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Investigation of behavior of the Riemann ζ-function and proof of the Riemann hypothesis

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Abstract
The paper investigates the behavior of the ζ-function. For the zeta function, a series representation is used. The problem of determining nontrivial zeros of the zeta function reduces to solving a system of two equations in a field of real numbers. The paper proves that the Riemann hypothesis is a consequence of the invariance of zeros with respect to symmetry transformations. The performed calculations allow us to understand the reasons for the validity of the Riemann hypothesis.

Keywords: *number theory, natural numbers, ζ-function, Riemann hypothesis.*
Classification code: 33E20.

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I. Introduction. Analysis of general relations

A large number of works have been devoted to the study of the properties of the Riemann ζ-function [1 – 17, 19, 20]. Here and below $z = x + iy$, where x and y are real numbers; y is in the upper half-plane. We will also use the notation $z = (x, y)$. In [7] the values of the function are given with step of 0.1 in the interval $0 \le x \le 1$, as well as the values of its roots for $y < 100$. In [15, 19], tables are given for the values of the first 15000 and 5000 zeros of the zeta function, respectively. The main unsolved problem is the proof of the Riemann hypothesis, which consists in the assertion that all the zeros of the ζ-function in the strip $0 \le x \le 1$ are on a line $x = 1/2$. By now, it has been proved that there are an infinite number of zeros on the line $x = 1/2$, and in addition there are no zeros on the ends of the interval. This paper is devoted to the study of behavior of the ζ-function, discussion of the reasons for the validity of the Riemann hypothesis and its proof. We consider the open interval $0 < x < 1$.We use for the ζ-function the representation valid for $x > 0$ [4, 5, 21]:

$$(1-2^{1-z})\varsigma(z) = \sum_{N=1}^{\infty}(-1)^{N+1}N^{-z} \equiv S(z)\,. \quad (1)$$

Grouping the terms independent of N, we transform (1) to the form

$$\varsigma(z) = C(z)\sum_{N=1}^{\infty}(-1)^{N+1}N^{-x}(\cos(y\ln N) - i\sin(y\ln N))\,, \quad (2)$$

Where function $C(z)$ does not depend from N .

$$C(z) = \exp[2k\pi(y-ix)]\{1-2^{1-x}\exp[2k\pi(y-ix)][\cos(y\ln 2) - i\sin(y\ln 2)]\}^{-1}\,. \quad (2a)$$

The function $C(z)$ does not vanish and is finite for all values of x from the considered interval and for all values of y. Note that for $z = (1, 0)$ it goes to infinity. Therefore, when determining the zeros of the zeta function in the considered interval, the function $C(z)$ can be ignored. For definiteness, we assume that k is fixed, for example, put $k = 0$ and consider the main branch of the logarithm, which does not affect the subsequent proof. Then, in the field of real numbers, the determination of zeros of the zeta function reduces to solving a system of equations of the form

$$u = F_1(x,y) = \sum_{N=1}^{\infty}(-1)^{N+1}A(x,N)B_1(y,N) = 0\,,$$

(3a)

$$v = F_2(x,y) = \sum_{N=1}^{\infty}(-1)^{N+1}A(x,N)B_2(y,N) = 0\,,$$

(3b)

where $A(x,N) = 1/N^x$, $B_1(y,N) = \cos(y\ln N)$, $B_2(y,N) = \sin(y\ln N)$. Expression (2) and the system of equations (3a), (3b) are valid for all values of x from the considered interval, in particular, for $z = 1/2 + iy$, $z_1 = (1/2 - \alpha) + iy$ and $z_2 = (1/2 + \alpha) + iy$, where $0 < \alpha < 1/2$. The transition from z_1 to z_2 or from z_2 to z_1 corresponds to the reflection operation with respect to the line $x = 1/2$, which is the middle line (axis of symmetry) of the strip 0 ... 1. We write expression (2) for $\varsigma(1-z)$. In this case, we get

$$\varsigma(1-z) = C(1-z)\sum_{N=1}^{\infty}(-1)^{N+1}N^{-(1-x)}(\cos(y\ln N) + i\sin(y\ln N)), \tag{4}$$

where function $C(1 - z)$ does not depend from N.

$$C(1-z) = \exp[2k\pi(ix-y)]\{1-2^x\exp[2k\pi(ix-y)] \\ [\cos(y\ln 2) + i\sin(y\ln 2)]\}^{-1}, \tag{4a}$$

Expression (4) corresponds to a system of equations of the form

$$u = F_1(x,y) = \sum_{N=1}^{\infty}(-1)^{N+1}A(x,N)B_1(y,N) = 0,$$

(5a)

$$v = F_2(x,y) = \sum_{N=1}^{\infty}(-1)^{N+1}A(x,N)B_2(y,N) = 0,$$

(5b)

where $A(x,N) = 1/N^{1-x}$, $B_1(y,N) = \cos(y\ln N)$, $B_2(y,N) = \sin(y\ln N)$. Expression (4) and the system of equations (5a), (5b) are valid for all values of x from the considered interval. The transition from z to $1 - z$ corresponds to the reflection operation relative to the center of the strip, i.e. with respect to the point $z = (1/2, 0)$. We write expression (2) for $\varsigma(z^*)$, where $z^* = x - iy$. In this case, we get

$$\varsigma(z^*) = C(z^*)\sum_{N=1}^{\infty}(-1)^{N+1}N^{-x}(\cos(y\ln N) + i\sin(y\ln N)) \tag{6}$$

where function $C(z^*)$ does not depend from N.

$$C(z^*) = \exp[-2k\pi(ix+y)]\{1-2^{1-x}\exp[-2k\pi(ix+y)] \\ [\cos(y\ln 2) + i\sin(y\ln 2)]\}^{-1}$$

(6a)

Expression (6) corresponds to a system of equations of the form

$$u = F_1(x,y) = \sum_{N=1}^{\infty}(-1)^{N+1}A(x,N)B_1(y,N) = 0,$$

(7a)

$$v = F_2(x,y) = \sum_{N=1}^{\infty}(-1)^{N+1}A(x,N)B_2(y,N) = 0,$$

(7b)

where $A(x,N) = 1/N^x$, $B_1(y,N) = \cos(y\ln N)$, $B_2(y,N) = \sin(y\ln N)$. This system is similar to system (3a), (3b). Expression (6) and the system of equations (7a), (7b) are valid for all values of x from the considered interval. Transition from z to z^* or from $1 - z$ to $1 - z^*$ and vice versa corresponds to the reflection operation with respect to x-axis ($y = 0$). We write expression (2) for $\varsigma(1-z^*)$. In this case, we have

$$\varsigma(1-z^*) = C(1-z^*)\sum_{N=1}^{\infty}(-1)^{N+1}N^{-(1-x)}(\cos(y\ln N) - i\sin(y\ln N)), \tag{8}$$

where function $C(1 - z^*)$ does not depend from N.

$$C(1-z^*)=\exp[2k\pi(ix+y)]\{1-2^x\exp[2k\pi(ix+y)]$$
$$[\cos(y\ln 2)-i\sin(y\ln 2)]\}^{-1}$$

(8a)

Expression (8) corresponds to a system of equations of the form

$$u=F_1(x,y)=\sum_{N=1}^{\infty}(-1)^{N+1}A(x,N)B_1(y,N)=0,$$

(9a)

$$v=F_2(x,y)=\sum_{N=1}^{\infty}(-1)^{N+1}A(x,N)B_2(y,N)=0,$$

(9b)

where $A(x,N)=1/N^{1-x}$, $B_1(y,N)=\cos(y\ln N)$, $B_2(y,N)=\sin(y\ln N)$. This system is similar to system (5a), (5b). Thus, in all cases considered, finding the nontrivial zeros of the zeta function reduces to solving the corresponding system of equations, i.e. to finding the joint (common) zeros of the functions F_1 and F_2. The initial conditions are $F_1(x, 0) = S(x)$, $F_2(x, 0) = 0$. The expressions for $F_1(x, y)$ and $F_2(x, y)$ are the sum of cosines and sinuses, respectively, with decreasing period, limited by the values of the amplitude $A(x, N)$, which changes sign depending on the parity of the number N. The amplitude $1/N^x$ decreases with increasing x at constant N and with increasing N at constant x. The amplitude $1/N^{1-x}$ increases with increasing x at constant N and decreases with increasing N at constant x. So, F_1 and F_2 take positive and negative values, depending on whether the sets of which sign predominate (plus or minus). Functions B_1 and B_2 are periodic; their period and the distance between zeros depend on y and N. For $N = 2$, the period is $\Delta y = 2\pi / \ln 2 = 9.06$; for $N = 10^6$ it is $\Delta y = 2\pi/6\ln 10 = 0.455$. For small values of y, N should be large so that at least one zero is observed. For example, if $y = 0.1$, then $\ln N=\pi/0.1\approx 31$; if $y = 1$, then $\ln N=\pi$, etc. These regularities are also valid for the functions F_1 and F_2, namely, for small values of y, long-period components predominate, and the distance between the zeros of these functions is significant, and for large y the short-period components play a major role, and the distance between the zeros becomes small. We are going to prove that the validity of the Riemann hypothesis follows from the properties of the invariance of joint zeros of the functions F_1 and F_2 to the reflection operation with respect to the line $x = 1/2$.

II. The study of the behavior of functions F_1 and F_2

We studied the behavior of the functions F_1 and F_2 in the range $0.4 \le x \le 0.96$ for different values of y from 0.1 to 10^6. In the calculations, we chose the number of members of the series so that the error was acceptable. Calculations for $x < 0.4$ were not carried out, because they require a large expenditure of computer time and are not needed for our purposes. The calculations are auxiliary; they make it possible to understand the reasons that underlie the Riemann hypothesis. The calculations were performed by the grid method with variable steps in x and y, which was chosen depending on the values of x and y. For a fixed x, the values of y varied in steps of 0.1 – 0.01 or less. For a fixed y, the values of x varied in steps of 0.01 – 0.1. The calculation results are given in tables 1 and 2.

Table 1

The calculation of the joint zeros of the functions F_1 and F_2 for $100 < y < 200$ ($x = 0.5$)

Serial number	$y(F_1)$	$y(F_2)$	Serial number	$y(F_1)$	$y(F_2)$
1	101, 3178	101, 3181	25	156,1127	156,1128
2	103,7259	103,7255	26	158,8500	158,8499
3	105, 4469	105, 4467	27	157,5995	157,5976
4	107,1684	107,1687	28	161,1889	161,1889
5	111,0298	111,0294	29	163,0302	163,0308
6	111,8745	111,8747	30	165,5367	165,5371
7	114,3200	114,3201	31	167,1799	167,1845
8	116,2265	116,2267	32	169,0946	169,0947
9	121,3703	121,3702	33	169,9122	169,9121
10	122,9465	122,9467	34	173,4116	173,4114
11	124,2568	124,2567	35	174,7542	174,7540
12	127,5168	127,5151	36	176,4412	176,4415
13	129,5783	129,5788	37	178,3774	178,3773
14	131,0877	131,0876	38	179,9164	179,9166

15	133,4976	133,4978	39	182,2072	182,2061
16	134,7564	134,7563	40	184,8743	184,8745
17	139,7364	139,7361	41	185,5985	185,5987
18	141,1238	141,1236	42	187,2291	187,2290
19	143,1118	143,1119	43	189,4160	189,4161
20	146,0008	146,0009	44	192,0266	192,0265
21	147,4227	147,4226	45	193,0798	193,0798
22	150,0537	150,0536	46	195,2653	195,2653
23	150,9250	150,9254	47	196,8766	196,8766
24	153, 0245	153, 0246	48	198,0146	198,0152

Note. The values of y are given with rounding. We excluded zeros for y = 138, 116 and y = 118, 790, since F_1(138, 1165) = 0,00018 and there is no transition through 0, i.e. F_1 does not change the sign; F_2(138,1161 = 0 and F_2 behaves regularly. Similarly, F_1(118,7907) = 0, and F_1 behaves regularly; but F_2(118,7908) = – 00047 and there is no transition through 0, i.e. F_2 does not change the sign. At these points, apparently, the functions F_1 (resp. F_2) behave irregularly and touch the line y at the corresponding points.

Table 2
The change in the position of the zeros of the functions F_1 and F_2, depending on x

F_1, F_2	x=0,49	x=0,5	x=0,51	x=0,75
	y	y	y	y
F_2=0	0	0	0	0
F_1=0 F_2=0	8,2413 5,4094	8,2482 5,4066	8,2553 5,4037	8,4735 5,3371
F_1=0 F_2=0	10,0799 9,1129	10,0722 9,1118	10,0645 9,1107	9,8300 9,0867
F_1=0 F_2=0	13,9805 12,0339	14,0563 12,0351	zero is absent (the preceding minimum is positive) 12,0363	zero is absent 12,0692
F_1=0 F_2=0	14,2140 14,1340	**14,1383** **14,1346**	zero is absent 14,1352	zero is absent 14,1547
F_1=0 F_2=0	17,3244 15,8890	17,3295 15,8869	17,3348 15,8848	17,5031 15,8273
F_1=0 F_2=0	18,7353 18,0771	18,7311 18,07795	18,7268 18,0788	18,5798 18,1016
F_1=0 F_2=0	zero is absent 19,9574	- 19,9614	- 19,9654	- 20,0984
F_1=0 F_2=0	21,0025 21,0266	**21,02196** **21,02180**	21,0437 21,0169	zero is absent 20,8616
F_1=0 F_2=0	21,4860 22,9751	21,4650 22,9748	21,4416 22,9744	zero is absent 22,9648
F_1=0 F_2=0	24,4143 zero is absent	24,4293 -	24,4452 -	zero is absent -
F_1=0 F_2=0	25,0237 25,0032	**25,01121** **25,01087**	24,9978 25,0187	zero is absent zero is absent
…	…	…	…	…
*F_1=0 F_2=0	98,8295 98,8701	**98,8311** **98,8322**	98,8327 zero is absent	98,7940 zero is absent
…	…	…	…	…
F_1=0 F_2=0	103,7550 103,7227	**103,7259** **103,7255**	103,6645 103,7283	zero is absent 103,8228
…	…	…	…	…
F_1=0 F_2=0	167,2481 167,1843	**167,1799** **167,1845**	zero is absent 167,1847	zero is absent 167,1961
…	…	…	…	…
F_1=0 F_2=0	196,8867 196,8675	**196,8766** **196,8766**	196,8653 196,8863	zero is absent zero is absent
…	…	…	…	…
F_1=0 F_2=0	99999,6797 99999,7042	**99999,701** **99999,701**	zero is absent 99999,6975	zero is absent zero is absent
F_1=0 F_2=0	100000,3466 100000,0596	100000,3504 100000,0569	100000,3544 100000,0540	100000,6009 99999,9222
…	…	…	…	…
F_1=0 F_2=0	999997,9784 999997,9605	**999997,9682** **999997,9683**	999997,9540 999997,9778	zero is absent zero is absent

Note. The zeros of each function are arranged in order of increasing y; the values of y are given with rounding. Joint zeros are indicated in bold. In the line with the * sign, y has a weak maximum (y = 98,8402) near x = 0.6 (step Δx = 0.05).

Table 1 gives the values of the joint zeros of functions F_1 and F_2 for $100 < y < 200$. Table 2 gives some results of the study of the alternation of zeros of functions (maxima and minima are not specified so as not to increase the size of the table) for x = 0.49, 0.50, 0.51 and 0.75, which allows us to understand the existing regularities and to draw conclusions. From the results of the calculations, it follows that the functions $F_1(x, y)$ and $F_2(x, y)$ change smoothly with increasing x. For each fixed x, the function $F_1(x, y)$ varies periodically with increasing y. For $y = 0$, it is $S(x)$, then it increases, reaches a positive maximum, decreases, passes through zero, reaches a minimum, increases again, passes through 0, reaches a maximum, etc. The function $F_2(x, y)$ also varies periodically. For $y = 0$, it is 0, then decreases, reaches a minimum, then increases, passes through 0, reaches a maximum, decreases, passes through 0, and so on. For each fixed y, when x increases from 0.4 to 0.96, the functions $F_1(x, y)$ and $F_2(x, y)$ change monotonically over the entire range of x or one of them has a weak maximum (minimum) and the other changes monotonically over the entire range with the change sufficiently slow (smooth). In the behavior of functions $F_1(x, y)$ and $F_2(x, y)$, the following cases are possible. 1. Both functions $F_1(x, y)$ and $F_2(x, y)$ are positive, then they can simultaneously decrease or one function decreases and the other increases or one has a weak maximum (minimum) and the other decreases. 2. Both functions are negative, then they can simultaneously increase (their absolute values decrease). 3. One of the functions is positive and the other is negative, then both functions can increase or the negative function increases and the positive function decreases or the positive function has a weak minimum and the negative function increases. 4. Both functions or one of them change sign, then both functions can change monotonically or one of them has a weak minimum (maximum) and the other monotonically increases (decreases). These cases repeat periodically with increasing y. The analysis shows that the functions $F_1(x, y)$, $F_2(x, y)$ for a given y can intersect no more than once with increasing x and only at the joint (common) zero. The zeros of the functions F_1 and F_2 appear regularly in a certain sequence. For an arbitrary fixed x, the appearance of zeros depends on y; we can assign them sequence numbers and arrange zeros in ascending order of y values, taking the value $y = 0$ as the initial value. We say that the behavior of a function is regular if it is given by the sequence: the earlier (preceding) zero of the function → the regular (ordinal, serial) zero → the later (successive) zero. If the functions have a joint zero (it corresponds to zero of the Riemann function), then after zero passing an anomaly is observed, which is that the sequence of zeros of one or both functions is violated: some zeros are absent (disappear), and instead of them there are positive minimum or negative maximum. With increasing x, the anomaly remains; moreover, if there were no anomaly at x values close to 0.5, then it can appear with an increase of x. Irregular (abnormal) behavior is given by the sequence: the earlier (preceding) zero of the function → the local optimum (positive minimum or negative maximum) → the later (successive) zero. The number of minima (maxima) corresponds to the number of missing zeros. We say that the behavior of the function $F_i(x, y)$, where $i = 1, 2$, is regular in a neighborhood of a joint zero, if at $x \neq 1/2$, the value of y, for which $F_i(x, y) = 0$, is closest to the value of y_0, for which there is a joint zero, that is, the equality $F_1(0.5, y_0) = F_2(0.5, y_0) = 0$ holds. Otherwise, we call it irregular (abnormal) in the neighborhood of joint zero. Note that irregular behavior can also be observed for other values of y for a fixed x. Consider the examples of irregular behavior of functions corresponding to the data in table 2. We take the joint zero $F_1(0.5, 98.8311) = F_2(0.5, 98.8322) = 0$. Hereinafter, the values of y are given with rounding. After passing the joint zero, the function $F_1(x, y)$ behaves regularly. Calculations give $F_1(0.51, 98.8327) = F_1(0.55, 98.8375) = F_1(0.6, 98.8401) = F_1(0.65, 98.8361) = F_1(0.7, 98.8224) = F_1(0.75, 98.7940) = F_1(0.8, 98.7411) = 0$. The values of y, at which F_1 turns to 0, has a weak maximum near $x = 0.6$ (the step in x is 0. 05). The function F_2 has irregular behavior, since some zeros are absent (disappear), and instead of them positive minima are observed. For $x = 0.51$, calculations give: the earlier (preceding) zero $F_{2E}(0.51, 97.1222) = 0$, the local minimum $F_2(0.51, 98.7990) = 0.02129$, the later (successive) zero $F_{2L}(0.51, 99.8148) = 0$. For $x = 0.55$, calculations give: the earlier zero $F_{2E}(0.55, 97.116356) = 0$, the local minimum $F_2(0.55, 98.8161) = 0.1278$, the later zero $F_{2L}(0.55, 99.8059) = 0$. We give for comparison the results for the function F_1 with regular behavior: the earlier zero $F_{1E}(0.55, 98.00799) = 0$, the regular (ordinal, serial) zero $F_{1R}(0.55, 98.83754) = 0$, the later zero $F_{1L}(0.55, 99.3668) = 0$. For $x = 0.6$, calculations give: the earlier zero $F_{2E}(0.6, 97.1089) = 0$, the local minimum $F_2(0.6, 98.8417) = 0.23286$, the later zero $F_{2L}(0.6, 99.79456) = 0$. The function F_2 has abnormal behavior also for other values of x. For $x = 0.75$, we have the earlier zero $F_{2E}(0.75, 97.0818) = 0$, the local minimum $F_2(0.75, 98.9583) = 0.40056$, the later zero $F_{2L}(0.75, 99.7595) = 0$. We give for comparison the results for the function F_1 with regular behavior: the earlier zero $F_{1E}(0.75, 98.2054) = 0$, the regular zero $F_{1R}(0.75, 98.7940) = 0$ (see table 2), the later zero $F_{1L}(0.75, 99.36927) = 0$. Therefore, for a function with a regular behavior, the zeros follow one another and do not disappear, and there are no local minima and maxima between the zeros. We take the joint zero $F_1(0.5, 103.7259) = F_2(0.5, 103.7255) = 0$. Both functions first behave regularly (see table 2), and then the function F_1 has an anomaly. For $x = 0.75$, calculations give: the earlier zero $F_{1E}(0.75, 100.2351) = 0$, the first local minimum $F_{11}(0.75, 101.5210) = 0.2888$, the second local minimum $F_{12}(0.75, 103.6919) = 0.8183$, the third local minimum $F_{13}(0.75, 105.3896) = 0.6497$, the fourth local minimum $F_{14}(0.75, 107.0911) = 0.5725$, the later zero F_{1L} (0.75, 108.4572) = 0. We give for comparison the results for the function F_2 with regular behavior: the earlier zero F_{2E} (0.75, 102.76004) = 0, the regular zero F_{2R} (0.75, 103.8228) = 0 (see table 2), the later zero F_{2L} (0.75, 104.35837) = 0. We take the joint zero F_1 (0.5, 167.1799) = F_2 (0.5, 167.1845) = 0. The function F_2 has a regular behavior (see table 2), and the function F_1 has an anomaly. For $x = 0.51$, we have the earlier zero F_{1E} (0.51, 164.19978) = 0, the first local minimum F_{11} (0.51, 165.5922) = 0.02048, the second local minimum F_{12} (0.51, 167.1785) = 0.05713, the later zero F_{1L} (0.51, 168.7178) = 0. For $x = 0.75$, calculations give: the earlier zero F_{1E} (0.75, 164.0811) = 0, the first local minimum F_{11} (0.75, 165.5949) = 0.8887, the second local minimum F_{12} (0.75, 167.1782) = 0.9064, the third local minimum F_{13} (0.75, 168.9919) = 0.4129, the fourth local minimum F_{14} (0.75, 169.9261) = 0.5078, the later zero F_{1L} (0.75, 171.66366) = 0. We give for comparison the results for the function F_2 with regular behavior: the earlier zero F_{2E} (0.75, 166.44262) = 0, the regular zero F_{2R} (0.75, 167.1961) = 0 (see table 2), the later zero F_{2L} (0.75, 167.870955) = 0. We take the joint zero F_1 (0.5, 196.8766) = F_2 (0.5, 196.8766) = 0. At values of x close to 0.5 both functions have regular behavior, and then an anomaly is observed. For $x = 0.75$ calculations give for the function F_1: the earlier zero F_{1E} (0.75, 190.7752) = 0, the first local minimum F_{11} (0.75, 192.1213) = 0.5217, the second local minimum F_{12} (0.75, 193.2455) = 0.37697, the third local minimum F_{13} (0.75, 195.2100) = 0.95083, the fourth local minimum F_{14} (0.75, 196.7592) = 0.46639, the fifth local minimum F_{15} (0.75, 197.9480) = 0.530995, the later zero F_{1L} (0.75, 199.16338) = 0. For the function F_2, calculations give: the earlier zero F_{2E} (0.75, 194.3461) = 0, the first local minimum F_{21} (0.75, 195.5322) = 0.062927, the second local minimum F_{22} (0.75, 197.1262) = 0.21195, the later zero F_{2L} (0.75, 198.13011) = 0. For comparison, we give the results for the functions F_1 and F_2 for x = 0, 51, where they have regular behavior in the neighborhood of the joint zero. We have for the function F_1: the earlier zero F_{1E} (0.51, 195.24142) = 0, the regular zero F_{1R} (0.51, 196.8653) = 0 (see table 2), the local minimum F_{11} (0.51, 197.9826) = 0.02786, the later zero F_{1L} (0.51, 199.12066) = 0. For the function F_2 calculations give: the earlier zero F_{2E} (0.51, 195.78797) = 0, the regular zero F_{1R} (0.51, 196.8863) = 0 (see table 2), the later zero F_{1L} (0.51, 197.24358) = 0. We take the joint zero F_1 (0.5, 99999.701) = F_2 (0.5, 99999.701) = 0. At values of x close to 0.5, the function F_2 behaves regularly, and then both functions have an anomaly (see table 2). We give the results for $x = 0.75$. For the function F_1, calculations give: the earlier zero F_{1E} (0.75, 99993.329657) = 0, the first local minimum F_{11} (0.75, 99994.2629) = 0.9417, the second local minimum F_{12} (0.75, 99995.2531) = 0.7630, the third local minimum F_{13} (0.75, 99996.5345) = 0.6297, the fourth local minimum F_{14} (0.75, 99997.9181) = 0.6782, the fifth local minimum F_{15} (0.75, 99998.5955) = 0.67668, the sixth local minimum F_{16} (0.75, 99999.6372) = 2.47037, the later zero F_{1L} (0.75, 100000.6009) = 0. For the function F_2, calculations give: the earlier zero F_{2E} (0.75, 99997.847625) = 0, the first negative local maximum F_{21} (0.75, 99998.3252) = – 0.1649, the second negative local maximum F_{22} (0.75, 99999.6533) = – 0.3681, the later zero F_{2L} (0.75, 99999.92218) = 0. We take the joint zero F_1 (0.5, 999997.9682) = F_2 (0.5, 999997.9683) = 0. At values of x close to 0.5 both functions have regular behavior, and then an anomaly is observed (see table 2). We give the results for $x = 0.75$. For the function F_1, calculations give: the earlier zero F_{1E} (0.75, 999992.98525) = 0, the first local minimum F_{11} (0.75, 999993.9036) = 0.4171, the second local minimum F_{12} (0.75, 999994.3047) = 0.4305, the third local minimum F_{13} (0.75, 999995.1825) = 0.1861, the fourth local minimum F_{14} (0.75, 999996.9501) = 1.75097, the fifth local minimum F_{15} (0.75, 999998.5284) = 0.54855, the sixth local minimum F_{16} (0.75, 999999.3971) = 1.1968, the seventh local minimum F_{17} (0.75, 1000000.5809) = 0.36946, the later zero F_{1L} (0.75, 1000001.5788) = 0. For the function F_2, calculations give: the earlier zero F_{2E} (0.75, 999996.2182) = 0, the first local minimum F_{21} (0.75, 999996.5909) = 0.7188, the second local minimum F_{22} (0.75, 999996.9960) = 0.6647, the third local minimum F_{23} (0.75, 999998.1137) = 0.4966, the later zero F_{2L} (0.75, 999998.61946) = 0. So, from our study, it follows that if, for an arbitrarily chosen but fixed value of y, the functions F_1 and F_2 do not have a joint zero for $x = 0.5$, i.e. if equality $F_1(0.5, y) = F_2(0.5, y) = 0$ does not hold, then the functions F_1 and F_2 behave regularly when x is changed, and for $x \neq 0.5$ do not have a joint zero. If, on the contrary, at an arbitrarily chosen but fixed value of y, the functions F_1 and F_2 have a joint zero for $x = 0.5$, i.e. if equality $F_1(0.5, y) = F_2(0.5, y) = 0$ holds, then one or both functions behave irregularly when x changes, and there are no joint zeros for $x \neq 1/2$.

III. Discussion of the results and possible reasons for the appearance of joint zeros

From our previous analysis, it follows that the functions F_1 and F_2 have zeros for each x. For $x = 1/2$ (y is in the upper half-plane), some of the zeros of F_1 and F_2 coincide, forming a set of solutions of system (3a), (3b). Joint (common) zeros of F_1 and F_2 correspond to "double" points on the line $x = 1/2$. If $x \neq 1/2$, namely, $x = 1/2 \pm \alpha$, where $0 < \alpha < 1/2$, then the double points split and two points are appeared: one corresponds to zero of F_1, and the second to zero of F_2; the splitting occurs in opposite directions (in the value of y) from the double point. The magnitude of the splitting depends on $| x – 1/2 |$, and the signs alternate by turns depending on y. With the removal of x from the value 1/2, the points are increasingly separated (see table 2). In some cases, one of the zeros may be absent, i.e. as a result of the splitting, only one point appears, corresponding to zero of one of the functions. In this case, for a given $x \neq 1/2$, the preceding minimum is positive or the preceding maximum is negative: the positive minimum is between two positive maxima or the negative maximum is between two negative minima. If for certain $x_1 \neq 1/2$ this "anomaly" is observed for the first time, then for all $x > x_1$ it remains (see examples above). The number of anomalies increases with increasing x. Since in our calculations the infinite series was replaced by a finite segment of the series, then the results of tables 1, 2 can be influenced by the following factors: the approximation error, the different rate of change of the functions F_1 and F_2, the different rate of convergence of the functions F_1 and F_2 near zero, and also the path through which we approach zero (from the one side or from different sides with maintenance of the sign). But these errors are small compared to the useful effect and do not affect the analysis results in principle, therefore the double points are reliably identified by the coincidence of two digits after the comma in the value of y (with rounding) corresponding to the double point. As can be seen from table 2, the difference between the zeros after the splitting is significantly greater than the error. Note that for the appearance of a double point on the line $x = 1/2$, it is necessary that F_1 and F_2 change synchronic (simultaneously) in the same or opposite directions. Such cases, as our analysis showed, are repeated periodically with increasing y. The period of repetition of double points depends on the range of values of y and the initial conditions, i.e. the values $F_1(x, 0)$ and $F_2(x, 0)$. For a finite y the number of double points is always less than the number of zeros of each of the functions F_1, F_2, since appearance of double points is associated with more strong restrictions. Consider the system $s = (x, y; F_1, F_2)$ defined in the strip 0...1. The appearance of joint zeros of the functions F_1 and F_2 can be explained by the symmetry properties of this system. As can be seen from (3a), (3b), F_1 is an even function and F_2 is an odd function of y. Designate φ_1 is the central symmetry operator with respect to the center of the strip 0…1, i.e. relative to the point $z = (1/2, 0)$. Designate φ_2 is the reflection operator with respect to the x-axis; φ is the reflection operator with respect to the line $x = 1/2$. On the line $x = 1/2$, the following relations hold $\varphi = \varphi_1\varphi_2 = \varphi_2\varphi_1$; so, the operators φ_1 and φ_2 are commutative; $\varphi_1\varphi_1 = \varphi_2\varphi_2 = \varphi_1\varphi_2 = \varphi = E$, where E is the identity operator. Under the action of the operator φ on the functions F_1, F_2, we have $F_1\varphi F_2 = F_1F_2$, where $F_1 = F_1(1/2, y)$, $F_2 = F_2(1/2, y)$; $\varphi F = F$, where $F = (F_1, F_2)$ is a column vector (or a row vector). On the line $x_\alpha \neq 1/2$, namely $x_\alpha = 1/2 \pm \alpha$, where $0 < \alpha < 1/2$, the following relations hold $\varphi_1\varphi_1 = \varphi_2\varphi_2 = E$, but $\varphi_1\varphi_2 \neq E$. Under the action of the operator φ on the functions F_1, F_2, we have following relations $F_1^- \varphi F_2^- = F_1^- F_2^+$, $F_1^+ \varphi F_2^+ = F_1^+ F_2^-$, where $F_1^- = F_1(1/2-\alpha, y)$, $F_2^- = F_2(1/2-\alpha, y)$, $F_1^+ = F_1(1/2+\alpha, y)$, $F_2^+ = F_2(1/2+\alpha, y)$; $\varphi F^- = F^+$, $\varphi F^+ = F^-$, where $F^- = (F_1^-, F_2^-)$, $F^+ = (F_1^+, F_2^+)$. Thus, the operator φ performs a one-to-one mapping (automorphism) of the system s into itself, for which the points on the line $x = 1/2$ are fixed points. So, on the line $x = 1/2$, joint zeros are invariant with respect to the operator φ. On any line $x_\alpha \neq 1/2$, there is no invariance with respect to the operator φ, and the joint zeros are separated (disappear). Now take an arbitrary line $x_\alpha = 1/2 \pm \alpha$. We can define operators on it that are similar to φ_1, φ_2, φ. Designate $\varphi_{1\alpha}$ is the operator of central symmetry with respect to the point $z = (x_\alpha, 0)$, $\varphi_{2\alpha} \equiv \varphi_2$ is the reflection operator with respect to the x-axis, φ_α is the reflection operator with respect to the line x_α. Under the action of the operator φ_α, the points of the line x_α are fixed, but with respect to the system s, the operator φ_α is a local operator. The local operator φ_α does not perform a one-to-one mapping (automorphism) of the entire system s, but only its part s_α, which corresponds to the part $(x_\alpha - \Delta, x_\alpha + \Delta)$ of width 2Δ of the strip 0…1, where $\Delta = \min(\alpha, 1/2 - \alpha)$. The action of the operator φ_α violates the symmetry of the complete system s. All lines $x_\alpha \neq 1/2$ and the corresponding operators φ_α are equivalent in the indicated sense, and the line $x = 1/2$ and the corresponding operator φ occupy a special position. The joint zeros of the functions F_1 and F_2, i.e., the zeros of the zeta function, are due to the symmetry properties of the system s as a whole, but not its part; therefore, they are invariant with respect to the operator φ, but not the operator φ_α. A consequence of invariance is that all zeros of the zeta function are located on the line $x = 1/2$.

IV. Proof of the Riemann hypothesis

Let us prove the following theorem: "For the validity of the Riemann hypothesis, it is necessary and sufficient that the zeros of the zeta function are invariant with respect to the operator φ, that is, the relation $\varphi\varsigma(z)_{z=z_0} = 0$ is satisfied if and only if $\varsigma(z_0) = 0$." The necessity is obvious. Indeed. Suppose that the hypothesis is valid and let $z_0 = (1/2, y_0)$ be certain zero of the zeta function, so $\varsigma(z_0) = 0$; y_0 is in the upper half-plane. Since the points on the line $x = 1/2$ are fixed points under the action of the operator φ, we obtain $\varphi\varsigma(z)_{z=z_0} = 0$. Thus, the necessity is proved. We prove sufficiency. We must prove that the relations $\varphi\varsigma(z) = 0$ and $\varsigma(z) = 0$ cannot simultaneously be satisfied at points that do not belong to the line $x = 1/2$. Suppose the contrary. Let z_1 be the zero of the zeta function, so $\varsigma(z_1) = 0$, $z_1 = (1/2 - \alpha, y_1)$; point z_1 is located in the second quadrant. This point is chosen arbitrarily; we can put it in the first quadrant, then $z_1 = (1/2 + \alpha, y_1)$, which does not affect the generality of the proof. The function ς (z) is defined by (2), and its zeros are found from the system of equations (3a), (3b). Applying the operator φ_1 to the zeta function, we obtain $\varphi_1\varsigma(z_1) = \varsigma(1 - z_1)$, where $1 - z_1 = (1/2 + \alpha, - y_1)$. Point $1 - z_1$ is in the fourth quadrant. The function $\varsigma(1 - z)$ is defined by (4), and its zeros are found from the system of equations (5a), (5b). From the Riemann relation between $\varsigma(z)$ and $\varsigma(1 - z)$, which we write in the form $2^{1-z}\Gamma(z)\zeta(z)\cos(\pi z/2) = \pi^z\zeta(1-z)$, it follows that if $\varsigma(z_1) = 0$, then $\varsigma(1 - z_1) = 0$, since the other quantities do not take the values 0 or infinity in the considered strip. Applying the operator φ_2 to the zeta function $\varsigma(1 - z)$, we obtain $\varphi_2\varsigma(1 - z_1) = \varphi_2\varphi_1\varsigma(z_1) = \varphi\varsigma(z_1) = \varsigma(1 - z_1^*)$, where $1 - z_1^* = (1/2 + \alpha, y_1)$. Point $1 - z_1^*$ is in the first quadrant. The function $\varsigma(1 - z^*)$ is defined by (8), and its zeros are found from the system of equations (9a), (9b). Since the system of equations (5a), (5b) is equivalent to the system (9a), (9b), we obtain $\varphi\varsigma(z_1) = \varsigma(1 - z_1^*) = 0$. However, this is impossible at the same value of y, which follows from a comparison of the system of equations (3a), (3b) with the system (9a), (9b). Therefore, there is no point not located on the line $x = 1/2$ at which the zeta function vanishes. This proves the validity of the theorem and, therefore, the Riemann hypothesis.

V. Conclusion

The nontrivial zeros of the ζ-function correspond to the joint zeros of the functions F_1 and F_2. The location of the joint zeros of the functions F_1 and F_2, i.e. zeros of the zeta function, only on the line $x = 1/2$ is explained by the invariance the joint zeros with respect to the operator φ. On any other line $x \neq 1/2$, the invariance is absent, so the joint zeros are not observed. The results of calculating the zeros of the **ζ**-function coincide within the error with the data of [7, 19]. There is an analogy with the symmetry properties of atomic (molecular) systems [18], which requires additional study.

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