}$$

where
$$G(t,s)=diag[G_1(t,s),G_2(t,s),\cdots,G_n(t,s)],$$
and
$$G_j(t,s)=\frac{\int_t^s\exp[\int_t^u q_j(v)dv+\int_u^s p_j(v)dv]du+\int_s^{t+T}\exp[\int_t^u q_j(v)dv+\int_u^{s+T}p_j(v)dv]du}{[\exp(\int_0^T p_j(u)du)-1][\exp(\int_0^T q_j(u)du)-1]}.$$

So Eq.(1) has a T -periodic solution, it can be expressed by
$$x(t)=\int_t^{t+T}G(t,s)\lambda C(s)f(s,x(s),x(s-\tau(s)))ds, \tag{5}$$
and by (H1), we have
$$G_j(t,s)\lambda c_j(s)f_j(s,x(s),x(s-\tau(s)))\ge 0,\ j=1,2,\cdots n,(t,s)\in R^2.$$

Corollary 2.1. Green's function $G(t,s)$ satisfies the following properties:
$$G_j(t,t+T)=G_j(t,t),\ \ G_j(t+T,s+T)=G_j(t,s),$$
$$\frac{\partial}{\partial s}G_j(t,s)=p_j(s)G_j(t,s)-\frac{\exp\int_t^s q_j(v)dv}{\exp\int_0^T q_j(v)dv-1},$$
$$\frac{\partial}{\partial t}G_j(t,s)=-q_j(s)G_j(t,s)+\frac{\exp\int_t^s p_j(v)dv}{\exp\int_0^T p_j(v)dv-1},\quad j=1,2,\cdots,n.$$

Lemma 2.3. Let $H_j=\int_0^T a_j(u)du, I_j=T^2\exp(\frac{1}{T}\int_0^T\ln b_j(u)du)$. If $H_j^{\ 2}\ge 4I_j,$ (6)

then
$$\min\left\{\int_0^T p_j(u)du,\int_0^T q_j(u)du\right\}\ge\frac{1}{2}(H_j-\sqrt{H_j^{\ 2}-4I_j}):=l_j,$$
$$\max\left\{\int_0^T p_j(u)du,\int_0^T q_j(u)du\right\}\le\frac{1}{2}(H_j+\sqrt{H_j^{\ 2}-4I_j}):=m_j,\quad j=1,2,\cdots,n.$$

Therefore the function $G_j(t,s)$ satisfies
$$0<N_j=:\frac{T}{(e^{m_j}-1)^2}\le G_j(t,s)\le\frac{T\exp(\int_0^T a_j(u)du)}{(e^{l_j}-1)^2}:=M_j, s\in[t,t+T],$$
$$1\ge\frac{G_j(t,s)}{M_j}\ge\frac{N_j}{M_j}\ge\sigma:=\min\left\{\frac{N_j}{M_j},j=1,2,\cdots,n\right\}>0,$$
and we denote
$$l=\min_{1\le j\le n}l_j,\quad m=\max_{1\le j\le n}m_j,\quad N=\min_{1\le j\le n}N_j,\qquad M=\max_{1\le j\le n}M_j.$$

Now, before presenting our main results, we give the following assumptions.

(H2) $f(t,\phi(t),\phi(t-\tau(t)))$ is a continuous function of t for each $\phi\in BC(R,R_+^{\ n})$.

(H3) For any $L>0$ and $\varepsilon>0$, there exists $\delta>0$, such that
$$\{\phi,\psi\in BC,\|\phi\|\le L,\|\psi\|\le L,\|\phi-\psi\|<\delta,0\le s\le T\}$$
imply $\left|f(s,\phi(s),\phi(s-\tau(s)))-f(s,\psi(s),\psi(s-\tau(s)))\right|_0<\varepsilon$.

III. Main Results

Now we define a mapping $T:K\to K$,

$$(Tx)(t)=\int_t^{t+T}G(t,s)\lambda C(s)f(s,x(s),x(s-\tau(s))ds.$$

We denote $(Tx)=(T_1x,T_2x,\cdots T_nx)^T$.

Lemma 3.1. $T:K\to K$ is well-defined.

Proof. For each $x\in K$, by (H2) we have $(Tx)(t)$ is continuous in t and

$$\begin{aligned}(Tx)(t+T)&=\int_{t+T}^{t+2T}G(t,s)\lambda C(s)f(s,x(s),x(s-\tau(s))ds\\&=\int_t^{t+T}G(t+T,v+T)\lambda C(v+T)f(v+T,x(v+T),x(v+T-\tau(v+T))dv\\&=\int_t^{t+T}G(t,v)\lambda C(v)f(v,x(v),x(v-\tau(v))dv\\&=(Tx)(t).\end{aligned}$$

Thus, $Tx\in X$, since

$N_j\le G_j(t,s)\le M_j, s\in[t,t+T].$

Hence, for $x\in K$, we have

$$\|T_jx\|\le M_j\int_0^T\left|\lambda c_j(s)f_j(s,x(s),x(s-\tau(s))\right|ds, \tag{7}$$

and

$$\begin{aligned}(T_jx)(t)&\ge N_j\int_0^T\left|\lambda c_j(s)f_j(s,x(s),x(s-\tau(s))\right|ds\\&=\frac{N_j}{M_j}M_j\int_0^T\left|\lambda c_j(s)f_j(s,x(s),x(s-\tau(s))\right|ds\\&\ge\sigma\|T_jx\|.\end{aligned}$$

Therefore, $Tx\in K$. This completes the proof.

Lemma 3.2. $T:K\to K$ is completely continuous.

Proof. We first show that T is continuous.

By (H3), for any $L>0$ and $\varepsilon>0$, there exists a $\delta>0$ such that $\{\phi,\psi\in BC,\|\phi\|\le L,\|\psi\|\le L,\|\phi-\psi\|\le\delta\}$ imply

$$\sup_{0\le s\le T}|f(s,\phi(s),\phi(s-\tau(s))-f(s,\psi(s),\psi(s-\tau(s))|_0<\frac{\varepsilon}{\lambda MTC},$$

where $C=\max_{1\le j\le n}\|c_j\|$.

If $x,y\in K$ with $\|x\|\le L,\|y\|\le L,\|x-y\|\le\delta$, then

$$\begin{aligned}|(Tx)(t)-(Ty)(t)|_0&\le\int_t^{t+T}|G(t,s)||\lambda C(s)f(s,x(s),x(s-\tau(s))-\lambda C(s)f(s,y(s),y(s-\tau(s))|_0ds\\&\le\int_0^T|G(t,s)||\lambda C(s)f(s,x(s),x(s-\tau(s))-\lambda C(s)f(s,y(s),y(s-\tau(s))|_0ds\\&<M\lambda TC\frac{\varepsilon}{M\lambda TC}=\varepsilon\end{aligned}$$

for all $t\in[0,T]$, where $|G(t,s)|=\max_{1\le j\le n}|G_j(t,s)|$, this yields $\|Tx-Ty\|<\varepsilon$, thus T is continuous.

Next we show that T maps any bounded sets in K into relatively compact sets. Now we first prove that f maps bounded sets into bounded sets. Indeed, let $\varepsilon=1$, by (H3), for any $\mu>0$, there exists $\delta>0$ such that $\{x,y\in BC,\|x\|\le\mu,\|y\|\le\mu,\|x-y\|\le\delta,0\le s\le T\}$ imply

$$|f(s,x(s),x(s-\tau(s))-f(s,y(s),y(s-\tau(s))|_0<1.$$

Choose a positive integer N such that $\frac{\mu}{N} < \delta$. Let $x \in BC$ and define

$x^k(t) = \frac{x(t)k}{N}, k = 0, 1, 2 \cdots, N$.

If $\|x\| < \mu$, then

$$\|x^k - x^{k-1}\| = \sup_{t\in R} \left| \frac{x(t)k}{N} - \frac{x(t)(k-1)}{N} \right| \le \|x\| \frac{1}{N} \le \frac{\mu}{N} < \delta.$$

Thus,

$$\left| f(s, x^k(s), x^k(s-\tau(s)) - f(s, x^{k-1}(s), x^{k-1}(s-\tau(s)) \right|_0 < 1$$

for all $s \in [0, T]$, this yields

$$\begin{aligned} &\left| f(s, x(s), x(s-\tau(s)) \right|_0 = \left| f(s, x^N(s), x^N(s-\tau(s)) \right| \\ &\le \sum_{k=1}^{N} \left| f(s, x^k(s), x^k(s-\tau(s)) - f(s, x^{k-1}(s), x^{k-1}(s-\tau(s)) \right|_0 + \left| f(s, 0, 0) \right|_0 \qquad (8) \\ &< N + \|f\| =: W. \end{aligned}$$

It follows from (7) that

$$\|Tx\| = \sup_{t\in R} \sum_{j=1}^{n} \left|(T_j x)(t)\right| \le \sum_{j=1}^{n} M_j \lambda C \int_0^T \left| f_j(s, x(s), x(s-\tau(s)) \right| ds \le M \lambda CTW.$$

Finally, for $t \in R$, we have

$$(T_j x)'(t) = \int_t^{t+T} \left[-q_j(s) G_j(t,s) + \frac{\exp \int_t^s p_j(v) dv}{\exp \int_0^T p_j(v) dv - 1} \right] \lambda c_j(s) f_j(s, x(s), x(s-\tau(s)) ds, \qquad (9)$$

$$j = 1, 2, \cdots, n.$$

Combine (7), (8), (9) and Corollary 2.1, we obtain

$$\begin{aligned} &\left| \frac{d}{dt}(Tx)(t) \right|_0 = \sum_{j=1}^{n} \left|(T_j x)'(t)\right| \\ &\le \sum_{j=1}^{n} \int_t^{t+T} \left| \lambda c_j(s) f_j(s, x(s), x(s-\tau(s)) \right| \left| -q_j(s) G_j(t,s) + \frac{\exp \int_t^s p_j(v) dv}{\exp \int_0^T p_j(v) dv - 1} \right| ds \\ &\le \sum_{j=1}^{n} \lambda C (M_j \|Q\| + \frac{e^m}{e^l - 1}) \int_t^{t+T} \left| f_j(s, x(s), x(s-\tau(s)) \right| ds \\ &\le \lambda C (M \|Q\| + \frac{e^m}{e^l - 1}) TW, \end{aligned}$$

where $\|Q\| = \max_{1\le j\le n} |q_j|$.

Hence $\{Tx : x \in K, \|x\| \le \mu\}$ is a family of uniformly bounded and equicontinuous functions on $[0, T]$. By a theorem of Ascoli-Arzela, the function T is completely continuous.

Theorem 3.1. Suppose that (H1)-(H3), (2) and (6) and that there are positive constants R_1 and R_2 with $R_1 < R_2$ such that

$$\sup_{\|\phi\|=R_1, \phi\in K} \int_0^T \left| f(s, \phi(s), \phi(s-\tau(s)) \right|_0 ds := P_1, \qquad (10)$$

and

$$\inf_{\|\phi\|=R_2,\phi\in K}\int_0^T|f(s,\phi(s),\phi(s-\tau(s))|_0\,ds:=P_2, \tag{11}$$

for each λ satisfy

$$\frac{R_2}{MCP_2}<\lambda<\frac{R_1}{MCP_1}. \tag{12}$$

Then Eq.(1) has a positive T -periodic solution x with $R_1\le\|x\|\le R_2$.

Proof. Let $x\in K$ and $\|x\|=R_1$. By (10) and (12), we have

$$\begin{aligned}|(Tx)(t)|_0 &\le M\int_t^{t+T}|\lambda C(s)f(s,x(s),x(s-\tau(s))|_0\,ds\\ &\le\lambda MC\int_t^{t+T}|f(s,x(s),x(s-\tau(s))|_0\,ds\\ &<\frac{R_1}{MCP_1}MC\int_t^{t+T}|f(s,x(s),x(s-\tau(s))|_0\,ds\le R_1\end{aligned}$$

for all $t\in[0,T]$. This implies that $\|Tx\|\le\|x\|$ for $x\in K\bigcap\partial\Omega_1,\Omega_1=\{x\in X,\|x\|<R_1\}$.

If $x\in K$ and $\|x\|=R_2$. By (11) and (12), we have

$$\begin{aligned}|(Tx)(t)|_0 &\ge N\int_t^{t+T}|\lambda C(s)f(s,x(s),x(s-\tau(s))|_0\,ds\\ &\ge\lambda NC\int_t^{t+T}|f(s,x(s),x(s-\tau(s))|_0\,ds\\ &>\frac{R_2}{NCP_2}NC\int_t^{t+T}|f(s,x(s),x(s-\tau(s))|_0\,ds\ge R_2\end{aligned}$$

for all $t\in[0,T]$. Thus, $\|Tx\|\ge\|x\|$ for $x\in K\bigcap\partial\Omega_2,\Omega_2=\{x\in X,\|x\|<R_2\}$.

By Krasnoselskii's fixed point theorem, T has a fixed point in $K\bigcap(\overline{\Omega_2}\setminus\Omega_1)$. It is easy to say that Eq.(1) has a positive T -periodic solution x with $R_1\le\|x\|\le R_2$. This completes the proof.

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