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Equality of Internal Angles and Vertex Points in Conformal Spherical Triangles

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Abstract
In this study, by using the conformal structure in Euclidean space, the conformal structures in spherical space and the equality of the internal angles and vertex points of conformal triangles in spherical space are given. Especially in these special conformal triangles, the conformal spherical equilateral triangle and the conformal spherical isosceles triangle, the internal angles and vertices are shown.
Keywords: *Conformal spherical triangle, Conformal spherical isosceles triangle, Conformal spherical equilateral triangle*

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I. Introduction

The set $S_1^n = \{x \in R_1^{n+1} : <x,x>=1\}$ is also called the n-dimensional unit pseudo-spherical space. Standard model for n-dimensional spherical geometry, $S^n = \{x \in IR^{n+1} : \|x\| = 1\}$ defined as S^n is the unit sphere of R^{n+1}. Euclidean metric over S^n as follows $d_E(x,y) = \|x-y\|$ [1,2,8].

Firstly, we remember the concepts of lines and triangles in the spherical plane.

As for $\alpha : IR \to S^n$ *ve* $x, y \in S^n$

curve $\alpha(t) = \cos t\, x + \sin t \dfrac{(y - \langle x,y\rangle x)}{\|y - \langle x,y\rangle x\|}$ is called S^n *'s line through* x,y [9].

Similarly for $\alpha : IR \to S^n$ *ve* $x, y \in S^n$,

$$\alpha(t) = \cos t\, x + \sin t \frac{(y - \cos t_1\, x)}{\sin t_1}, \quad t \in [0, t_1]$$

curve segment is called *the line segment of* S^n *limited to* x,y [9].

x,y,z, three of which are three points are not on the same spherical line;

$$\alpha(t) = \cos t\, x + \sin t \frac{(y - \cos t_1\, x)}{\sin t_1}, \quad t \in [0, t_1]$$

$$\beta(s) = \cos s\, y + \sin s \frac{(z - \cos s_1\, y)}{\sin s_1}, \quad s \in [0, s_1]$$

$$\gamma(u) = \cos u\, z + \sin u \frac{(x - \cos u_1\, z)}{\sin u_1}, \quad u \in [0, u_1]$$

The combination of the $\alpha(t_1) = \beta(0), \beta(s_1) = \gamma(0)$ *ve* $\gamma(u_1) = \alpha(0)$ segmented line segments is called the spherical triangle, and the spherical zone bounded by the triangle is called the *spherical triangular zone* [9].

Ω isspherical triangle with P_1, P_2, P_3 vertex points;

$$M = \begin{bmatrix} 1 & \cos\varphi_{12} & \cos\varphi_{13} \\ \cos\varphi_{12} & 1 & \cos\varphi_{23} \\ \cos\varphi_{13} & \cos\varphi_{23} & 1 \end{bmatrix}$$

matrix is called *egde matrix*of Ω [4]. P_i, P_j Ω's two vertices;

$$\cos\varphi_{ij} = \langle P_i, P_j \rangle$$

the real number φ_{ij} in the property $\cos\varphi_{ij} = \langle P_i, P_j \rangle$ is called Ω *'s edge length limited by* P_i, P_j [4].

If the edges of the P_i, P_j, P_k -pointed Ω spherical triangle through P_k point are also

$\alpha : IR \to S^n$,

$\beta : IR \to S^n$;

the θ_{ij} angle, which is to be $\langle \alpha'(t)\big|_{P_k}, \langle \beta'(s)\big|_{P_k} \rangle = \cos\theta_{ij}$, is called *the internal angle of* Ω *at point* P_k [9].

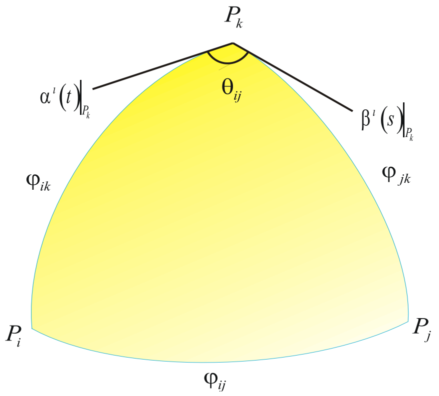

Figure 1. Triangle with internal angle in Spherical Space

II. Conformal Triangles in Spherical Space

Definition 2.1. The set $\{P \in S^2 : \langle m, P \rangle = \cos r\}$, as $m \in S^2$ and $r \in IR^+$, is called *the m-centered r spherical circle in* S^2 [9].

Definition 2.2. Let Ω be the spherical triangle with P_1, P_2, P_3 vertex points.If there are real numbers $r_1, r_2, r_3 \in IR^+$ as $0 < \varphi_{ij} = r_i + r_j \leq \frac{\pi}{2}$ with an edge length φ_{ij} limited to P_i, P_j ; Ω is called *conformal spherical triangle* [9].

Theorem 2.1. Let Ω bespherical triangle with P_1, P_2, P_3 vertex points. Ω to be conformal if and only if

$$If\ \ r_1 \in \left(0, \frac{\pi}{4}\right),\ r_2 \in (0, r_1)\ and\ r_3 \in \left(0, \frac{\pi - 2r_1}{2}\right) \qquad (2.1)$$

or

$$If\ \ r_1 \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right),\ r_2, r_3 \in \left(0, \frac{\pi - 2r_1}{2}\right)$$

where $r_1, r_2, r_3 \in IR^+$ [9].

Now, we give egde matricies for conformal sphericaltriangles. These matricies play very important roles throughout the paper for calculations.

Lemma 2.1. Edge matrix of conformal spherical triangles, edge matrix of conformal spherical equilateral triangles and edge matrix of conformal spherical isosceles triangles as follows

$$M = \begin{bmatrix} 1 & \cos(r_1 + r_2) & \cos(r_1 + r_3) \\ \cos(r_1 + r_2) & 1 & \cos(r_2 + r_3) \\ \cos(r_1 + r_3) & \cos(r_2 + r_3) & 1 \end{bmatrix} (2.2)$$

$$\tilde{M} = \begin{bmatrix} 1 & \cos(r_1 + r_2) & \cos(r_1 + r_2) \\ \cos(r_1 + r_2) & 1 & \cos(r_1 + r_2) \\ \cos(r_1 + r_2) & \cos(r_1 + r_2) & 1 \end{bmatrix} (2.3)$$

$$\widehat{M} = \begin{bmatrix} 1 & \cos(r_1 + r_2) & \cos(r_1 + r_2) \\ \cos(r_1 + r_2) & 1 & \cos(r_2 + r_3) \\ \cos(r_1 + r_2) & \cos(r_2 + r_3) & 1 \end{bmatrix} (2.4)$$

respectively [9].

From[4]

$$\cos\theta_{ij} = \frac{-M_{ij}}{\sqrt{M_{ii}M_{jj}}} \quad , i \neq j;\ i, j = 1,2,3 \ (2.5)$$

and from equation (14) in [5], we can define

$$\sin\theta_{ij} = \frac{\sqrt{|M|}}{\sqrt{M_{ii}M_{jj}}} \quad , i \neq j;\ i, j = 1,2,3\ .$$

(2.6)

III. Equality of Internal Angles and Vertex Points in Conformal Spherical Triangles

In this section, using the expressions of the internal angles and vertex points we have defined earlier, the equations of internal angles to vertex points of the conformal spherical triangle and special conformal spherical triangles will be shown.

Now, in Eq. 2.5

$$\cos\theta_{ij} = \frac{-M_{ij}}{\sqrt{M_{ii}M_{jj}}} \quad , i \neq j;\ i, j = 1,2,3$$

was given.

As $sin\,P_k = \dfrac{\sqrt{|M|}}{\sqrt{(M_{ii})(M_{jj})}}$ $\quad ,i \neq j, i \neq k, j \neq k;\ i, j, k = 1,2,3$.(3.1)

It is

$$cos\,\theta_{12} = \frac{-M_{12}}{\sqrt{M_{11}M_{22}}}\ .$$

If M_{11}, M_{12} and M_{22} from Eq. 2.2 are calculated and replaced,

$$\cos\theta_{12} = \frac{\cos(r_1+r_3)\cos(r_2+r_3)-\cos(r_1+r_2)}{\sqrt{\sin^2(r_2+r_3)\sin^2(r_1+r_3)}}$$

is obtained.

Similarly, if M_{11}, M_{22} and $|M|$ are used at Eq 3.1, calculated from Eq 2.2,

$$sin\,P_3 = \frac{\sqrt{|M|}}{\sqrt{M_{11}\,M_{22}}}$$

$$sin\,P_3 = \frac{\sqrt{4\,sin\,r_1\,sin\,r_2\,sin\,r_3\,sin(r_1+r_2+r_3)}}{\sqrt{sin^2(r_2+r_3)\,sin^2(r_1+r_3)}}$$

would be. From here

$$\theta_{12} = \arccos\left(\frac{\cos(r_1+r_3)\cos(r_2+r_3)-\cos(r_1+r_2)}{\sqrt{\sin(r_2+r_3)\sin(r_1+r_3)}}\right),$$

$$P_3 = arcsin\left(\frac{\sqrt{4\,sin\,r_1\,sin\,r_2\,sin\,r_3\,sin(r_1+r_2+r_3)}}{\sqrt{sin(r_2+r_3)\,sin(r_1+r_3)}}\right) (3.2)$$

areobtained.

We calculate the cosine of the right side of Eq 3.2. It would be

$$cos\left(arcsin\left(\frac{\sqrt{4\,sin\,r_1\,sin\,r_2\,sin\,r_3\,sin\left(r_1+r_2+r_3\right)}}{\sqrt{sin\left(r_2+r_3\right)sin\left(r_1+r_3\right)}}\right)\right)$$

$$=\sqrt{1-sin^2\left(arcsin\left(\frac{\sqrt{4\,sin\,r_1\,sin\,r_2\,sin\,r_3\,sin\left(r_1+r_2+r_3\right)}}{\sqrt{sin\left(r_2+r_3\right)sin\left(r_1+r_3\right)}}\right)\right)}$$

$$=\sqrt{1-\left(\frac{\sqrt{4\,sin\,r_1\,sin\,r_2\,sin\,r_3\,sin\left(r_1+r_2+r_3\right)}}{\sqrt{sin\left(r_2+r_3\right)sin\left(r_1+r_3\right)}}\right)^2}$$

$$=\frac{\sqrt{sin\left(r_1+r_2\right)sin\left(r_1+r_3\right)-4\,sin\,r_1\,sin\,r_2\,sin\,r_3\,sin\left(r_1+r_2+r_3\right)}}{sin\left(r_1+r_2\right)sin\left(r_1+r_3\right)}$$

When necessary calculations are made, we get

$$sin\left(r_1+r_2\right)sin\left(r_1+r_3\right)-4\,sin\,r_1\,sin\,r_2\,sin\,r_3\,sin\left(r_1+r_2+r_3\right)=\left(cos\left(r_1+r_3\right)cos\left(r_2+r_3\right)-cos\left(r_1+r_2\right)\right)^2.$$

Thus,

$\theta_{12}=P_3$

equation is obtained.By using similar method

$\theta_{23}=P_1$

and

$\theta_{13}=P_2$

are obtained [6].

III.1. Equality of Internal Angles and Vertex Points in the Conformal Spherical Equilateral Triangle

Let Ω be a spherical triangle with P_1,P_2,P_3 vertex points, dihedral angles and $\varphi_{12},\varphi_{13},\varphi_{23}$ edge lengths. Let $\Omega\in S^2$; if $\theta_{12}=\theta_{13}=\theta_{23}$, $\varphi_{12}=\varphi_{13}=\varphi_{23}$ and $\theta_{12}>\frac{\pi}{3}$, Ω is called *equilateralspherical triangle* [7].

Now, in Eq. 2.5

$$\cos\theta_{ij}=\frac{-\tilde{M}_{ij}}{\sqrt{\tilde{M}_{ii}\tilde{M}_{jj}}}\quad ,\ i\neq j;\ i,j=1,2,3$$

was given.

Including

$$sin\,P_k=\frac{\sqrt{\left|\tilde{M}\right|}}{\sqrt{\left(\tilde{M}_{ii}\right)\left(\tilde{M}_{jj}\right)}}\quad ,i\neq j,i\neq k,j\neq k;\ i,j,k=1,2,3\ .(3.3)$$

If $\tilde{M}_{11},\tilde{M}_{12}$ and $\tilde{M}_{22}$ are calculated and replaced from Eq. 2.3;

$$cos\,\theta_{12}=\frac{cos\left(r_1+r_2\right)\left(cos\left(r_1+r_2\right)-1\right)}{\sqrt{sin\left(r_1+r_2\right)}}$$

is obtained.

Similarly, if $\tilde{M}_{11},\tilde{M}_{22}$ and $\left|\tilde{M}\right|$ calculated from Eq. 2.3 used in Eq. 3.3 , it becomes as

$$sin\,P_3=\frac{\sqrt{\left|\tilde{M}\right|}}{\sqrt{\tilde{M}_{11}\,\tilde{M}_{22}}}$$

$$sin\,P_3=\frac{\sqrt{\left(cos\left(r_1+r_2\right)-1\right)^2\left(cos\left(r_1+r_2\right)+1\right)}}{\sqrt{sin\left(r_1+r_2\right)}}.$$

Here,

$$\theta_{12} = \arccos\left(\frac{\cos(r_1+r_2)(\cos(r_1+r_2)-1)}{\sqrt{\sin(r_1+r_2)}}\right),$$

$$P_3 = arcsin\left(\frac{\sqrt{(cos(r_1+r_2)-1)^2(cos(r_1+r_2)+1)}}{\sqrt{sin(r_1+r_2)}}\right) \quad (3.4)$$

areobtained.We calculate the cosine of the right side of Eq. 3.4. It is

$$cos\left(arcsin\left(\frac{\sqrt{(cos(r_1+r_2)-1)^2(cos(r_1+r_2)+1)}}{\sqrt{sin(r_1+r_2)}}\right)\right) = \sqrt{1-sin^2\left(arcsin\left(\frac{\sqrt{(cos(r_1+r_2)-1)^2(cos(r_1+r_2)+1)}}{\sqrt{sin(r_1+r_2)}}\right)\right)}$$

$$= \sqrt{1-\left(\frac{\sqrt{(cos(r_1+r_2)-1)^2(cos(r_1+r_2)+1)}}{\sqrt{sin(r_1+r_2)}}\right)^2} = \frac{\sqrt{sin(r_1+r_2)-(cos(r_1+r_2)-1)^2(cos(r_1+r_2)+1)}}{sin(r_1+r_2)}.$$

We get

$$sin(r_1+r_2)-(cos(r_1+r_2)-1)^2(cos(r_1+r_2)+1) = cos(r_1+r_2)(cos(r_1+r_2)-1)^2$$

when necessary calculations are made. Thus

$$\theta_{12} = P_3$$

equality is obtained. By using similar method

$$\theta_{23} = P_1$$

and

$$\theta_{13} = P_2$$

are obtained [6].

III.2 Equality of Internal Angles and Vertex Points in the Conformal Spherical Isosceles Triangle

Let Ω be a spherical triangle with P_1, P_2, P_3 vertex points, dihedral angles and $\varphi_{12}, \varphi_{13}, \varphi_{23}$ edge lengths. Let $\Omega \in S^2$; if $\theta_{12} = \theta_{13}$ and $2\theta_{12} > \pi - \theta_{23}$, Ω is called *isosceles spherical triangle* [7].

Now, in Eq. 2.5

$$\cos\theta_{ij} = \frac{-\widehat{M}_{ij}}{\sqrt{\widehat{M}_{ii}\widehat{M}_{jj}}} \quad , i \neq j;\ i, j = 1,2,3$$

was given. Including

$$sin\,P_k = \frac{\sqrt{|\widehat{M}|}}{\sqrt{(\widehat{M}_{ii})(\widehat{M}_{jj})}} \quad , i \neq j, i \neq k, j \neq k;\ i, j, k = 1,2,3 \,.(3.5)$$

If $\widehat{M}_{11}, \widehat{M}_{12}$ and $\widehat{M}_{22}$ are calculated and replaced from Eq. 2.4;

$$cos\,\theta_{12} = \frac{cos(r_1+r_2)(cos(r_2+r_3)-1)}{\sqrt{sin(r_2+r_3)\,sin(r_1+r_2)}}$$

is obtained.Similarly, if $\widehat{M}_{11}, \widehat{M}_{22}$ and $|\widehat{M}|$ calculated from Eq. 2.4 used in Eq. 3.5 , it becomes as

$$sin\,P_3 = \frac{\sqrt{|\widehat{M}|}}{\sqrt{\widehat{M}_{11}\,\widehat{M}_{22}}}$$

$$sin\,P_3 = \frac{\sqrt{4\,sin\,r_1\,sin\,r_2\,sin(r_1+r_2)}}{\sqrt{sin(r_2+r_3)\,sin(r_1+r_2)}}.$$

Here,

$$\theta_{12} = \arccos\left(\frac{\cos(r_1+r_2)(\cos(r_2+r_3)-1)}{\sqrt{\sin(r_2+r_3)\sin(r_1+r_2)}}\right),$$

$$P_3 = arcsin\left(\frac{\sqrt{4\,sin\,r_1\,sin\,r_2\,sin(r_1+r_2)}}{\sqrt{sin(r_2+r_3)\,sin(r_1+r_2)}}\right) \quad (3.6)$$

areobtained.We calculate the cosine of the right side of Eq. 3.6. It is

$$cos\left(arcsin\left(\frac{\sqrt{4\,sin\,r_1\,sin\,r_2\,sin(r_1+r_2)}}{\sqrt{sin(r_2+r_3)\,sin(r_1+r_2)}}\right)\right) = \sqrt{1-sin^2\left(arcsin\left(\frac{\sqrt{4\,sin\,r_1\,sin\,r_2\,sin(r_1+r_2)}}{\sqrt{sin(r_2+r_3)\,sin(r_1+r_2)}}\right)\right)}$$

$$= \sqrt{1-\left(\frac{\sqrt{4\,sin\,r_1\,sin\,r_2\,sin(r_1+r_2)}}{\sqrt{sin(r_2+r_3)\,sin(r_1+r_2)}}\right)^2} = \frac{\sqrt{sin(r_2+r_3)\,sin(r_1+r_2)-4\,sin\,r_1\,sin\,r_2\,sin(r_1+r_2)}}{sin(r_2+r_3)\,sin(r_1+r_2)}.$$

We get

$$sin(r_2+r_3)\,sin(r_1+r_2)-4\,sin\,r_1\,sin\,r_2\,sin(r_1+r_2) = cos(r_1+r_2)(cos(r_2+r_3)-1)^2$$

when necessary calculations are made. Thus

$$\theta_{12} = P_3$$

equality is obtained. By using similar method

$$\theta_{23} = P_1$$

and

$$\theta_{13} = P_2$$

are obtained [6].

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