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An Explicit One-Step Method of an Order Eight Rational Integrator

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Abstract
In this work we derived and analyzed the convergence and consistency of an order eight rational integrator wherein our numerator and denominator is 4 ($i.e\ m = n = 4$) for the solution of problems in ordinary differential equations. A demonstration of the implementation of our integrator was also carried out; the result shows that our integrator is stable computationally. The integrator was observed to be A-stable, consistence and hence convergence.

Key Words: *Convergence, Consistency, Gaussian Elimination, Simultaneous Linear Algebra.*

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I. Introduction

Scientific Computing is the Mathematical Subject that deals with the use of computer to solve mathematical problems. The process involves

i. Analyzing the problem into a computable form
ii. Developing the Analysis into an algorithm
iii. Writing a Computer Programme in a Computer Programming Language based on the algorithm
iv. Running the programme to obtain Output Results and
v. Analysing the output for the work.

Luke et al (1975) opened the main stream researches in the use of rational approximating functions of the form:

$$R(x) = \frac{P_m(x)}{Q_n(x)} \tag{1.1}$$

where $P_m(x)$ and $Q_n(x)$ are polynomial functions of the same variable x, whose denominator degrees m and numerator degree n need not be unique for developing Rational Integrators. Herein we desire to avoid one of the methods that use the determinant of the matrix equation in arriving at the solution to our Simultaneous Linear Algebraic Equations (SLAE) where in this case the unknown variables are not very many to handle. We must herein state that whenever the unknown variables are more than three; the Gaussian Elimination Method (GEM) becomes much more attractive to users Derrick and Grossman (1987). For the purpose of this research we have 4 unknownq_1, q_2, q_3, q_4 and so we are employing the services of the Gaussian Elimination Method (GEM) (also known as row reduction). It is usually understood as a sequence of row operations performed on the associated matrix of coefficients. This method can also be used to find the rank of a matrix, to calculate the determinant of a matrix, and to calculate the inverse of an invertible square matrix. The method is named after Carl Friedrich Gauss (1777-1855), Wikipedia (2015).

According to Aashikpelokhai (1991), Elakhe and Aashikpelokhai (2010, 2011, 2013) this method represents an important family of implicit and explicit iterative methods for approximation of (ODEs) in numerical analysis especially in solving (IVPs) in (ODEs) of the form

$$y' = f(x, y), y(x_0) = y_0, a \le x \le b \tag{1.2}$$

For any 4×4 matrix of coefficients such as represented in (2.1) we employ the GEM by following the work in Elakhe (2011) and also Elakhe et al (2011) whose work on order 4 based denominator with $m = 0$ arrived with a new formula after a very exhaustive detailed analysis. Tejumola (1971, 1975, 1988), Tejumola and Ezeilo (1979) Tejumola and Afuwape (1990), Ebiendele (2010, 2011, 2013), Ebiendele and Okodugha (2013), Afuwape et al (2007) concentrated their work on the theoretical solutions in (ODEs) whose result on lurie systems we will follow to achieve our goal, here their major aim was centered on the nonlinearities of the equilibrium state of the degenerate systems. Here in this research we wish to derive a singulo-stiff numerical rational integrator, study its stability and determine the nature of the stability function.

As our work is concerned with the stability function of the eight order rational integrator, we would ensure that there is a theoretical guarantee of its workability before future testing, this assurance is obtained by proving consistency and convergence. Works of Inegbedion (1995), Tian and Kuang (1996), Papakostas et all (1996), Schroll (1996), In'tHout (1996), Barry et al (1996), David et al (1996), Vozovol et al (1996), Momodu (1997), Irhumudomon (1997), Enaholo (2000), Uzor (2000), did not carry out implementation. They concentrated on the theoretical aspect of their works. We cite just a few here to justify this non-implementation work.

In an attempt to extend the approximation method of Euler, Runge in 1895 elaborated on Euler method to give a more elaborate scheme which was capable of greater accuracy. The requirement of evaluating the derivative at the midpoint or endpoint of a step not yet completed was achieved by first performing an Euler type of calculation to obtain a preliminary approximation to the solution at one of these points. Exponential integrators are among the integrators that have become an active area of research, which originally was developed for solving stiff differential equations and also partial differential equations which include hyperbolic as well as parabolic problems such as heat. They are a class of numerical methods for the solution of partial and ordinary differential equations. This deals with the exact integration of the linear part of the initial value problem from numerical analysis. They can be constructed to be explicit or implicit for numerical ordinary differential equations or serve as the time integrator for numerical partial differential equations. Examples of published works in this area include the work of Fatunla (1978, 1980).

This research work however, is aimed at creating and applying a new integration approach to solve these classes of problems. We shall also be examining the stability structure of the new integration method.

Terms and Notations

Definition: (Dahlquist, 1963)
A numerical method is said to be A-Stable if its Region of Absolute Stability (RAS) contains the whole of the left-hand half of the complex plane i.e. $Re(\bar{h}) < 0$.

Definition: (Lambert, 1973, 1974)
A numerical integrator is said to be Absolutely Stable if the absolute value of the stability function $\varsigma(\bar{h})$ is less than unity. That is,

$$\left|\varsigma(\bar{h})\right| = \left|\varsigma(u+iv)\right| < 1, \quad i=\sqrt{-1} \tag{1.3}$$

Definition: Region of Absolute Stability (RAS) (Lambert, 1976)
A region D of the complex plane is said to be a Region of Absolute Stability (RAS) of a given method, if the method is absolutely stable for $\bar{h} \in D$.

Definition: (Lambert, 1973)
A given one-step method is said to be L-stable if it is A-stable and in addition,

$$\lim_{Re(\bar{h}) \to -\infty} \left|\varsigma(\bar{h})\right| = 0. \tag{1.4}$$

Definition: (Aashikpelokhai 1991)
The function $f(x,y)$ is said to satisfy a Lipschitz condition in y, over the region D, if there exist a constant L such that

$$\| f(x,y_1) - f(x,y_2) \| \leq L \| y_1 - y_2 \| \tag{1.5}$$

In this case, L is called the Lipschitz constant and $f(x,y)$ is said to be Lipschitzian.
By virtue of the relation

$$\frac{\partial f(x,y)}{\partial y} = \lim_{(y_1-y_2)\to 0} \frac{f(x,y_1)-f(x,y_2)}{y_1-y_2} \tag{1.6}$$

Consequently, $\frac{\partial f(x,y)}{\partial y}$ becomes a ready tool for the computation of L. Thus, we can simply write

$$L = \frac{||\partial f(x,y)||}{\partial y} \tag{1.7}$$

II. Existence and Uniqueness of IVPs (ODEs) Solutions

Theorem: (Aashikpelokhal et al 2010b)
The initial value first order linear differential equation

$$a_1(x)\frac{dy}{dx} + a_0(x)y = h(x), a_1(x) \not\equiv 0$$

$$y(x_0) = y_0, a \leq x \leq b$$

has a unique solution in the interval $[a,b]$ in which it is defined on the real line.

Proof: (Aashikpelokhai et al 2010b)
By the method of integrating factor, we obtain the general solution

$$y = \left[exp\left(-\int \frac{a_0(x)}{a_1(x)}dx\right)\right]\left[A + \int\left\{\frac{h(x)}{a_1(x)}exp\left(\int\frac{a_0(x)}{a_1(x)}dx\right)\right\}dx\right] \quad (2.1)$$

where A is the integrator constant.

Existence

Select any point $x = x_0 \ in \ [a,b]$ and the value $y = y_0$ along the y-axis. Substitute the pair (x_0, y_0) into (1.6.1.1), solve for the constant A.

This value of A yields a particular solution $y = y(x)$ obtained from (2.1) for every choice of arbitrary $x = x_0$in the interval $[a,b]$ and any $y = y_0$values chosen along the y-axis, when the pair (x_0, y_0) is substituted into the result (2.1) we obtain a new particular A which in turn yields a corresponding new solution.

Hence, every initial value problem (ivp) above has at least one solution in the interval $[a,b]$.

Uniqueness

To prove uniqueness, we wish to prove that if any two solutions are given, then they must be identical. Well then, let y_1, y_2be such solutions of the given ivp. In this case we have for each $i = 1, 2$.

$$a_1(x)\frac{dy_i}{dx} + a_0(x)y_i = h(x), a_1(x) \not\equiv 0, \quad (2.2)$$

implying, by linearity of the differential operator

$$a_1(x)\frac{d(y_1-y_2)}{dx} + a_0(x)(y_1 - y_2) = 0 \ a_1(x) \not\equiv 0 \quad (2.3)$$

and $(y_2 - y_1)(x_0) = y_2(x_0) - y_1(x_0) = 0$ (2.4)

Hence $y_2 - y_1$ is a solution of the homogenous ivp

$$a_1(x)\frac{dy}{dx} + a_0(x)y = 0, \qquad a_1(x) \not\equiv 0$$
$$y(x_0) = 0$$

But by $y_h = Ae^{-\int p(x)dx}$the solution to this homogenous ivp is given by

$$y = A \ exp\left(-\int\frac{a_0(x)}{a_1(x)}dx\right) \quad (2.5)$$

where A is our arbitrary constant of integration.

Hence, therefore

$$(y_2 - y_1) = A \ exp\left(-\int\frac{a_0(x)}{a_1(x)}dx\right) \quad (2.6)$$

$$\therefore y = y_2 - y_1 \quad (2.7)$$

Substituting $y(x_0) = 0$ into our equation (2.5) we obtain

$$A \ exp\left(-\int\frac{a_0(x)}{a_1(x)}dx\right) = 0$$

but then,

$A \ exp\left(-\int\frac{a_0(x)}{a_1(x)}dx\right) \not\equiv 0$for every value of x on the real line. Hence, $A = 0$, meaning that in (2.5) we now have $y = 0$ as the solution to the ivp.

But by (2.5), $y = y_2 - y_1$

Hence, $y_2 - y_1 = 0$ and so $y_2 = y_1$

A Multinomial Result

Result for all real a_i

Algebraic

$$\left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \quad 1 \le i,j \le n \quad (2.8)$$

$$e.g. (a_1 + a_2 + a_3 + a_4 + a_5)^2$$
$$= a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + 2[a_1a_2 + a_1a_3 + a_1a_4 + a_1a_5 + a_2a_3 + a_2a_4 + a_2a_5$$
$$+ a_3a_4 + a_3a_5 + a_4a_5 \quad (2.9)$$

Theorem1.2.2.1: (Aashikpelokhai et al 2010a)

Let $a_i, \ i = 1(1)n$ be any real or complex function, then$(a_1 + a_2 ... + a_n)^2 = 2\sum a_i a_j$, $1 \le i < j \le n$.

Proof

We employ the method on n. for $n = 1$ we have

$$LHS = a_1^2$$

$RHS = a_1^2 + 2\sum a_i a_j$, $1 \le i < j \le n$.

But $1 \nless 1 \therefore$ there is no a_j which make $i < j$

$\therefore a_j = 0$and so $RHS = a_1^2$

$$ie \ LHS = a_1^2 = RHS = a_1^2$$

So the formula is true for $n = i$
Assume the formula true for arbitrary positive integer $k > 1$ then
$(a_1 + a_2 + \cdots + a_k)^2 = (a_1^2 + a_2^2 + \cdots + a_k)^2 + 2\sum a_i a_j \text{ , } 1 \le i < j \le n.$ (2.10)
meaning that

$$(a_1 + a_2 + \cdots + a_k)^2 = a_1^2 + a_2^2 + \cdots + {a_k}^2 + 2[(a_1a_2 + a_1a_3 + \cdots + a_1a_k) + (a_2a_3 + a_2a_4 + \cdots + a_2a_k) + \cdots + a_{k-1}a_k]$$

Next we consider

$$(a_1 + a_2 + \cdots + a_k + a_{k+1})^2 = (A + a_{k+1})^2 \text{ , } where\ A = (a_1 + a_2 + \cdots + a_k)^2.$$

$and\ (A + a_{k+1})^2 = A^2 + {a_{k+1}}^2 + 2Aa_{k+1}$ (2.11)

$$ie\ (a_1 + a_2 + \cdots + a_k + a_{k+1})^2 = (a_1 + a_2 + \cdots + a_k)^2 + {a_{k+1}}^2 + 2(a_1 + a_2 + \cdots + a_k)a_{k+1}$$

By induction step
$= \sum_{i=1}^{k} a_i^2 + 2\sum_{i<j} a_i a_j + {a_{k+1}}^2 + 2(a_1a_{k+1} + a_2a_{k+1} + \cdots + a_ka_{k+1})$ (2.12)
Observe that, the last term here represents the extra terms, each needed to bring them to $(k+1)th$ terms, namely
i) $\sum_{i=1}^{k} a_i^2 + a_{k+1} = \sum_{i=1}^{k+1} a_i^2$ (2.13)
ii) $2\sum_{i<j}^{k} a_i a_j + 2(a_1a_{k+1} + a_2a_{k+1} + \cdots + a_ka_{k+1})$
$= 2[(a_1a_2 + a_1a_3 + \cdots + a_1a_k + a_1a_{k+1}) + (a_2a_3 + a_2a_4 + \cdots + a_2a_k + a_2a_{k+1})$
(1.2.2.7)
$(a_3a_4 + a_3a_5 + \cdots + a_3a_k + a_3a_{k+1}) + \cdots + a_ka_{k+1}] = 2\sum_{i<j}^{k+1} a_i a_j$ (2.14)
$\therefore$ The induction step k has made true
$(a_1 + a_2 + \cdots + a_k + a_{k+1})^2 = \sum_{i=1}^{k+1} a_i^2 + 2\sum_{1\le i<j\le k} a_i a_j$ (2.15)
but$k$ was chosen arbitrarily.
$\therefore$the multinomial theorem is true for all $n \in \mathbb{Z}^+$.

***Padé* Approximants (Aashikpelokhai et al 2010)**

The subject of *Padé* approximants dates back to as early as Cauchy (1759-1857) and Jacobi (1804-1851). Frobenius gave a detailed investigation of the algebraic properties of *Padé* approximants in 1881. *Padé* in his dissertation of the Ecole Nor male Suprrieure, classified these rational fraction approximants (now known as *Padé* approximants) arranged them in a table now known as the *Padé* table and investigated the structure of the table as well as special properties of the approximants to e^x.
The Linear *Padé* Approximant according to Aashikpelokhai (1991) has the form (1.1).
where$P_m(x)$ is the polynomial $\sum_{i=0}^{m} p_i x^i$
and$Q_n(x)$ is the polynomial $\sum_{i=0}^{m} q_i x^i$.
The important aspect of *Padé* approximants is the *Padé* Table.
By *Padé* Table we mean the array

$$\begin{bmatrix} [1/1] & [1\ \ /2] & [1/3] & [1/4] \\ [2/1] & [2/2] & [2/3] & [2/4] \\ [3/1] & [3/2] & [3/3] & [3/4] \\ [4/1] & [4/2] & [4/3] & [4/4] \end{bmatrix}$$

III. Derivation Of Our Method

The GEM and Our Rational Integrator

The work done in the first major stages shows that row 1 of the given matrix equation was left untouched while the elements in column 1 of row 2, row 3 and row 4 became zero each.
Consequently, by noting result (1.6) a comprehensive picture of the resulting matrix expressed above as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & d_{22} & d_{23} & d_{24} \\ 0 & d_{32} & d_{33} & d_{34} \\ 0 & d_{42} & d_{43} & d_{44} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \quad (3.1)$$

which represents the matrix equation form at the end of MAJOR STEP 1 and as kept in the computer memory. As we carry this computer picture in our minds while moving to the beginning of the second major step, we focus, as demanded by the GEM on the 3x3 matrix to stand in the place of the original matrix equation (3.1).
in this chapter we are handling the implementation of the derivation work done in chapter 2 precisly. We now employ the derived results in (2.1.7 – 2.1.14) to state our implementation algorithm for obtaining the matrix form (2.1.6) above.

ALGORITHM FOR d_{ij} and e_i IN TERMS OF a_{ij} and b_i

For row i from 1 to 3 do
  For column j from 1 to 3 do
  P_j ivot = $\frac{a_{1,j+1}}{a_{11}}$
  $d_{ij} = a_{i+1,j+1} - a_{i+1,1} * P_j$ ivot
  End do
  P_4ivot = $\frac{b_1}{a_{11}}$
  $e_i = b_{i+1} - a_{i+1,1} * P_4$ivot
End do

Next, we pick on the derivation work done in the second major stages of the GEM derivation and note that while we focus on the matrix results, we recall that the derivation shows us that at the end of the second major step, row 1remained untouched while row 2 and row 3 entries in its column 1 become zero each. A comprehensive matrix picture that results at the end of the second major step is given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & d_{11} & d_{12} & d_{13} \\ 0 & 0 & f_{11} & f_{12} \\ 0 & 0 & f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ e_1 \\ g_1 \\ g_2 \end{bmatrix} \qquad (3.2)$$

This is the form now in the computer memory. This form is carried over to the beginning of 3rd major step. This then leads us to obtain our implementation algorithm for obtaining f_{ij} and g_i in terms of d_{ij} and e_i as state here under.

ALGORITHM FOR f_{ij} and g_i IN TERMS OF d_{ij} and e

For row i from 1 to 2 do
  For column j from 1 to 2 do
  $P_j ivot = \frac{d_{1,j+1}}{d_{11}}$
  $f_{ij} = d_{i+1,j} - d_{i+1,1} * P_j ivot$
  End do
  $P_3 ivot = \frac{e_1}{d_{11}}$
  $g_i = e_{i+1} - d_{i+1} * P_3 ivot$
End do

We at this last stage pick on the derivation work done in the third major stage for our implementation as follows. As we carried the result (3.2) with us from the last major step. The GEM left row 1 of the initial matrix equation is untouched while the transformation in the GEM left column 1 of its row 2 with zero entry. The resulting final matrix equation at the end of the third major stage is stated here as; this form is what now exists in the computer memory – our Upper Triangular Matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & d_{11} & d_{12} & d_{13} \\ 0 & 0 & f_{11} & f_{12} \\ 0 & 0 & 0 & \left\{f_{22} - f_{21}\frac{f_{12}}{f_{11}}\right\} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ e_1 \\ g_1 \\ g_2 - f_{21}\frac{g_1}{f_{11}} \end{bmatrix} \qquad (3.3)$$

Note that the matrix equation (3.3) obtained here is the required Upper Triangular Matrix (UTM). At this stage, knowing fully well that the matrix equation entries for the UTM are fully known, we use the last row to obtain the value directly for q_4 and the use the back substitution to get q_4, q_3, q_2 and $q_1$in that order.

Indeed, employing the fourth (last) row, we get

$$\left(f_{22} - \frac{f_{21}f_{12}}{f_{11}}\right) g_2 = g_2 - \frac{f_{21}f_{12}}{f_{11}}$$

which then yields

$$q_4 = \frac{g_2 f_{11} - g_1 f_{21}}{f_{22}f_{21} - f_{21}f_{12}} \qquad (3.4)$$

row 3 gives

$$q_3 = \frac{g_1 - f_{12}q_4}{f_{11}} \qquad (3.5)$$

row 2 gives

$$q_2 = \frac{e_1 - d_{12}q_3 - d_{13}q_4}{e_{11}} \qquad (3.6)$$

row 1 gives

$$q_1 = \frac{b_1 - a_{12}q_2 - a_{13}q_3 a_{14}q_4}{a_{11}} \qquad (3.7)$$

From these results, one can at any time in that order produce $q_i x_{n+1}^i$ $i = 4,3,2,1$. In order to obtain the final form of the numerator of the primitive form of the integrator

$$y_{n+1} = \frac{p_0 + p_1 x_{n+1} + p_2 x^2{}_{n+1} + p_3 x^3{}_{n+1} + p_4 x^4{}_{n+1}}{1 + q_1 x_{n+1} + q x^2{}_{n+1} + q_3 x^3{}_{n+1} + q_4 x^4{}_{n+1}} \quad (3.8)$$

Re-write (3.2.40) – (3.2.44) on page 22 as follows

$$p_0 = y_n \quad (3.9)$$

$$p_1 x_{n+1} = y_n q_1 x_{n+1} + h y_n^{(1)} \quad (3.10)$$

$$p_2 x_{n+1}^2 = y_n q_2 x_{n+1}^2 + h y_n^{(1)} q_1 x_{n+1} + \frac{h^2 y_n^{(2)}}{2!} \quad (3.11)$$

$$p_3 x_{n+1}^3 = y_n q_3 x_{n+1}^3 + h y_n^{(1)} q_2 x_{n+1}^2 + \frac{h^2 y_n^{(2)}}{2!} q_1 x_{n+1} + \frac{h^3 y_n^{(3)}}{3!} \quad (3.12)$$

$$p_4 x_{n+1}^4 = y_n q_4 x_{n+1}^4 + h y_n^{(1)} q_3 x_{n+1}^3 + \frac{h^2 y_n^{(2)}}{2!} q_2 x_{n+1}^2 + \frac{h^3 y_n^{(3)}}{3!} q_1 x_{n+1} + \frac{h^4 y_n^{(4)}}{4!} \quad (3.13)$$

The sum $\sum_{i=0}^{4} p_i x_{n+1}^i$usinng (3.3.9) – (3.3.13) yields

$$p_0 + p_1 x_{n+1} + p_2 x_{n+1}^2 + p_3 x_{n+1}^3 + p_4 x_{n+1}^4$$

$$= \sum_{r=0}^{4} \frac{h^r y_n^{(r)}}{r!} + \left(\sum_{r=0}^{3} \frac{h^r y_n^{(r)}}{r!}\right) q_1 x_{n+1} + \left(\sum_{r=0}^{2} \frac{h^r y_n^{(r)}}{r!}\right) q_2 x_{n+1}^2 + \left(\sum_{r=0}^{1} \frac{h^r y_n^{(r)}}{r!}\right) q_3 x_{n+1}^3 + y_n q_4 x_{n+1}^4$$

$$ie \sum_{i=0}^{4} p_i x_{n+1}^i = T_4 + T_3 q_1 x_{n+1} + T_2 q_2 x_{n+1}^2 + T_2 q_3 x_{n+1}^3 + y_n q_4 x_{n+1}^4$$

$\therefore$ The numerator $\sum_{i=0}^{4} p_i x_{n+1}^i$ of the primitive form of the integrator is given by

$$\sum_{i=0}^{4} p_i x_{n+1}^i = T_4 + T_3 q_1 x_{n+1} + T_2 q_2 x_{n+1}^2 + T_2 q_3 x_{n+1}^3 + y_n q_4 x_{n+1}^4 \quad (3.14)$$

Consequently, we write our final form of the eight order rational integrator

$$y_{n+1} = \frac{T_4 + T_3 q_1 x_{n+1} + T_2 q_2 x_{n+1}^2 + T_2 q_3 x_{n+1}^3 + y_n q_4 x_{n+1}^4}{1 + q_1 x_{n+1} + q_2 x_{n+1}^2 + q_3 x_{n+1}^3 + q_4 x_{n+1}^4}$$

where the $q_i, i = 4, 3, 2, 1$ are obtained in that order by the GEM results (3.4) – (3.7).
Next, we produce our algorithm relating the matrix equation equivalent entries as follows.
For neat programming we separate the coefficient work as follows

For r from 1 to m do

$$c_r \leftarrow \left(\frac{h}{x_{n+1}}\right)^r \frac{y_n^{(r)}}{r!}$$

End do

Employing our equivalent matrix equations, we observe that

$$a_{ij} = c_{9-(i+j)} \; and \; b_i = -c_{9-i}$$

Therefore, having completed the coefficients $c_{r's}$ above we can conviently use them for the matrix **A** and vector **b** entries as follows

Algorithm for Matrix A and Vector b

For i from 1 to 4 do
  For j from 1 to 4 do
    $IRC = 9 - (i + j) or\; 9 - i - j\; (choose\; any\; of\; them)$
   $a_{ij} = c(IRC)x$
  End do
  $IR = 9 - i$
  $b_i = -c(IR)$
End do

Algorithm Relations A(Ij), B(I) With The Coefficients

By the equation results gotten, we have

$$a_{ij} = c_{9-(1+j),} i, j = 1(1)4 \quad (3.15)$$

Examples: $a_{11} = c_7 = c_{9-2,} a_{13} = c_{9-5} = c_4$
while

$$b_i = -c_{9-i}, i = 1(4) \quad (3.16)$$

The result stated in gave us

$c_r = \frac{h^r y_n^{(r)}}{r!\, x_{n+1}^r}$for each r a nonnegative integer.

Note that the 9 in the relations (3.15) and (3.16) is related to the integrator rational approximant degrees sum $= 4$ from numerator and 4 from denominator which we all order $p = 8$ $\therefore$

$$a_{ij} = c_{p+1-(i+j),} (i, j) = 1(1)4\,. \quad (3.17)$$

similarly for $b_i,\; i = 1(1)4$.

From the foregoing one can put down the algorithm for entering the coefficients for $\boldsymbol{Ax} = \boldsymbol{b}$ where $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{b} = (b_i)$ $i = 1(1)4$

Algorithm for Matrix A and Vector b Entries

For row i from 1 to 4 do

$$b_i = -\frac{h^{9-i} y_n^{(9-i)}}{(9-i)!\, x_{n+1}^{9-i}}$$

For column j from 1 to 4 do

$$a_{ij} = \frac{h^{9-(i+j)} y_n^{(9-(i+j))}}{(9-(i+j)!\, x_{n+1}^{9-(i+j)}}$$

End do

End do

For direct hand or desk computation we write

$$a_{ij} = \frac{h^{9-(i+j)} y_n^{(9-(i+j))}}{(9-(i+j)!\, x_{n+1}^{9-(i+j)}}\ i, j = 1(1)4. \quad (3.18)$$

$$b_i = -\frac{h^{9-i} y_n^{(9-i)}}{(9-i)!\, x_{n+1}^{9-i}}\ i = 1(1)4 \quad (3.19)$$

We conclude this chapter on implementation strategy by drawing up our Algorithm

Implementation Algorithm

1. Enter or initialize initial entries x_0, y_0, ϵ, big etc
2. Set $x_1 = x_0 + h$
3. Obtain higher derivatives of $y^{(1)} = f(x, y), y(x_0) = y_0$
4. Obtain the coefficients $c_{r's}$
5. Enter $a_{ij} \in \boldsymbol{A}$ *and* $b_i \in \boldsymbol{b}$
6. Compute $d_{ij} \in \boldsymbol{D}$ and $e_i \in \boldsymbol{e}$
7. Compute $f_{ij} \in \boldsymbol{F}$ *and* $g_i \in \boldsymbol{g}$
8. Use the GEM results (3.4) – (3.7) to compute q_4, q_3, q_2, q_1 in that order by back substitution.
9. Obtain Taylor, T_n, series for y_{n+1}
10. Compute your denominator $1 + \sum_{i=1}^{4} q_i x_{n+1}^i = Denom$
11. Compute your numerator

a. $T_4 + T_3 q_1 x_{n+1} + T_2 q_2 x_{n+1}^2 + T_2 q_3 x_{n+1}^3 + y_n q_4 x_{n+1}^4$

12. If $|denom| > \epsilon$ then compute $y_1 = \frac{Num}{Denom}$ else make a comment you are at a singular point or simply leave the zone without comment
13. Print the set result if desired
14. Update your variables: $x_0 \leftarrow x_1$, $y_0 \leftarrow y_1$ return to step 2
15. End

IV. Convergence and Consistency Analysis of the Integrator

Every one-step method for numerical solution of ivp

$$y_{n+1} = y_n + h\emptyset(x_n, y_n; h) \quad (4.1)$$

is convergent if and only if

$$\lim_{h\to 0}\left\{\frac{y_{n+1} - y_n}{h}\right\} = \lim_{h\to 0} \emptyset(x_n, y_n;\ h) = f(x_n, y_n) \quad (4.2)$$

1. where $\emptyset(x_n, y_n;\ h)$ is commonly referred to as the potential function of the numerical method Momodu (1997).

Theorem

An Explicit One-Step Method of Rational Integrators is consistent and convergent.

Proof: Method 1 (Direct)

From the preliminary discussion above, we must show that

$$f(x_n, y_n) = \emptyset(x_n, y_n;\ 0) = \lim_{h\to 0}\left\{\frac{y_{n+1} - y_n}{h}\right\}$$

$$= \lim_{h\to 0} \emptyset(x_n, y_n; h) = \emptyset(x_n, y_n;\ 0) = f(x_n, y_n)$$

for our rational integrator.

Indeed, since

$$y_{n+1} = \frac{p_0 + p_1 x_{n+1} + p_2 x^2{}_{n+1} + p_3 x^3{}_{n+1} + p_4 x^4{}_{n+1}}{1 + q_1 x_{n+1} + q x^2{}_{n+1} + q_3 x^3{}_{n+1} + q_4 x^4{}_{n+1}}$$

where

$q_4 = \frac{g_2 f_{11} - g_1 f_{21}}{f_{22} f_{21} - f_{21} f_{12}} q_3 = \frac{g_1 - f_{12} q_4}{f_{11}}$

$q_2 = \frac{e_1 - d_{12} q_3 - d_{13} q_4}{e_{11}}, \quad q_1 = \frac{b_1 - a_{12} q_2 - a_{13} q_3 a_{14} q_4}{a_{11}}$

so that:

$$y_{n+1} - y_n = \frac{(y_n + p_1 x_{n+1} + p_2 x^2{}_{n+1} + p_3 x^3{}_{n+1} + p_4 x^4{}_{n+1}) - y_n(1 + q_1 x_{n+1} + q x^2{}_{n+1} + q_3 x^3{}_{n+1} + q_4 x^4{}_{n+1})}{1 + q_1 x_{n+1} + q x^2{}_{n+1} + q_3 x^3{}_{n+1} + q_4 x^4{}_{n+1}} \quad (4.3)$$

that is

$$y_{n+1} - y_n = \frac{(p_1 - y_n q_1) x_{n+1} + (p_2 - y_n q_2) x^2{}_{n+1} + (p_3 - y_n q_3) x^3{}_{n+1} + (p_4 - y_n q_4) x^4{}_{n+1}}{1 + q_1 x_{n+1} + q x^2{}_{n+1} + q_3 x^3{}_{n+1} + q_4 x^4{}_{n+1}} \quad (4.4)$$

Dividing through by h

$x_{n+1} = (n+1)h$,we obtain

$$\frac{y_{n+1} - y_n}{h} = \frac{(p_1 - y_n q_1)(n+1) + (p_2 - y_n q_2)(n+1)^2 h + (p_3 - y_n q_3)(n+1)^3 h^2 + (p_4 - y_n q_4)(n+1)^4 h^3}{1 + q_1 h(n+1) + q_2 h^2 (n+1)^2 + q_3 h^3 (n+1)^3 + q_4 h^4 (n+1)^4} \quad (4.5)$$

In the limit $h \to 0$, gives

$$\lim_{h \to 0} \left(\frac{y_{n+1} - y_n}{h} \right) = (p_1 - y_n q_1) n + 1$$

But recalling

$p_1 - y_n q_1 = \frac{h\, y_n{}^{(1)}}{x_{n+1}}$hence,

$= \frac{y_n{}^{(1)}}{n+1}$

Which gives

$\lim_{h \to 0} \left(\frac{y_{n+1} - y_n}{h} \right) = y_n{}^{(1)} = f(x_n, y_n)$as required.

Conclusively, we report herein that our order eight rational integrator is convergent to the initial value problem and hence by Lambert (1973) every one-step numerical integrator is consistent if and only if it is convergent.

Hence, our order eight rational integrator is convergent and hence consistent.

$h = 0.0250$						
STEP	H	X0	X1	Y1	TSOL	INTE ERROR
1	0.0250	0.0000	0.0250	1.0513D+00	1.0513D+00	6.3800D-10
2	0.0250	0.0250	0.0500	1.1054D+00	1.1054D+00	1.8630D-09
3	0.0250	0.0500	0.0750	1.1625D+00	1.1625D+00	3.9634D-09
4	0.0250	0.0750	0.1000	1.2230D+00	1.2230D+00	7.2088D-09
5	0.0250	0.1000	0.1250	1.2874D+00	1.2874D+00	1.2093D-08
6	0.0250	0.1250	0.1500	1.3561D+00	1.3561D+00	1.9564D-08
7	0.0250	0.1500	0.1750	1.4296D+00	1.4296D+00	3.2288D-08
8	0.0250	0.1750	0.2000	1.5085D+00	1.5085D+00	5.7186D-08
9	0.0250	0.2000	0.2250	1.5936D+00	1.5936D+00	1.1820D-07
10	0.0250	0.2250	0.2500	1.6858D+00	1.6858D+00	5.6754D-07
11	0.0250	0.2500	0.2750	1.7861D+00	1.7861D+00	-3.0384D-07

Table 1: Demonstrating using $y^{(1)} = 1 + y^2, y(0) = 1 \; 0 \leq x \leq 1$

V. Conclusions

Using GEM, we have established an effective order 8 rational integrator. We have determined the stability nature of our formula. We also had proved that the method is convergent, consistent and A - stable in which case it can be used to carry out implementation. Hence, this integrator is fully recommended for users who are currently working in this area of research.

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Elakhe, Aliu and Ebiendele together worked at every stage of this research to produce this beautiful work.

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