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Why does the classical Benders decomposition algorithm not work well for the b-Complementary Multisemigroup Dual Problem?

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Abstract
We present an adaptation of the classical algorithm of the decomposition of Bender to the b-complementary multisemigroup dual problem. Despite this decomposition has been shown in literature as a good tool for dealing with high dimensional mixed-integer linear programming, that is not the case for the presented one in this paper, which is better to be solved by the simplex algorithm without partitioning. We present results from computer experiments to show that conclusion.
Keywords*: Multisemigroup; Complementary; Duality; Benders Decomposition.*

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I. Introduction

Aráoz in 1973 defined the Semigroup Problem (SP), characterizes the polyhedra, and shows the relation between the minimal system of linear inequality of the polyhedra and extreme points and rays. Aráoz and Johnson, in 1982, presented the polyhedra of a multivalued additive system problem. A particular case of multivalued additive systems is the b-complementary Multisemigroups (b-CMS). A b-CMS is an associative, an abelian, a b-consistent, and a b-complementary additive system. Madriz, in 2016, constructed the dual problem associated with a b-CMS problem, extending the duality result of semigroup by Johnson in 1980. Madriz's work is based on the theorem presented by Ar´aoz and Johnson in 1989, where they determine that, given a base of the subadditive cone, it is possible to establish a system of equations and inequalities that define the polyhedron associated with a multivalued associative additive system. Aráoz and Johnson in 1982 defined the finite b-complementary multisemigroup as an associative, commutative, consistent, and complementary additive multivalued system and they showed the following characterization of the faces of the convex hull of multisemigroup solutions.

Let $(A,\hat{+})$ be an b-complementary multisemigroup and $A_+ = A - \{\hat{0},\infty\}$. The **master corner polyhedron** is defined as

$$P(A_+,b) = conv.\,hull\{t \in Z^{|A_+|} : b \in \widehat{\sum_{g\in A_+}} t(g)g\},$$

for some fixed right-hand side element $b \in A_+$ and $|A_+|$ denotes the cardinality of the set A_+ and Z is the set of integer numbers.

Let $C(A_+)$ be the subadditive cone associated with $P(A_+,b)$ (see [1]), Aráoz and Johnson in [2] showed the following characterization of the $P(A_+,b)$.

Theorem 1.1 (Theorem 3.8 in [2])} Let (L,E) be a base of $C(A_+)$. The following system defined a $P(A_+,b)$

$$\sum_{g\in A_+} \rho(g)t(g) = \rho(b), \text{ for all } \rho \in L$$

$$\sum_{g\in A_+} \pi(g)t(g) \geq \pi(b), \quad \pi \in E$$

$$t(g) \geq 0, g \in A_+.$$

In general, given a finite b-complementary multisemigroup A , the **b-complementary multisemigroup master problem** is defined as

$$P_{ms} : min \sum_{g \in A_+} c(g)t(g)$$
$$\text{s.t: } b \in \widehat{\sum_{g \in A_+} t(g)g}$$
$$t(g) \in Z_+, \quad g \in A_+$$

where $c(g) \in R^{|A+|}$, and $\boldsymbol{R}$ is the set of real numbers..
Using theorem 3.8 proved by Aráoz and Johnson in 1982, Madriz in 2016 showed that the P_{ms} problem is equivalent to the following problem

$$P_p : min \sum_{g \in A_+} c(g)t(g)$$
$$\text{s.t.: } \sum_{g \in A_+} \rho(g)t(g) = \rho(b), \quad \rho \in L;$$
$$\sum_{g \in A_+} \pi(g)t(g) \geq \pi(b), \quad \pi \in E;$$
$$t(g) \geq 0, \quad g \in A_+,$$

where (L, E) is a base for $C(A_+)$and $c \in R^{|A+|}$. In addition, Madriz in 2016 calculates the dual problem of P_p and proves the duality theorem for P_p and P_d (see Theorem 4.3 of Madriz 2016 [4]).

$$P_d : max \sum_{\rho \in L} \rho(b)\upsilon(\rho) + \sum_{\pi \in E} \pi(b)w(\pi)$$
$$\text{s.t.: } \sum_{\rho \in L} \rho(g)\upsilon(\rho) + \sum_{\pi \in E} \pi(g)w(\pi) \leq c(g), \quad g \in A_+;$$
$$\upsilon(\rho) \text{ unrestricted}, \quad \rho \in L;$$
$$w(\pi) \geq 0, \quad \pi \in E.$$

Let (L,E) be a base of C(A) and $\upsilon \in L$, we denote with P_υ the following problem,

$$P_\upsilon : max \sum_{\pi \in E} \pi(b)w(\pi)$$
$$\text{s.t.: } \sum_{\pi \in E} \pi(g)w(\pi) \leq K_\upsilon(g), \quad g \in A_+$$
$$w(\pi) \geq 0, \ \pi \in E.$$

where

$$K_\upsilon(g) = c(g) - \sum_{\rho \in L} \rho(g)\upsilon(\rho)$$

for all $g \in A_+$.
The dual problem of the P_υ is the problem

$$P_{\upsilon d} : min \sum_{g \in A_+} K_\upsilon(g)t(g)$$
$$\text{s.t.: } \sum_{g \in A_+} \pi(g)t(g) \geq \pi(b), \ \pi \in E$$
$$t(g) \geq 0, \quad g \in A_+,$$

therefore the P_d problem is equivalent to the problem

$$max \sum_{\rho \in L} \rho(b)\upsilon(\rho) + Z^*_{P_{\upsilon d}}$$
$$\text{s.t.: } \upsilon(\rho) \text{ unrestricted}, \quad \rho \in L,$$

where $Z^*_{P_{\upsilon d}}$ is the solution of the problem $P_{\upsilon d}$.

Now, let be *X* the polyhedron

$$X = \{t \in R_+^{|A+|} : \sum_{g \in A_+} \pi(g)t(g) \geq \pi(b), \pi \in E\}$$

and *V(X)* and *R(X)* the sets of vertices and the extreme rays of the *X*.

Note that in the case where X is empty, the dual problem P_{vd} is infeasible. Moreover, from duality theory, the primal problem P_v has no feasible or it is unbounded. Therefore, we can assume that the set X is non-empty. On the other hand, since the convex polyhedron X is independent of $\rho \in L$, the **Benders inner problem for the dual** of $P_d$is defined as

$$B_{id} : max \quad \beta$$
$$\textbf{s.t:} \quad \beta \leq \sum_{\rho \in L} \rho(b)v(\rho) + \sum_{g \in A_+} K_v(g)t(g)$$
$$t \in V(X)$$

Therefore, the **Benders master problem for the dual** $P_d$is the problem

$$B_{md} : max \sum_{\rho \in L} \rho(b)v(\rho) + Z^*_{B_{id}}$$
$$\text{s.t.:} \quad \sum_{g \in A_+} K_v(g)t(g) \leq 0,$$
$$t \in R(X),$$
$$v(\rho) \text{ unrestricted, for all } \rho \in L,$$

where $Z^*_{B_{id}}$ is the solution of the associated B_{id} problem.

In this paper, we apply the Bender decomposition algorithm to solve the problem P_d.The paper is divided as follows. In section 2 we describe an algorithm using Benders decomposition for the dual problem associated with the problem of b-complementary multisemigroup. Finally, in section 3 we present experimental results for this algorithm, and conclusions in section 4.

1. Material and methods.

Give a base (L,E) of $C(A_+)$. since A_+, L and E are finite sets, we can consider $A_+ = \{g_1, \ldots, g_r\}$ where $r = |A_+|$and $g_1 \hat{+} g_j = g_j$ for all $j \in \{1, \ldots, r\}$;
$L = \{\rho_1, \ldots, \rho_l\}$where $l = |L|$;
$E = \{\pi_1, \ldots, \pi_e\}$where $e = |E|$.
In addition, we consider:
$t = (t_1, \ldots, t_r)$ where $t_i = t(g_i)$, for all $i \in \{1, \ldots, r\}$;
$\rho_r = (\rho_1(g_r), \ldots, \rho_l(g_r)); \pi_r = (\pi_1(g_r), \ldots, \pi_e(g_r)$
$\Gamma = [\rho_{ij}]_{l \times r}$, where $\rho_{ij} = \rho_i(g_j)$ for $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, r\}$;
$\Pi = [\pi_{kj}]_{e \times r}$, where $\pi_{kj} = \pi_k(g_j)$ for $k \in \{1, \ldots, e\}$and $j \in \{1, \ldots, r\}$;
$C = \{(t_0, t) \in R \times R^r \mid \Pi t \geq \pi_r t_0, t \geq 0, t_0 \geq 0\}$;
$C_0 = \{t \in R^r \mid \Pi t \geq 0, t \geq 0\}$;
$P = \{t \in R^r \mid \Pi t \geq \pi_r, t \geq 0\}$.
For the point $(t_0, t) \in C$, we associate the subset of R^{r+1}defined by
$\{(x_0, v) \in R \times R^l \mid x_0 t_0 + (t\Gamma^T - t_0 \rho_r)v \leq tc\}$.
Finally, for a non-empty $Q \subseteq C$, we have

$$H(Q) = \bigcap_{(t_0,t) \in Q} \{(x_0, v) \in R \times R^l \mid x_0 t_0 + t(\Gamma^T v) - t_0(\rho_r v) \leq tc\}.$$

Benders decomposition algorithm for P_d problem is presented in Algorithm [1] which is the original Benders' method presented in [3] with some modifications. To bound region space, we consider in line 16 only the vectors contained in hypercube $[0, M]^{l+1}$for a given large real number M, because the space of subproblem (1) must be bounded as assumed in [3]. At line 28, dual of subproblem (2) can be obtained by a terminated simplex table. The extreme direction d^nin line 37 can be obtained by terminating the simplex table of subproblem (2) at line 22. Additionally, we put a condition at line 45 to avoid infinite loops when there is no modification in x_0^n and v^n.

We performed experiments using a notebook Lenovo g400s, with an Intel Core i3-3110M CPU @2.40GHz 64-bit processor, and 4GB RAM. The operational system was Ubuntu 18.04.1 LTS, and Kernel 4.15.0-36-generic. For Benders' method and random numbers generators algorithms, we respectively used Python 2.7.15rc1 and Python 3.6.5 with the following libraries: NumPy 1.15.1, SciPy 1.1.0, Pandas 0.23.4, and Matplotlib 2.2.3.

II. Experiments and Results

We performed experiments to compare the execution of Benders' method and the simplex without partitioning running only the simplex to solve P_d. So, we aim to find the best algorithm for the proposed problems.

In Figures 1,2,3 and 4, we plotted the objective function values of each 100 random test. In these figures, we ordered the tests by the absolute value of objective function values difference between Benders' decomposition and simplex without partitioning. So, we see almost 60% of tests for each *r* value used has little difference in objective functions. The simplex algorithm used in the experiments was *linprog* which is found in the *optimize* package of SciPy library. The classical Benders' method algorithm was developed in Python[1]

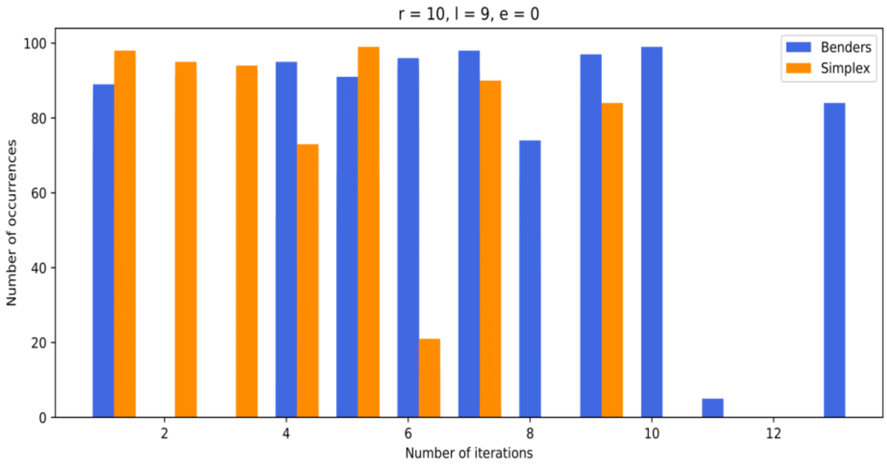

Figure 1. Number of occurrences versus number of iterations for different matrices sizes with r = 10.

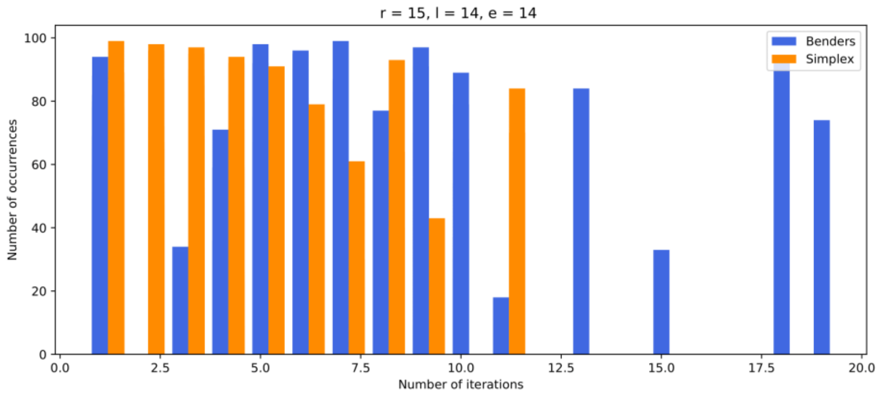

Figure 2. Number of occurrences versus number of iterations for different matrices sizes with r = 15.

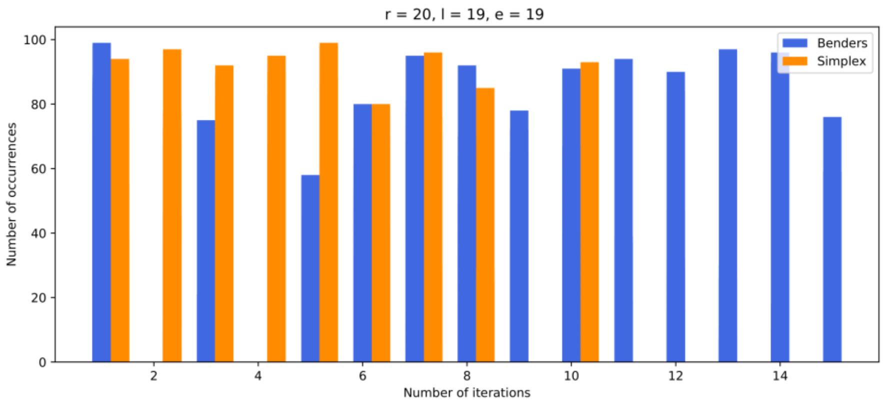

Figure 3. Number of occurrences versus number of iterations f or different matrices sizes with r = 20.

1https://github.com/yuri-tavares/benders-decomposition.

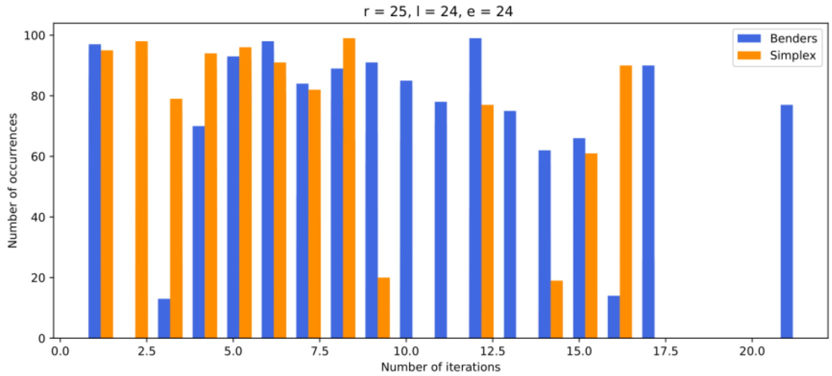

Figure 4. Number of occurrences versus number of iterations for different matrices sizes with r = 25

In Figures 5,6,7 and 8, we plotted the objective function values of each 100 random tests. In these figures, we ordered the tests by the absolute value of objective function values difference between Benders' decomposition and simplex without partitioning. So, we see almost 60 of the tests for each r value used has little difference in the objective function. However, we expected that no difference at all should exist.

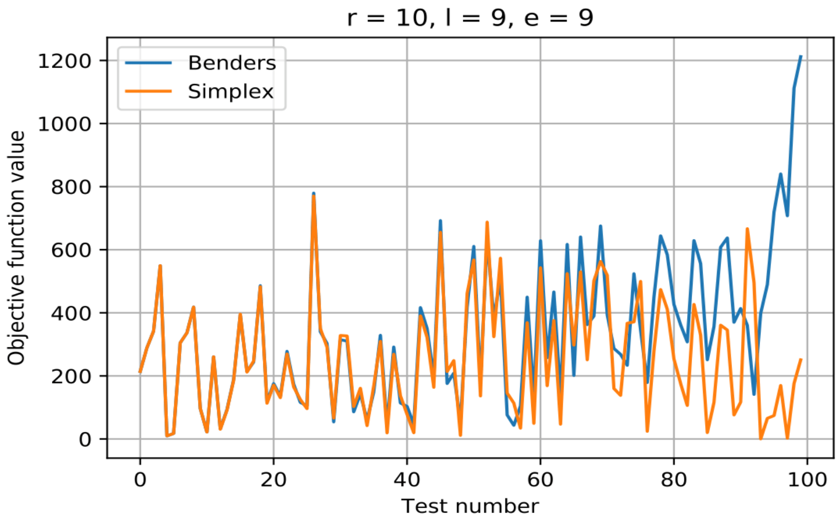

Figure 5. Objective function values ordered by absolute difference with r = 10.

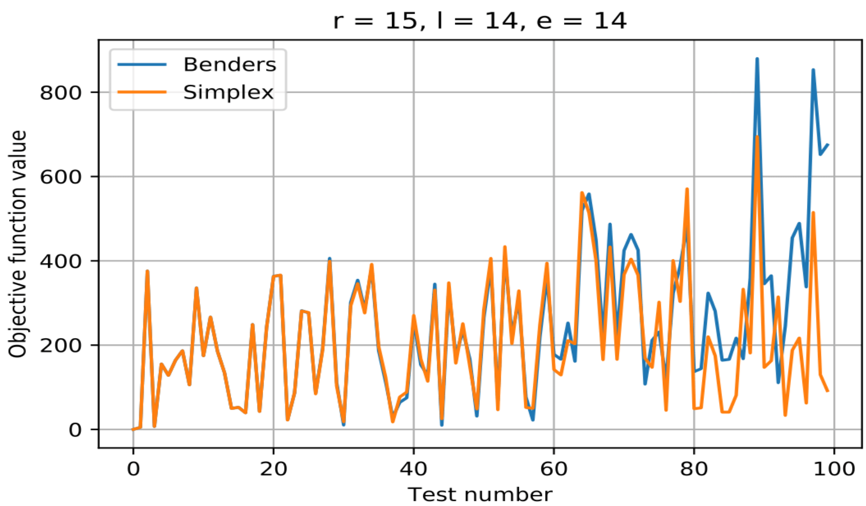

Figure 6. Objective function values ordered by absolute difference with r = 15

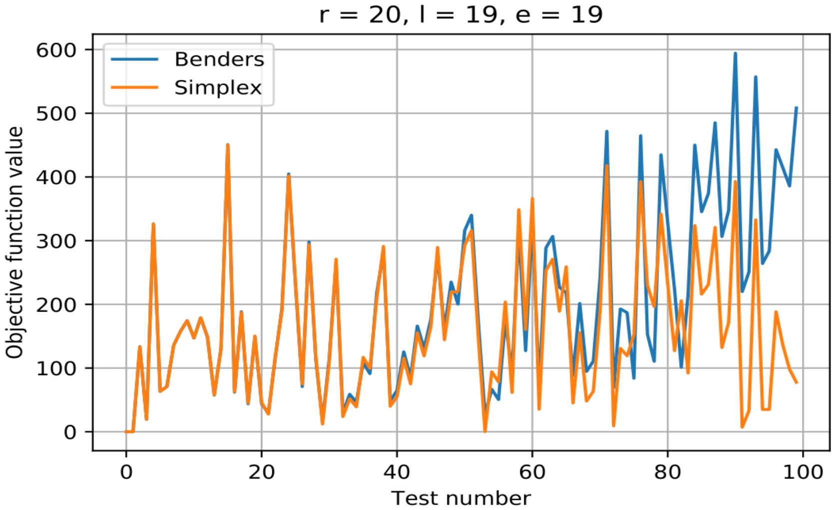

Figure 7. Objective function values ordered by absolute difference with r = 2.

Figures 9, 10, and 11 show the theoretical number of iterations in the worst case for r = 10, 20, and 30. We see for up to a certain number n, Benders' method has fewer number of iterations than the simplex. However, in almost all cases optimal objective function values differ, and, consequently, solution vectors do so. As shown in Figures 5 and 8 optimal objective function value difference significantly varies.

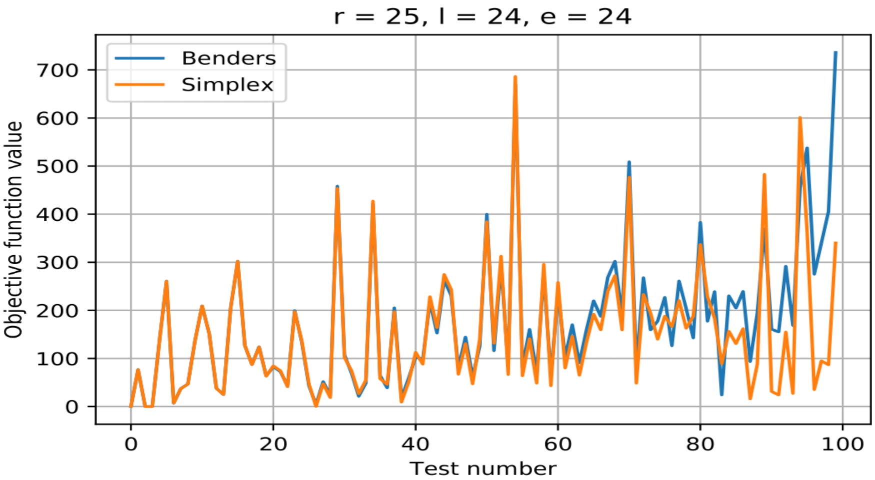

Figure 8. Objective function values ordered by absolute difference with r = 25.

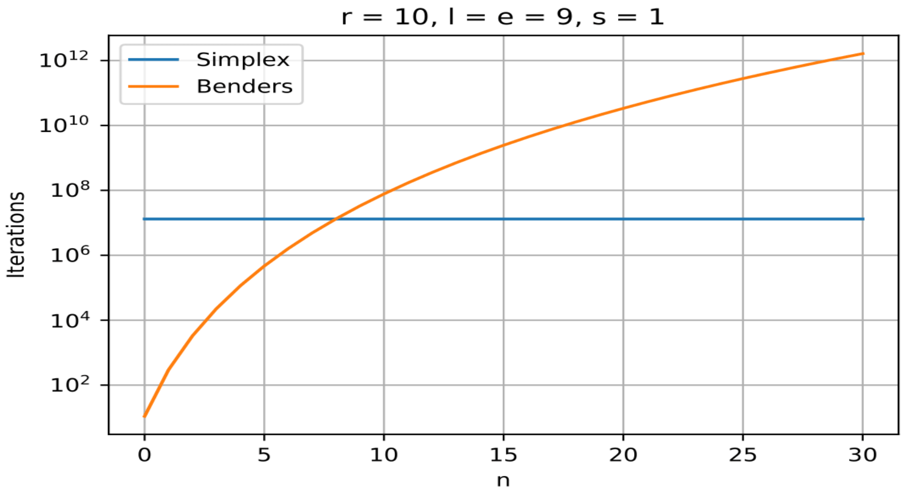

Figure 9. Theoretical number of iterations in the worst case for simplex without partitioning and Benders' method for r = 10.

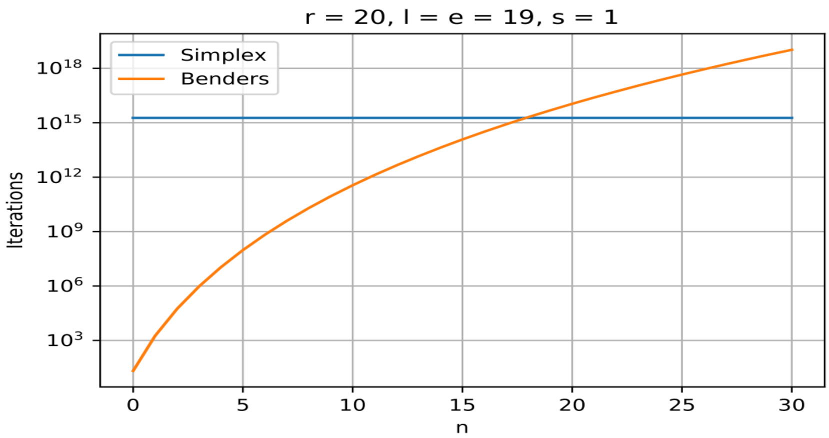

Figure 10. Theoretical number of iterations in the worst case for simplex without partitioning and Benders' method for r = 20.

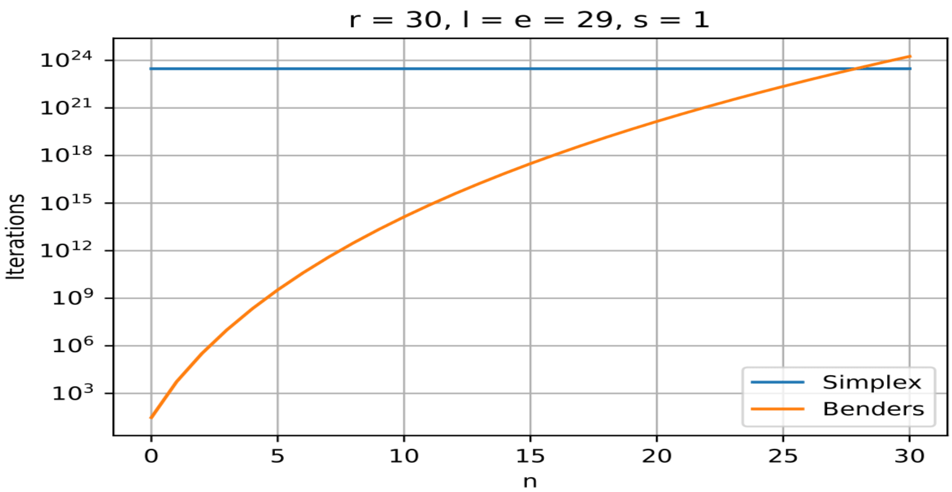

Figure 11. Theoretical number of iterations in the worst case for simplex without partitioning and Benders' method for r = 30.

Additionally, Benders' decomposition algorithm returns no feasible solution for some matrices, while simplex without partitioning returns a finite solution for these. One example of such matrices is shown in Figure 12. However, when we swap the roles of Γ and Π, Benders' decomposition algorithm gives the correct answer! From these experiments, we also infer that if the optimal value of the Benders coincides with the one of

Simplex, it is executed with a lesser or equal number of iterations, up to a certain *n*. For better optimization of the algorithm, this maximum *n* could be used as the stopping criterion, however, the random choice of submatrices can yield a different result by using Benders' decomposition. So, the simplex algorithm is better suited for the presented problem. All experiments were carried out with the premise that $e = l = r-1$. We executed 100 random tests for each r in {10, 15, 20,25}.

$$c = \begin{bmatrix} 66 \\ 334 \\ 745 \\ 595 \\ 409 \\ 854 \\ 465 \\ 91 \\ 380 \\ 893 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 908 & 965 & 583 & 45 & 883 & 455 & 184 & 816 & 398 & 198 \\ 57 & 589 & 935 & 141 & 191 & 410 & 889 & 419 & 458 & 813 \\ 947 & 500 & 454 & 354 & 254 & 113 & 661 & 469 & 297 & 583 \\ 983 & 855 & 951 & 857 & 984 & 664 & 632 & 547 & 370 & 954 \\ 218 & 230 & 268 & 550 & 31 & 67 & 893 & 418 & 764 & 646 \\ 615 & 259 & 21 & 664 & 482 & 577 & 375 & 440 & 411 & 962 \\ 419 & 92 & 595 & 428 & 383 & 189 & 158 & 552 & 713 & 302 \\ 696 & 205 & 27 & 336 & 768 & 186 & 328 & 460 & 513 & 539 \\ 707 & 753 & 934 & 782 & 766 & 205 & 793 & 756 & 478 & 916 \end{bmatrix}$$

$$\Pi = \begin{bmatrix} 983 & 823 & 652 & 720 & 354 & 497 & 514 & 296 & 135 & 6 \\ 548 & 934 & 534 & 472 & 802 & 262 & 587 & 889 & 497 & 678 \\ 77 & 148 & 913 & 273 & 596 & 399 & 455 & 991 & 74 & 73 \\ 44 & 687 & 696 & 451 & 861 & 554 & 336 & 17 & 743 & 381 \\ 605 & 511 & 162 & 568 & 709 & 638 & 412 & 512 & 249 & 796 \\ 586 & 10 & 308 & 581 & 15 & 857 & 473 & 325 & 394 & 695 \\ 618 & 586 & 567 & 66 & 668 & 713 & 89 & 58 & 291 & 457 \\ 132 & 157 & 358 & 858 & 478 & 949 & 223 & 954 & 216 & 338 \\ 679 & 170 & 332 & 187 & 187 & 473 & 80 & 402 & 226 & 810 \end{bmatrix}.$$

Figure 12. Example of c, Γ and Π which yields a contradictory result by using Benders' decomposition.

III. Conclusions

We presented an algorithm for solving b-complementary multisemigroup problems using Benders' decomposition and simplex algorithm. We assumed that the basis of the convex cone was given for this problem to solve it. It is still an open problem on how to get a basis of the convex cone from a b-complementary multisemigroup problem. We aim to solve this problem as future work.

Besides this open problem, we implemented computational experiments for comparing Benders' method and simplex without partitioning by assuming a basis of the convex cone was given. Two kinds of experiments were made, one for verifying closeness of solution and the number of iterations to obtain it, and others for analyzing the theoretical number of iterations in the worst case.

We observed in the first kind of experiment that results were not close to the solution using simplex without partitioning and almost all cases gave different results. Also, we have found out that the order of parameters given to Benders decomposition algorithm matters.

As observed from our tests for the theoretical number of iterations, we noted that as the size of matrices increases, there are numbers of iterations that the classical Benders' method returns the result faster than using simplex without partitioning in the worst case. By these experiments, Benders decomposition is faster up to a certain value that depends on input size (i.e., restriction matrices and objective function vector). Above this value, the results have shown that it is better to use simplex without partitioning.

Although the Benders' decomposition is faster for these values, there are inconsistencies in the results from the decomposition algorithm against the simplex algorithm, as shown in the first kind of experiments. So, we conclude that using the classical Benders decomposition algorithm does not work well for the b-complementary multisemigroup dual problem.

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Algorithm 1 The Algorithm from the P_d problem, part 1.

Input: $\rho_r \in \mathbb{R}^l$; $\pi_r \in \mathbb{R}^e$; $\Gamma \in \mathbb{R}^{l\times r}$; $\Pi \in \mathbb{R}^{e\times r}$; $c \in \mathbb{R}^r$; $M \in \mathbb{R}$, a large value.

Output: $(v^n, w^n) \in \mathbb{R}^l \times \mathbb{R}^e_+$: solution of P_d.

$n = 0$;
Select a finite set $Q^n \subset C$;
if $H(Q^n) = \emptyset$ **then**
    Algorithm terminates;
else
    if there are $t_0 > 0$ and $t \in \mathbb{R}^r_+$ such that $(t_0, t) \in Q^n$ **then**
        $solveMaxProblem = True$;
    else
        put $x_0 = +\infty$ and take any v^n such that $(x', v^n) \in H(Q^n)$ for any x';
        $solveMaxProblem = False$;
    end if
end if
$terminated = False$;
repeat
    if $solveMaxProblem$ **then**
        Solve the problem $max\{x_0 \mid (x_0, v) \in H(Q^n) \cap [0, M]^{l+1}\}$; (1)
        if problem (1) is not feasible **then**
            Algorithm terminates;
        end if
        Take (x_0^n, v^n) the optimal solution of the problem (1);
    end if
    Solve the problem $min\{(c - \Gamma^T v^n)t \mid \Pi t \geq \pi_r, t \geq 0\}$; (2)
    if problem (2) is not feasible **then**
        $terminated = True$;
    else $\triangleright$ continues in Algorithm 2.

Algorithm 2 The Algorithm from the P_d problem, part 2.

        if problem (2) has a finite optimal solution t^n; **then**
            if $(c - \Gamma^T v^n)t^n = x_0^n - \rho_r v^n$ **then**
                Solve the dual of the problem (2) and store this solution in w^n;
                Take (v^n, w^n) the optimal solution of problem P_d;
                $terminated = True$;
            else
                if $(c - \Gamma^T v^n)t^n < x_0^n - \rho_r v^n$ **then**
                    $Q^{n+1} = Q^n \cup \{(1, t^n)\}$;
                end if
            end if
        else
            Select a vertex $t^n \in P$ and extreme direction $d^n \in C_0$ such that $t^n + \lambda d^n \to \infty$, where $\lambda \to +\infty$;
            if $(c - \Gamma^T v^n)t^n \geq x_0^n - \rho_r v^n$ **then**
                $Q^{n+1} = Q^n \cup \{(0, d^n)\}$;
            else
                $Q^{n+1} = Q^n \cup \{(1, t^n), (0, d^n)\}$;
            end if
        end if
    end if
    if $solveMaxProblem$ **and** $x_0^{n-1} = x_0^n$ **and** $v^{n-1} = v^n$ **then**
        Take (v^n, w^n) the optimal solution of problem P_d;
    end if
    $solveMaxProblem = True$;
    $n = n + 1$;
until $terminated$

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