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Characteristics on Baer ring with Von Neumann regular ring and Semi simple ring

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Abstract: *This paper has mainly focused on the relation among semisimple ring, Baer ring and Von-Neumann regular ring where it has been showed that " R is a ring then R is semi simple artinian if and only if $B(R)$ is a Baer ring for all infinite set Γ [Theorem 3.1] and " R is semisimple if and only if R is Von Neumann regular. On the other hand Von Neumann regular tends to exchange ring but conversely is not possible .In fact we have showed the chain with semi simple ring, Baer ring and Von Neumann regular ring. We have tried to connect the semi simple ring with the properties between Baer ring and Von Neumann regular ring in terms of row and column finite matrix rings.*

Result: The main result of this paper is "Let R be a ring. Then the following conditions are equivalent

1. *R is semisimple artinian*
2. *$B(R)$ is a Baer ring*
3. *R is von Neumann regular "*

Key Word: *Semi simple ring, Baer ring, Von Neumann regular ring.*

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I. Introduction

The study of the relationship between a ring and its matrix ring has a long history. Already many authors have contributed to the study of finite and infinite matrix rings. The study of these two topics was initiated in the 1940s with contributions of Baer, Jacobson, Mackey, Amitsur, O'Meara, Camillo, Simon and others.

In 1994, V. P. Camiilo and H P YU [6] have given a relationship between exchange ring, semi perfect ring. They proved that "a ring R is semi perfect if and only if R is an exchange ring."After that in 1998, J J Simon [1] has shown nice characterizations among semisimple ring, perfect ring and semiprimary ring in terms of finitely generated projective modules over row and column finite matrix rings. Mackey and Ornstein [7, 8] proved that if a ring R is semisimple then the row and column finite matrices over R is Baer ring**.** A ring with identity is a Baer ring if every left (equivalently every right) annihilator is generated by an idempotent. In 2001, V. Camillo, F. J**.** Costa-Cano and J. J. Simon [2] have shown that the converse of Mackey and Ornstein concepts is also true. They have proved that a ring R is semisimple artinian if and only if $RCFM(R)$ (row and column finite matrices over (R)) is a Baer ring. They have also explained that a ring (R) is left and right perfect if and only if $RCFM(R)/J(RCFM(R))$ is a Baer ring, where $J(RCFM(R))$ is the Jacobson radical of row and column finite matrix rings. In 2002, C. Busque and J. J. Simon [9] have mentioned that R is a semisimple artinian ring if and only if $\varepsilon_{\alpha\beta}(R)$ is a von Neumann regular ring where $\varepsilon_{\alpha\beta}(R)$ is an infinite matrix rings and α , β are infinite cardinals. After that in 2003, for any unital von Neumann regular ring R , $RCFM(R)$ is an exchange ring. The main aim of this paper is to illuminate the relationship among Von Neumann regular ring, Baer ring and semisimple ring in terms of row and column finite matrix rings.

Note 1: Clean rings are always exchange rings, and the converse is true when idempotents are central in R .
Note 2: Every semisimple ring is clean.

Definition 1.1: An element a in a ring R is said to be **von Neumann regular** if there exists $b \in R$ such that $a = aba$. The ring R is called a **von Neumann regular ring** if every element in R is von Neumann regular.

Remark 1.1: Regular rings are semiprimitive. Let R be a regular ring. Let $a \in J(R)$, the Jacobson radical of R, and choose $x \in R$ such that $a = axa$. Then $a(1 - xa) = 0$ and, since $1 - xa$ is invertible because a is in the Jacobson radical of R, we get $a = 0$.

Example1.1: Every direct product of regular rings is clearly a regular ring.

Example 1.2: If V is a vector space over a division ring D, then $End_D V$ is regular.

Example 1.3: Every semisimple ring is regular.

Proof: For a division ring D the ring $M_n(D) \cong End_D D^n$ is regular by example1. 2. Now by example 1.1 and the Wedderburn-Artin theorem.

II. Discussion

Theorem 2.1: A R ring is regular iff every finitely generated left ideal of R is generated by an idempotent.

Corollary 2.1: If the number of the idempotents of a regular ring R is finite, then R is semisimple.

Proof: By the theorem 1.1, R has only a finite number of left principal ideals. Since every left ideal is a sum of left principal ideals, it follows that R has only a finite number of left ideals and hence it is left artinian. Thus R is semisimple because is semiprimitive by Remark 1.1.

Theorem 2.2: Let R be a prime ring. If $B(R)$ is a Baer ring then R has an indecomposable decomposition $R = Ru_1 \oplus \cdots \oplus Ru_n$ where each Ru_i is a minimal left annihilator, and hence u_i is primitive and $u_i R u_i$ is a domain with $B(u_i R u_i)$ a Baer ring.

Proof: By [2, Corollary 7] we may choose an idempotent u_i such that Ru_1 is a minimal left annihilator. Take $K_1 = R(1 - u_1)$ and choose a minimal annihilator $Ru_2' \subseteq K_1$. Let $u_2 = (1 - u_1)u_2'$. Then u_1 and u_2 are orthogonal idempotents and $Ru_2' = Ru_2$. Now $R = Ru_1 \oplus \cdots \oplus Ru_2 \oplus K_2$ with $K_2 = R(1 - u_1 - u_2)$ and continue in this way. Since R has ACC on left annihilators the procedure must stop. Hence $R = Ru_1 \oplus \cdots \oplus Ru_n$.

Now we prove that $u_i R u_i$ is a domain. It is easy to calculate that if Re is a minimal annihilator with $e^2 = e$ then $ere \cdot ese = 0$ implies $ere = 0$ or $ese = 0$.

Lemma 2.1: If $B(R)$ is a Baer ring then R is semi-simple artinian.

Proof: By [2, Proposition 8 & 11] and Theorem 2.2, the ring R is semi-prime and has a decomposition $R = \oplus_{i=1}^{n} Ru_i$, where the u_i's are primitive pairwise orthogonal idempotents, with $u_i R u_i$ division ring. By [27, Corollary 4.4] each Ru_i is simple, and hence R is semi-simple.

III. Result

Theorem 3.1: Let R be a ring. Then is R semisimple artinian if and only if $B(R)$ is a Baer ring for all infinite set Γ.

Proof: By [10, Theorem 4], it is proved that if R is a semisimple ring then $B(R)$ is a Baer ring for all infinite set Γ. To prove the second part, lemma 2.1 is enough and by [13, Theorem 4], it is well known that being a Baer ring is a corner-preserving property.

Corollary 3.1: Let R be a ring. Then R is semisimple if and only if R is von Neumann regular.

Proof: It is direct impact by [11, Proposition 11] and the fact that R is semisimple if and only if is R von Neumann regular.

Proposition 3.1: Let R be a semi-simple ring and α be an infinite cardinal. If $RCFM_\alpha(R)$ is left semi hereditary then R is a semi-simple ring.

Proof: Suppose $RCFM_\alpha(R)$ is left semi hereditary. It is well-known that semi hereditary is a corner-preserving property, so our condition implies that $RCFM(R)$ (countable matrices) is left semi hereditary. Set

$B = RCFM(R)$. First we will see that if B is left semi hereditary then R is a von Neumann regular ring. To see this, take any nonzero element $d \in R$ and construct the matrix of Lemma 2.1; that is, $a \in B(R) = B$ is the matrix defined as $a(2n+1, n+1) = d$, $a(2n, n) = a(2n, n+1) = 1$ and $a(i, j) = 0$ otherwise.

$$a = \begin{pmatrix} d & 0 & \cdots & & \\ 1 & 1 & 0 & \cdots & \\ 0 & d & 0 & \cdots & \\ 0 & 1 & 1 & 0 & \cdots \\ & & \ddots & \ddots & \end{pmatrix}.$$

By the Lemma 2.1, we have that there is a direct summand K of $R^{(\mathrm{N})}$, and $j_0 \in \mathrm{N}$ such that $R^{(\mathrm{N})} = Ra_1 \oplus K$, and $a_{j_0+k} \in K$ for all $k \in \mathrm{N}$. Note that for definition of a, $da_{2n} = a_{2n-1} + a_{2n+1}$. By Proposition 2.2.3, let $\varphi : R^{(\mathrm{N})}a \to Ra_1$ be the projection for the decomposition $R^{(\mathrm{N})}a = Ra_1 \oplus K$ with $a_{j_0+k} \in K$ for all $k \in \mathrm{N}$. In general, $da_{2n} = a_{2n-1} + a_{2n+1}$ so that $\varphi(da_{2n}) = \varphi(a_{2n-1}) + \varphi(a_{2n+1})$ implies $dx_{2n}d = x_{2n-1}d + x_{2n+1}d$, and also $x_1 d = d$. Now we claim that j_0 is even. If not, $j_0 + 1$.is even so that $da_{j_0+1} = a_{j_0} + a_{j_0+2}$ and then $\varphi(a_{j_0}) = 0$ which is impossible. Hence j_0 is even, and $da_{j_0} = a_{j_0-1} + a_{j_0+1}$ which implies that $dx_{j_0}a_1 = x_{j_0-1}a_1$, and this, in turn, implies that $dx_{j_0}d = x_{j_0-1}d$. Now we will do decent argument. Suppose we have proved for $3 \le j \le j_0$, with j odd, that $x_j d = dy_j d$ for some $y_j \in R$. Then we may consider $da_{j-1} = a_{j-2} + a_{j+1}$ from which $dx_{j-1}d = x_{j-2}d + x_{j+1}d$. Since $x_j d = dy_j d$ then $x_{j-2}d = d(x_{j-1} - y_j)d$. Setting $y_{j-1} = x_{j-1} - y_j$ we are done. So, for all j odd, with $1 \le j \le j_0$, we have that $x_j d = dy_j d$. In particular, $x_1 d = dy_1 d$. Finally, since $x_1 d = d$, we have $d = dy_1 d$. This proves that R is von Neumann regular.

Now we have to prove that if R is von Neumann regular and B is left semi hereditary then R is semi-simple. Since left semi hereditary rings are left coherent then we may apply directly [29, proposition22] to get that R is von Neumann regular and has ACC on direct summands, which implies that R is semi simple.

Corollary 3.2: If α is an infinite cardinal and R is a ring, then R is semi simple artinian if and only if $B_\alpha(R)$ is von Neumann regular and $B_\alpha(R)$ is left coherent.

Proof: By [11, proposition 11], it is a direct consequence and the fact that R is semisimple artinian if and only if $B_\alpha(R)$ is von Neumann regular ring and satisfies acc on direct summands [12, Theorem 19.26 A].

Theorem 3.2: Let R be a ring. Then the following conditions are equivalent

1. R is semi simple artinian
2. $B(R)$ is a Baer ring
3. R is von Neumann regular

Proof: (1) $\Rightarrow$(2) It is clear by Theorem 3.1.

(2) $\Rightarrow$(3) This comes from Theorem 3.1 that if $B(R)$ is a Baer ring then R is semi simple and by corollary 3.1, if R is semi simple if and only if R is von Neumann regular.

(3) $\Rightarrow$ (1) By Corollary 3.1 it is obvious.

IV. Conclusion

Finally, the relation has been developed to focus on the characteristics of Baer ring, semi simple ring with Von Neumann regular ring in terms of row and column finite matrix ring which indicates “A ring with identity is a Baer ring if every left (equivalently every right) annihilator is generated by an idempotent”.

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