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Spin system on a hexagon and Riemann Hypothesis

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Abstract
The aim of this paper is to be a complement of a previous work published in the fall of 2019. We illustrate the relation between a truncation of the Xi function in the variable z = 1-1/s to order 6 where s is the usual complex variable and the partition function of a ferromagnetic model on a hexagon (6 spin ½ variables) with long range interaction, with a concrete look at the truth of a possible proof of the Riemann Hypothesis. Concerning Statistical Mechanics, we use an important Theorem of the last century on the locus (unit circle in z)of the zeros (z= $e^{-2\cdot\beta\cdot H}$ is the magnetic field variable).
The content of this note illustrate in detail our previous general results concerning the Riemann Hypothesis.
Key words: *Classical Spin ½ model, long range interaction, Lee-Yang Theorem, truncation of the Xi function, loci of the zeros first few Li-Keiper coefficients, Riemann Wave Background, and Riemann Hypothesis.*

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I. The spin system on the hexagon and the truncation of the Xi function to order 6 in z.

We start with the Equation for the zeros of the partition function of the spin model on the hexagon: 6 spin ½ variables with two-body interaction and magnetic field. We look at the three real zeros in the range w≥ -2 where w is the variable w=z+1/z, thus to the six zeros in the z-variable sitting on the unit circle. The Equation in the variable w is given by[1, 2]:

$$w^3 + w^2\cdot 6\cdot x^5 + w\cdot(15\cdot x^8-3) + 20\cdot x^9 - 12\cdot x^5 = 0. \quad (1)$$

The 6 zeros in z are on the unit circle for x <1 (Theorem of Lee -Yang (1952) ($x=e^{-2.K}$,K=β·J is the parameter of the two-body interaction between two spins variables and β is the inverse of the absolute temperature).
Below, we give the plot of the left hand side of Eq.(1) for various values of the ferromagnetic interaction's parameter x.

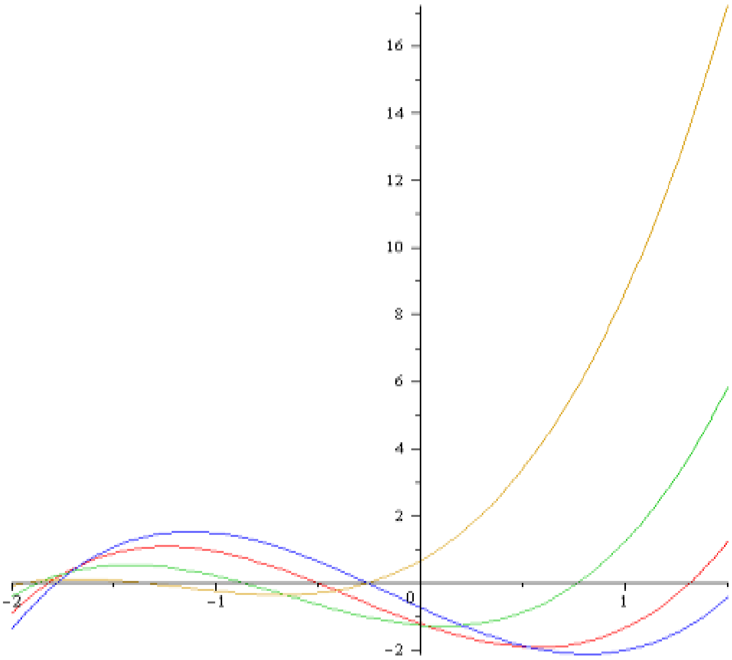

Fig.1. Plot of the left hand side of Eq.(1) with the w-zeros.

x=0·6 (blue), x=0·7 (red), x = 0·8 (green), x = 0·9 (maroon).

The three zeros in w are in the range $w > -2$ for every $x<1$ and this ensures that the six zeros in z ($w= z+1/z$) are on the unit circle.
The corresponding Equation for the truncated Xi function for 2N=6, where 2N is the degree of the truncation, [1, 2, 3] (see also below) is given by:

$$w^3+ w^2 \cdot(6-\lambda_1) + w\cdot(12-6\cdot\lambda_1 + \varphi_2)+ 8 -13\cdot \lambda_1 + 6\cdot \varphi_2 - \varphi_3 = 0. \quad (2)$$

where $\varphi_2 = (½)\cdot(\lambda_2 +\lambda_1^2)$ and $\varphi_3 = (1/3)\cdot(\lambda_3 +(3/2)\cdot\lambda_1\cdot\lambda_2 + \lambda_1^3/2)\cdot\lambda_1$ is the first Li-Keiper coefficient. For $\varphi_2 =2\cdot\lambda_1$ and for $\varphi_3 =3\cdot\lambda_1$, we have $w_1= -2$, $w_2= -2$, and $w_3=-2+ \lambda_1$ (4 zeros are then in $z= -1$ and the other two are given by the solution of the Equation $z+1/z = -2+\lambda_1$. It then appears the Koebe function K(z) as argument of the logarithm containing our Riemann Wave Background).
Then

$$f(z) = \log(1-\lambda_1 \cdot z/(1+z)^2) = \log(1- \lambda_1\cdot K(-z))$$

or with

$$z\to -z, \log(1+\lambda_1\cdot K(z))=\log(1+\lambda_1 \cdot z/(1-z)^2) \quad (3)$$

$$\to z^2+(2- \lambda_1)\cdot z+1= 0 \quad \to \quad w =-2+\lambda_1. \quad (4)$$

Below, we give the plots as in the Figure above -here-for the two values x=0.999 and x=0.9992, the last one corresponding to $\lambda_1= 0.0230957...$ (the value of the first Li-Keiper coefficient).

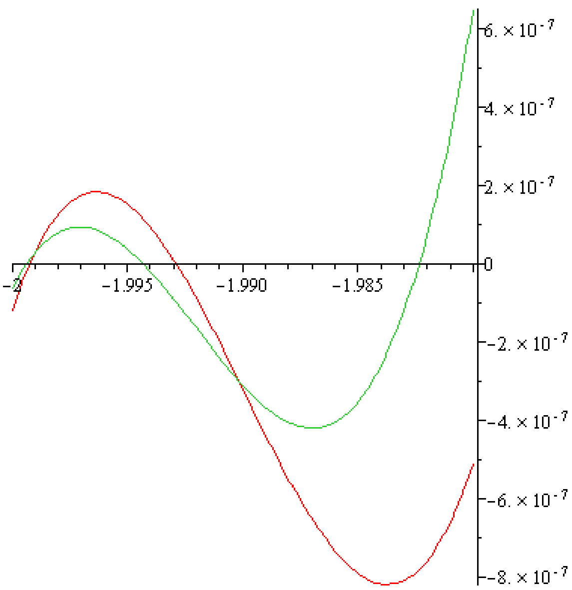

Fig.2. Left hand side of Eq.(1): the case x=0.999 (in red)and the case x=0.9992 (in green).

This explain in detail with the spin model on the hexagon (2·N=6) the possible proof of the RH from our last works in the IOSR-JM worked for general 2N [2, 3].

II. The movement of the real zeros

The w-zeros and also the values of the complex zeros in z on the unit circle are given on the Table and Figures below, where we also give values as a function of x (ferromagnetic case i.e. x<1) in the range x=0.6.. 0.999023043 (this corresponds to the range λ_1 =[5.53.. 0.0230...].
This captures the movement of the z-zeros on the unit circle as a function of the interaction parameter x for x<1 or λ_1.

x =0.6	w_1 = -1.779 w_2 = -0.261 w_3 = + 1.574	z'= -0.889±0.456·i z"= -0.130 ±0.991·i
x= 0.7	w_1 = -1.817 w_2 = -0.506 w_3 = 1.314	z'= -0.908±0.417·i z"= -0.657±0.753·i
x= 0.8	w_1 = -1.867 w_2 = -0.868 w_3 = 0.769	z'= -0.933±0.358·i z"= -0.384±0.923·i
x=0.9	w_1 = -1.928 w_2 = -1.362 w_3 = -0.252	z'= -0.964±0.265·i z"= -0.126±0.992·i
x=0.92	w_1 = -1.942 w_2 = -1.477 w_3 = -0.534	z'= -0.971±0.239·i z"= -0.267±0.963·i
x=0.94	w_1 = -1.955 w_2 = -1.599 w_3 = -0.848	z'= -0.977±0.210·i z"= -0.424±0.905·i
x=0.96	w_1 = -1.970 w_2 = -1.726 w_3 = -1.195	z'= -0.985±0.172·i z"= -0.597±0.801·i
x=0.98	w_1 = -1.984 w_2 = -1.860 w_3 = -1.578	z'= -0.992±0.126·i z"= -0.789±0.614.i
x=0.99	w_1 = -1.992 w_2 = -1.929 w_3 = -1.784	z'= -0.996±0.089·i z"= -0.892±0.452·i
x=0.999	w_1 = -1.999 w_2 = -1.992 w_3 = -1.977	z'= -0.999±0.0316·i z"= -0.988±0.151·i

Table 1. The w' and the z-zeros

On the Table 1, w_1 is the smallest zero in w,w_3 the greater one. The same for the complex zeros in z. Notice that 4 zeros in z (corresponding to w_1 and w_2, converge to w=-2, while for the last of the three, i.e. $w_3 \rightarrow -2+\lambda_1 \sim -2+ 0.023095.. = -1.977..$as x is near unity with Eq. (1).

In the Figure3, we represent the set (z') as well the set (z") in the range (0.6..0.999) of the interaction parameter $x=e^{(-2\cdot K)}$.
The set (z') are the zeros with the smallest abscissa (in red): they converge to the point $z'_0 = -1 +i\cdot 0$,as "x→1".The set (z") contains the zeros with the greater abscissa of the three zeros for every values of x (points in green): they converge (as "x→1") to z''_{0-} which is the solution with ($Im(z''_0)<0$) of the Equation above, i.e. of $1+z^2+(2-\lambda_1)\cdot z=0$ i.e. $w=-2+\lambda_1 \sim -1.977..$
$z''_{0-}= -0.988 -i \cdot 0.151$. This explain the movement of the zeros as a function of x or λ_1 up to $\lambda_1 \sim 0.0230957..$(the true value of λ_1).

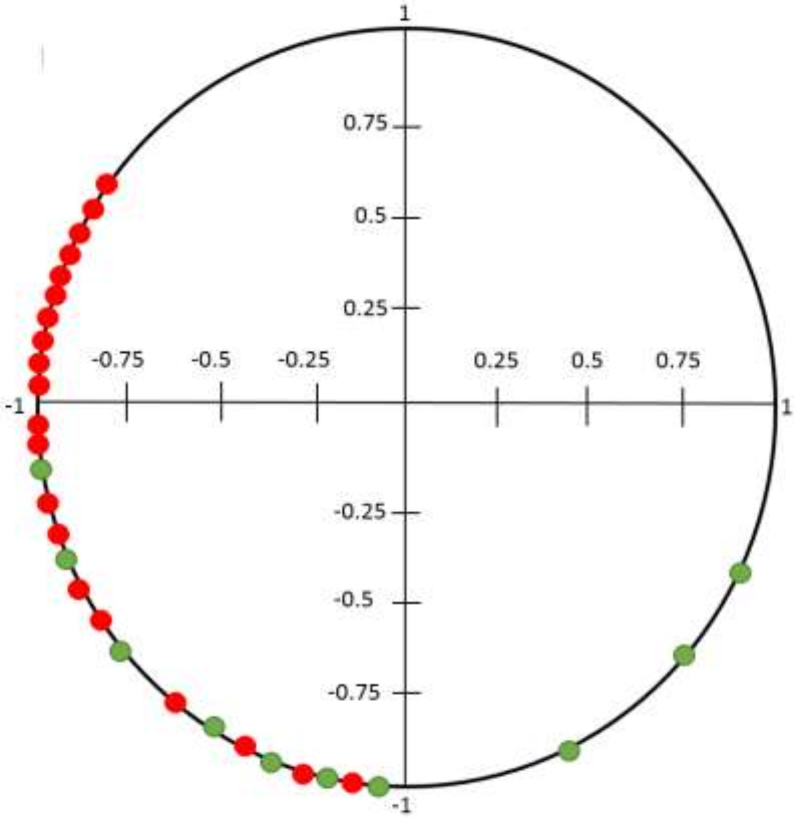

Fig.3. z' (red points) are the solutions with smallest RE(z*); z'' (green points) are the solutions with the greater RE(z*) for every x= e^{-2K} from x= 0.6 to x=0.999. Moreover, the red points near the green points are the z's corresponding to w_2's on the Table 1.

Here too, the w_2's converge to w= -2. In the above limit we have: $(1+z)^4\cdot(1+z\cdot(2-\lambda_1)+z^2)$. Since in the truncation we have multiplied by $(1+z)^{2N}$, here $(1+z)^6$ and changed z in -z, finally we have:

$$log\left[\frac{(1+z)^4\cdot(1+z(2-\lambda_1)+z^2)}{(1+z)^6}\right]=log\left[\frac{(1+z)^2-z\lambda_1}{(1+z)^2}\right]$$

Then z → -z to obtain

$$log\left[1+\frac{z\lambda_1}{(1-z)^2}\right]=log[1+\lambda_1\cdot K(z)]$$

For every N (Riemann Wave Background).

The relation between the spin model on the hexagon and the truncation of the Xi function (here only to order 6 in z!) is realized by means of the three Equations [1, 2]:

$$6-\lambda_1=6\cdot x^5 \qquad (5)$$
$$15-6\cdot\lambda_1+\varphi_2=15\cdot x^8 \qquad (6)$$
$$20-15\cdot\lambda_1+6\cdot\varphi_2-\varphi_3=20\cdot x^9 \qquad (7)$$

which make equal the left hand side of Eq.(1) with that of Eq.(2).
Since we have just one parameter x or λ_1, we may calculate λ_2 and λ_3 as a function of x i.e. of λ_1 in the interval of interest.
The Formulas and the plots are also given below (notice that for, every x <1, we have three values ($\lambda_1, \lambda_2, \lambda_3$).
With Eq.(5) in Eq. (6) we then have:

$$\varphi_2=15\cdot((1-\lambda_1/6)^{8/5}-1)+6\cdot\lambda_1 \text{ and since } \varphi_2=(½)\cdot(\lambda_2+\lambda_1^2),$$

then:

$$\lambda_2 = 30\cdot((1-\lambda_1/6)^{8/5} - 1) + 12\cdot\lambda_1 - \lambda_1^2 \qquad (8)$$

with Eq. (5), Eq.(6) and Eq. (7), we have:

$$\varphi_3 = 20\cdot(1-x^9) - 15\cdot\lambda_1 + 6\cdot\varphi_2(\lambda_1) = 20\cdot((1-(1-\lambda_1/6)^{9/5}) - 15\cdot\lambda_1 + 6\cdot\varphi_2.$$

Since $\varphi_3=(1/3)\cdot(\lambda_3 + (3/2)\cdot\lambda_1\cdot\lambda_2 + \lambda_1^3/2)$, then

$$\lambda_3 = 3\cdot(20\cdot((1-(1-\lambda_1/6)^{9/5}) - 15\cdot\lambda_1 + 3\cdot(\lambda_2 + \lambda_1^2)) - (3/2)\cdot\lambda_1\cdot\lambda_2 - \lambda_1^3/2 \quad (9)$$

Let us define: $\lambda_2' = 4\cdot\lambda_1 - \lambda_1^2$ and $\lambda_3' = 9\cdot\lambda_1 - 6\cdot\lambda_1^2 + \lambda_1^3$.

(10)

In the two Figures below we represent the pairs of functions (λ_2, λ_2') and (λ_3, λ_3') in the range of interest of the independent variable λ_1 = [0.02..0.025](notice still that the true value of the first Li-Keiper coefficient is λ_1= 0.0230957089...). As it is known, the Li-Keiper Equivalent to the truth of the Riemann Hypothesis is that all the Li-Keiper coefficients are non-negatives).

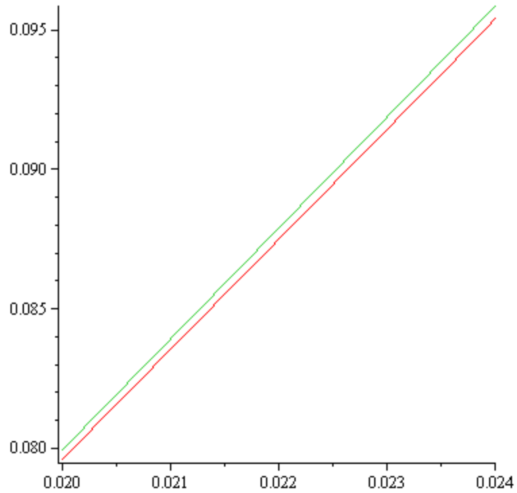

Fig. 4. λ_2 (in green, increased by 0.0002 for a better view) and the lower bound λ_2' (in red), in the range λ_1= [0.020..0.024]

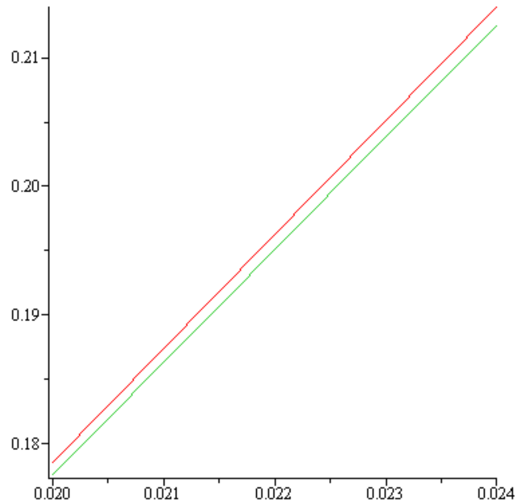

Fig. 5. λ_3 (in red) and the lower bound λ_3' (in green).In the same range [0.020..0.024] of λ_1.

We now notice that for λ_1 = 0.0230957... we have λ_2 = 0.09206289 with the lower bound λ_2' = 0.09184942. (The true value of λ_2 (i.e. for the complete Xi function in the "thermodynamic limit" is 0.0923467..) and we

obtain $\lambda_3 =$ 0.20594681 with the lower bound $\lambda_3' = 0.204673229$ (the true value of λ_3 for the complete Xi function is 0.20763...). See references in [1, 2].
λ_2' and λ_3' ($\lambda_1' = \lambda_1$ from definition) , are the coefficients of the first three terms of the expansion at z=0 of the above log-function i.e. containing the Riemann Wave background (Eq.(3)), explicitly:

$$\log(1+\lambda_1\cdot z/(1-z)^2) = \qquad (11)$$

$$= [\ \lambda_1\cdot z + (½)\cdot(4\cdot\lambda_1- \lambda_1^2)\cdot z^2 + (1/3)\cdot(9\cdot\lambda_1 - 6\cdot \lambda_1^2+ \lambda_1^3) +... +(1/4)\cdot(16\cdot\lambda_1 -20\cdot\lambda_1^2 +8\cdot\lambda_1^3- \lambda_1^4)\cdot z^4 + \ldots\ ] = \qquad (12)$$

$$= \sum \lambda_n' \cdot \frac{z^n}{n}$$

Notice that the lower bounds λ_2' and λ_n' are obtained setting $\varphi_k = k\cdot\varphi_1 = k\cdot \lambda_1$, k integer i.e. (in general here k=2 and k=3):

$$\varphi_2 =(1/2)\cdot(\lambda_2 + \lambda_1^2) = 2\cdot\lambda_1 \rightarrow \lambda_2 = 4\cdot\lambda_1- \lambda_1^2 = \lambda_2' \quad (13)$$

and $\varphi_3= (1/3)\cdot(\lambda_3 +(3/2)\cdot\lambda_1 \cdot\lambda_2 +\lambda_1^3/2) = 3\cdot\lambda_1$.
Still with $\lambda_2 = \lambda_2' \rightarrow \lambda_3= 9\cdot\lambda_1 - 6\cdot\lambda_1^2 + \lambda_1^3 = \lambda_3'$. (14)

For $2\cdot N > 6$, for example $2\cdot N= 8$,we continue: since

$$\varphi_4 = (¼)\cdot(\lambda_4 +(4/3)\cdot\lambda_1 \cdot \lambda_3+(1/2)\cdot \lambda_2^2 + \lambda_1^2\cdot\lambda_2+ (1/6)\cdot\lambda_1^4) =4\cdot\lambda_1 \quad (15)$$

Then, still with $\lambda_2 = \lambda_2'$ and $\lambda_3 = \lambda_3'$ we obtain the next term of the above expansion (See Eq.(12)) i.e.

$$\lambda_4 = \lambda_4' = 16\cdot\lambda_1-20\cdot\lambda_1^2 + 8\cdot\lambda_1^3 - \lambda_1^4. \qquad (16)$$

Notice now that the sequence of the lower bounds $\{\lambda_n'\}n=1..\infty$, is given by the "discrete" "wave" [3] embedded in the continuous wave (n real) given by:

$$\lambda_n' = 4\cdot\sin^2(\phi\cdot n/2) \quad n=1,2,... \qquad (17)$$

where ϕ is the solution of $4\cdot\sin^2(\phi\cdot 1/2) = \lambda_1=1+\gamma/2-(1/2)\cdot\log(4\cdot\pi)$i.e. $\phi = 0.1521193523..$,wave, with the period of 41.30431278 units, as represented below with the first two minima (n>0).

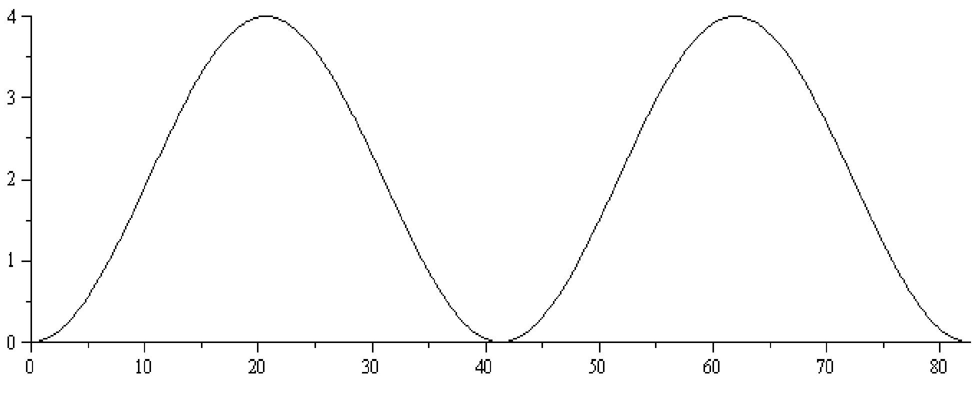

Fig. 6. The continuous wave (n real, Eq.(17)) containing the discrete sequence of our lower bounds (n integer), i.e. the Riemann Wave background [2, 3].

The black point near the first minimum of the continuous curve (n real>0) has coordinates (41, 0.00214255...) while the coordinates of the first minimum (for n>0**)** are (41.304312., 0).

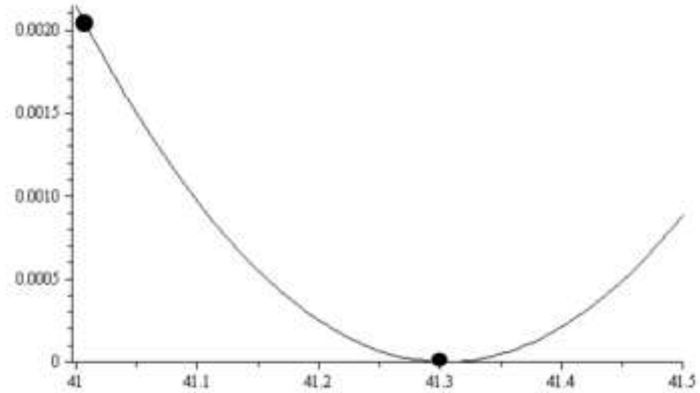

Fig. 7. The wave in the region n = [41..41.5]

Remark

The slope of the Wave has a maximum at $\cos(n\cdot\phi)=0$, i.e. at $n\sim 10.334$. At $n=10$ the value $\lambda_{10}' = 1.9..$ (notice that $\lambda_9 = 1.85..$) while for $n=10.334..$ the value of the wave is 2.004... (notice that $\lambda_{10}= 2.279..$) as may be seen on the first Figure above.

Moreover, from above, if $\lambda_2 = \lambda_2' = 4\cdot\lambda_1 - \lambda_1^2$ then

$$\varphi_2 = (\tfrac{1}{2})\cdot(\lambda_2 + \lambda_1^2) = (\tfrac{1}{2})\cdot(4\cdot\lambda_1 - \lambda_1^2 + \lambda_1^2) = 2\cdot\lambda_1.$$

In the same way, if additionally $\lambda_3 = \lambda_3' = 9\cdot\lambda_1 - 6\cdot\lambda_1^2 + \lambda_1^3$, then

$\varphi_3 = (1/3)\cdot(\lambda_3 + (3/2)\cdot\lambda_1\cdot\lambda_2 + (\lambda_1^3)/2) = (1/3)\cdot(9\cdot\lambda_1 - 6\cdot\lambda_1^2 + \lambda_1^3 +$
$+(3/2)..\lambda_1\cdot(4\cdot\lambda_1 - \lambda_1^2) + (\lambda_1^3)/2) = 3\cdot\lambda_1 + 0\cdot\lambda_1^2 + 0\cdot\lambda_1^3 = 3\cdot\lambda_1.$

To conclude these lines, if $\lambda_4 = \lambda_4' = 16\cdot\lambda_1 - 20\cdot\lambda_1 + 8\cdot\lambda_1^3 - \lambda_1^4$ then

$\varphi_4 = (\tfrac{1}{4})\cdot(\lambda_4' + (4/3)\cdot\lambda_1\cdot\lambda_3' + (1/2)\cdot\lambda_2'^2 + \lambda_1^2\cdot\lambda_2' + (1/6)\cdot\lambda_1^4) =$
$= (\tfrac{1}{4})\cdot(16\cdot\lambda_1 - 20\cdot\lambda_1^2 + 8\cdot\lambda_1^3 - \lambda_1^4 + (4/3)\cdot\lambda_1\cdot(9\cdot\lambda_1 - 6\cdot\lambda_1^2 + \lambda_1^3) +$
$+(1/2)\cdot(4\cdot\lambda_1 - \lambda_1^2)^2 + \lambda_1^2\cdot(4\cdot\lambda_1 - \lambda_1^2) + (1/6)\cdot\lambda_1^4)) =$
$= 4\cdot\lambda_1 + 0\cdot\lambda_1^2 + 0\cdot\lambda_1^3 + 0\cdot\lambda_1^4 = 4\cdot\lambda_1.$

Alternatively, if we consider the first three Equations of our infinite set, i.e. [3]

$$\varphi_2 - \varphi_1 = (1/(2!))\cdot\xi''(1), \quad \varphi_3 - 2\cdot\varphi_2 + \varphi_1 = (1/3!)\cdot\xi'''(1) \quad (18)$$
$$\text{and } \varphi_4 - 3\cdot\varphi_3 + 3\cdot\varphi_2 - \varphi_1 = (\tfrac{1}{4}!)\cdot\xi'''(1),$$

that we obtained, just from the definition of the Li-Keiper coefficients, and since we had rigorously proven that $\xi''(1) = 2\cdot\varphi_1 + 2\cdot\delta_2(\delta_2>0)$, then, since $\xi''(1)>0$, $\rightarrow \varphi_2 = 2\cdot\varphi_1 + \delta_2 = 2\cdot\lambda_1 + \delta_2$ where$\delta_2 = > 0$. Thus $\varphi_2 > 2\cdot\lambda_1$.
For φ_3 we have: $\varphi_3 = 2\cdot\varphi_2 - \varphi_1 + (1/3!).\xi'''(1)$ and from above:

$$\varphi_3 = 2\cdot(2\cdot\varphi_1 + \delta_2) - \varphi_1 + (1/3!)\cdot\xi'''(1) = 3\cdot\lambda_1 + 2\cdot\delta_2 + (1/3!)\cdot\xi'''(1) = 3\cdot\lambda_1 + \delta_3$$

Since $\xi'''(1) > 0$, $\rightarrow \delta_3 = 2\cdot\delta_2 + (1/3!)\cdot\xi'''(1) > 0, \rightarrow \varphi_3 > 3\cdot\lambda_1.$
For 2N> 6, for example 2N=8,we continue: the third Equation of our infinite set is given by: $\varphi_4 - 3\cdot\varphi_3 + 3\cdot\varphi_2 - \varphi_1 = (\tfrac{1}{4}!)\cdot\xi''''(1)$.

Then:
$\varphi_4 = 3\cdot\varphi_3 - 3\cdot\varphi_2 + \varphi_1 + (\tfrac{1}{4}!)\cdot\xi''''(1) = 4\cdot\lambda_1 + \delta_4.$

$\varphi_4 = 3\cdot(2\cdot\varphi_2 - \varphi_1 + (1/3!)\cdot\xi'''(1) - \varphi_2) + \varphi_1 + (\tfrac{1}{4}!)\cdot\xi''''(1) =$
$= 4\cdot\lambda_1 + \delta_4$, where $\delta_4 = 3\cdot(\delta_3 - \delta_2) + (\tfrac{1}{4}!)\cdot\xi''''(1)$
$\varphi_4 = 3\cdot(\varphi_2 - \varphi_1 + (1/3!)\cdot\xi'''(1)) + \varphi_1 = 3\cdot(\varphi_1 + \delta_2 + (1/3!)\cdot\xi'''(1)) + \varphi_1 + (\tfrac{1}{4}!)\cdot\xi''''(1)$
$= 4\cdot\varphi_1 + 3\cdot\delta_2 + (3/3!)\cdot\xi'''(1)) + (\tfrac{1}{4}!)\cdot\xi''''(1) = 4\cdot\lambda_1 + \delta_4$
Since:

$$\delta_4 = 3\cdot\delta_2 + (3/3!)\cdot\xi'''(1) + (1/4)\cdot\xi''''(1) = 3\cdot\delta_3 - 3\cdot\delta_2 + (1/4!)\cdot\xi''''(1) > 0$$

then

$$\varphi_4 > 4\cdot\lambda_1.$$

Conversely, from$\varphi_2 = 2.\ \varphi_1 + \ \delta_2(\ \delta_2 > 0)$, then

$(½)\cdot(\lambda_2 + \lambda_1^2\ ) = 2\cdot\lambda_1 + \delta_2 \ \rightarrow\ \lambda_2 = 4\cdot\lambda_1 - \lambda_1^2\ + 2\cdot\delta_2 \rightarrow\ \ \lambda_2 > \lambda_2'\ =\ 4\cdot\lambda_1 - \lambda_1^2.$

From $\varphi_3 = 2\cdot\varphi_2 - \varphi_1 + (1/3!)\cdot\xi'''(1)$

then since $\varphi_3 = \ (1/3)\cdot(\lambda_3 + (3/2)\cdot\lambda_1\cdot\lambda_2\ + (1/2)\cdot\lambda_1^3\ )$,

$\varphi_3\ \ = 2\cdot(2\cdot\ \varphi_1 + \delta_2) - \varphi_1 + (1/3!)\cdot\xi'''(1) = 3\cdot\lambda_1 + 2\cdot\delta_2 + (1/3!)\cdot\xi'''(1) =$
$= 3\cdot\ \lambda_1 + \ \delta_3$, where $\delta_3 = 2\cdot\delta_2 + (1/3!)\cdot\xi'''(1)$.

then $\ (1/3)\cdot(\lambda_3 + (3/2)\cdot\lambda_1\cdot\lambda_2\ + (1/2)\cdot\lambda_1^3\ ) = 3\cdot\lambda_1\ +\ \delta_3$ and

$$\lambda_3 = 9\cdot\lambda_1 + 3\cdot\ \delta_3 - (3/2).\cdot\lambda_1.(4\cdot\ \lambda_1 - \ \lambda_1^2\ + 2\cdot\delta_2) - (1/2)\cdot\lambda_1^{\ 3} =$$

$= \ 9\cdot\lambda_1\ - 6\cdot\lambda_1^2 + \lambda_1^3 + 3\cdot\delta_3 - 3\cdot\delta_2\cdot\lambda_1$
where

$$3\cdot\delta_3 - 3\cdot\delta_2\cdot\lambda_1 = \ \ 3\cdot(2\cdot\delta_2\ + (1/3!)\ \cdot\xi'''(1)\ ) - 3\cdot\ \delta_2\cdot\ \lambda_1 > 0.$$

$$\rightarrow \lambda_3 > \lambda_3' = 9\cdot\lambda_1 - 6\cdot\lambda_1^2 + \lambda_1^{\ 3}.$$

Additionally, from $\varphi_4 = 3.\varphi_3\ - 3.\ \varphi_2 + \varphi_1 + (¼!)\cdot\xi''''(1)$, we obtain:

$\varphi_4\ =\ 3\cdot(3\cdot\lambda_1 + \delta_3 - 2\cdot\lambda_1\ - \delta_2\ )\ +\ \ \lambda_1\ + (¼!)\cdot\xi''''(1) = 4\cdot\lambda_1\ + 3\cdot(\ \delta_3\ - \delta_2\ ) + (¼)\cdot\xi''''(1) = 4\cdot\lambda_1\ + \delta_4$.
where $\delta_4\ =\ 3\cdot\delta_3 - 3\cdot\delta_2\ + (¼!)\cdot\xi''''(1)$.

Then, with the definition of φ_4, we obtain:

$$\lambda_4 = 16\cdot\lambda_1\ + 4\cdot\delta_4\ - (4/3)\cdot\lambda_1\cdot\ \lambda_3\ \ - (1/2)\cdot\lambda_2^2\ -\ \lambda_1^2\cdot\lambda_2^2 - (1/6)\cdot\lambda_1^4.$$

With the expressions above for λ_2 and for λ_3, then:

$$\lambda_4 = 16\cdot\lambda_1 + 4\cdot\delta_4\ - (4/3)\cdot\lambda_1\cdot\lambda_3\ \ - (1/2)\cdot\ \lambda_2^2\ -\ \lambda_1^2\cdot\lambda_2^2\ - (1/6)\cdot\lambda_1^4 =$$

$= 16\cdot\lambda_1\ + 4\cdot\delta_4\ - (4/3)\cdot\lambda_1\cdot(\ 9\cdot\lambda_1\ - 6\cdot\lambda_1^2 +\ \lambda_1^3 + 3\cdot\delta_3 - 3\cdot\delta_2\cdot\lambda_1\ ) +$

$$- (½)\cdot(4\cdot\lambda_1 - \lambda_1^2 + 2\cdot\delta_2\ )^2 -\ \lambda_1^2\cdot(4\cdot\lambda_1 - \lambda_1^2 + 2\cdot\delta_2)\ -\ \lambda_1^4/6 =$$

$= 16\cdot\lambda_1 - 20\cdot\lambda_1^2 + 8\cdot\lambda_1^3\ -\ \lambda_1^4\ + A_4 =\ \lambda_4'\ + A_4$ where

$A_4 = 4\cdot(\ (¼!)\cdot\xi''''(1)) + (1/3!)\cdot\xi'''(1)\cdot(12 - 4\cdot\lambda_1\ ) +\ \delta_2\cdot(\ 12 - 16\cdot\ \lambda_1\ + 4\cdot\lambda_1^2 - 2\cdot\delta_2\ ).$

Since $A_4 > 0$ then

$$\lambda_4 >\ \lambda_4' =\ 16\cdot\lambda_1 - 20\cdot\lambda_1^2 + 8\cdot\lambda_1^3 - \lambda_1^4.$$

We now limit to calculate A_4; we obtain: $A_4 =\ 0.0098293...$ a number (as it should be)in agreement with
$\lambda_4 - \lambda_4' = 0.368790479492 - 0.358961379661 = 0.009829...$

(Notice that λ_4' is up to the factor of (¼) the coefficient of $(1/4)z^4$ in Eq.(12) (i.e. the four term of our Riemann Wave background)**.**The same for λ_5 where the fifth term (coefficient of $(1/5)\ \cdot z^5$ in Eq.(12) is obtained as $\lambda_5' = 25\cdot\lambda_1 - 50\cdot\lambda_1^2 + 35\cdot\lambda_1^3 - 10\cdot\lambda_1^4 + \lambda_1^5$ and $\lambda_5 = \lambda_5' + A_5$.
We also computed and found $A_5 = 0.0243$..a number equal up to some decimals with $\lambda_5 - \lambda_5' = 0.57554271446 - 0.551150480117 =\ 0.02439223..$
Our infinite set of Equations for the partial partition functions φ's are given by [3]

$$\sum_{k=0}^{n-1}(-1)^k \cdot \binom{n-1}{k} \cdot \varphi_{n-k} = \left(\frac{\xi^n(1)}{\Gamma(n+1)}\right) \tag{19}$$

where $\xi^n(1)$ is the n-ten derivative of ξ, at s=1, positive for each n, thanks to Riemann.
We then remark that if in the definition of the λ's with the φ's, we set in the φ_n, our lower bound λ_1', λ_2', ….λ_n' the above Equation becomes, for each n:

$$0 < \left(\frac{\xi^n(1)}{\Gamma(n+1)}\right) \tag{20}$$

a characterization of our Riemann Wave background; as illustration for n=3, we have:

$\varphi_3 - 2\cdot\varphi_2 + \varphi_1 = (1/3!)\cdot\xi'''(1) = (1/3)\cdot(\lambda_3'+(3/2)\cdot\lambda_1'\cdot\lambda_2'+(½)\cdot(\lambda_1')^3 - 2\cdot(1/2)\cdot(\lambda_2'+(\lambda_1')^2) + \lambda_1' =$
$=(1/3)\cdot(9\cdot\lambda_1 - 6\cdot\lambda_1^2 + \lambda_1^3 + (3/2)\cdot\lambda_1\cdot(4\cdot\lambda_1 - \lambda_1^2) + (1/2)\cdot\lambda_1^3) - 2\cdot(1/2)\cdot(4\cdot\lambda_1 - \lambda_1^2 + \lambda_1^2) + \lambda_1 = = 3\cdot\lambda_1 - 3\cdot\lambda_1 =$
0.

III. Concluding remark

In this work we have given the explicit computations (as a detailed illustration) concerning our recent works on the concreteness of the Riemann Wave background for our possible proof of the truth of Riemann Hypothesis. Our initial idea has been to use some important rigorous results of Statistical Mechanics of the last century concerning spin 1/2 lattice models on C_1(a circle) with ferromagnetic long range interactions: an important property is that the partition functions of some spin models have all the zeros (also) in the thermodynamic limit, on the unit circle in the external magnetic field variable $z = e^{-2.\beta.h}$ where β is the inverse of the absolute temperature. The canonical partition function in the variable z, contains the ferromagnetic interactions. Especially we carried out attemption to the high temperature region (β small).

At the same time we considered a special truncation of the Xi function in the variable z = 1-1/s and a connection with the model of Statistical Mechanics was obtained giving rise- to what we called-, the Riemann wave background i.e. a lower bound to the set of the Li-Keiper coefficients.

Then, in considering the definitions of them, we obtained alternatively an infinite set of Equations for the partial partitions functions which possesses a linear lower bound proportional to the first Li-Keiper coefficient -linear bounds-, which are connected with the corresponding lower bound for the same Li-Keiper coefficients.

These bounds refer to a function (argument of a logarithm) which contains the optimal function i.e. the Koebe function.

Such a function, is connected with the partition function of the smallest spin system (2N=2) and appears as a stability property also for the truncated Xi function in the thermodynamic limit.

We are thus in the presence of non negative periodic lower bounds for the infinite set of the Li-Keiper coefficients furnishing us a possible proof of the truth of the Riemann Hypothesis.

Finally, a summation of the members of the infinite set of Equations [3], results **in a shift of one unity** in the argument of the ξ function (s=1→s=2), and the connection with a function counting the number of zeros occouring in the binary list, i.e. in the binary representation of a integer, with the appearance of the well known Glaisher-Kinkelin constant.

References

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APPENDIX 1
Ursell and Ursell-Mayer functions
Remark on the Cumulants of a random variable

If X is a random variable, s_n the moments and u_n the so called cumulants then, indicating with E the expectation, we have:

$$E(e^{z\cdot X}) = \sum_n \frac{s_n \cdot z^n}{n!} = e^{\left(\sum_n \frac{u_n \cdot z^n}{n!}\right)} \tag{A_1}$$

$u_1(X)= E(X)$
$u_2(X)= E(X^2) - E^2(X)$
$u_3(X)= E(X^3)-3 \cdot E(X)\cdot E^2(X) + 2\cdot E^3(X)$
$u_4(X)= E(X^4) -4\cdot E(X)\cdot E(X^3) -3\cdot E(X^2)\cdot E(X^2)+12\cdot E^2(X)\cdot E(X^2)-6\cdot E^4(X)$

We just notice here that for the moments $s_n/n! = \varphi_n$ and that for the cumulants $u_n/n! = \lambda_n/n$ where λ_n are the Li-Keiper coefficients.
Explicitly, from Eq.(A_1):

$E(X) = u_1(X) = \lambda_1 = \varphi_1$.

$E(X^2)= u_2(X,X) + E^2(X))= (\lambda_2 + \lambda_1^2) = 2\cdot\varphi_2$.

$E(X^3)= u_3(X,X,X) +3\cdot E(X)\cdot E(X^2)+2\cdot E^3(X)=2\cdot\lambda_3+3\cdot\lambda_1\cdot 2\cdot\varphi_2-2\cdot\lambda_1^3 =$
$= 2\cdot(\lambda_3 + (3/2)\cdot\lambda_1 \cdot\lambda_2 + \lambda_1^3/2) = 6\cdot\varphi_3$.

$E(X^4) = u_4(X)+4\cdot E(X)\cdot E(X^3)+3\cdot E(X^2)\cdot E(X^2)-12\cdot E^2(X)\cdot E(X^2)+6\cdot E^4(X)$
$= 6\cdot\lambda_4 +4\cdot \lambda\cdot(\lambda_3 + (3/2)\cdot\lambda_1 \cdot\lambda_2 + \lambda_1^3/2) +3\cdot(\lambda_2 + \lambda_1^2)^2 -$
$-12\cdot\lambda_1^2\cdot(\lambda_2 +\lambda_1^2) +6\cdot\lambda_1^4 = 6\cdot(\lambda_4+(4/3)\cdot\lambda_1\cdot\lambda_3 +(1/2)\cdot\lambda_2^2 +$
$+ \lambda_1^2\cdot\lambda_2 + \lambda_1^4/6) = 4!\varphi_4$.

Ursell-Mayer expansion

In the Ursell-Mayer expansion one use the quantity

$1 + f(K) = e^{(-2\cdot K)}$.

For high temperature, f(K) is small and $f(K) \sim -2\cdot K$ (K small).

APPENDIX 2

2.a Partition function of the spin model.

$$Z(x,z) = exp\Bigg[(2\cdot N\cdot x^{2N-1}\cdot z) + \left(\binom{2N}{2}\cdot x^{2(2N-2)} - \left(\frac{1}{2}\right)\cdot(2\cdot N\cdot x^{2N-1})^2\right)\cdot z^2$$
$$+\Bigg[\Bigg[\binom{2N}{3}\cdot x^{3(2N-3)} - \frac{1}{3}\cdot\binom{2N}{1}\cdot x^{(2N-1)}\cdot\binom{2N}{21}\cdot x^{2(2N-2)} + \cdots$$
$$-\tfrac{1}{3}\cdot\binom{2N}{1}\cdot x^{(2N-1)}\cdot\left(2\cdot\binom{2N}{2}\cdot x^{2(2N-2)} - \binom{2N}{1}\cdot x^{2N-1}\right)^2\Bigg]\cdot z^3\Bigg]\dots\Bigg]\dots =$$
$$= \sum_{k=0}^{N}\binom{2N}{k}\cdot x^{k(2N-k)}\cdot z^k \dots \tag{1a}$$

2.b Partition function of the truncated Xi function.

$$Z(\{\lambda_n, z\}) = e^{\left(2N\log(1+z)+\sum_{k=1}^{\infty}(-1)^k\cdot\left(\frac{\lambda_k}{k}\right)\cdot z^k\right)} =$$

$$= e^{\left((2N-\lambda_1)\cdot z-(2N-\lambda_2)\cdot z^2+(2N-\lambda_3)\cdot z^3-(2N-\lambda_4)\cdot z^4+..\right)}$$

$$= 1 + (2N-\lambda_1)\cdot z + \left(\binom{2N}{2} - \binom{2N}{1}\cdot\lambda_1 + \binom{2N}{0}\cdot\left(\frac{\lambda_2+\lambda_1{}^2}{2}\right)\right)\cdot z^2 + \cdots =$$

$$= \sum_{i=0}^{N}\left[\sum_{k=0}^{i}(-1)^k\cdot\binom{2N}{N-k}\cdot\varphi_k\right]\cdot z^i = \sum_{i=0}^{N}\psi_i\cdot(z^i)[2,3].$$

(1.b)

$$\binom{2N}{1}\cdot x^{2N-1} = \binom{2N}{1} - \lambda_1$$

$$\binom{2N}{2}\cdot x^{2\cdot(2N-2)} = \binom{2N}{2} - \binom{2N}{1}\cdot\lambda_1 + \binom{2N}{0}\cdot\left(\frac{1}{2}\right)\cdot(\lambda_2+\lambda_1^2) = \binom{2N}{2} - \binom{2N}{1}\cdot\varphi_1 + \binom{2N}{0}\cdot\varphi_2$$

$$\binom{2N}{3}\cdot x^{3\cdot(2N-3)} = \binom{2N}{3} - \binom{2N}{2}\cdot\varphi_1 + \binom{2N}{1}\cdot\varphi_2 - \binom{2N}{0}\cdot\varphi_3$$

Now, from above: with x^{2N-1} ~1-2·k·(2N-1) (k small) we have $\varphi_1 = \lambda_1 = 4\cdot kN\cdot(2N-1)$and with the second Equation above, we obtain in the same limit $\varphi_2 = (½),(\lambda_2 + \lambda_1^2) \sim 2\cdot\lambda_1$ and so on for φ_n, n>2 [3].
As an example for n=3,

$$\binom{2N}{3}\cdot X^{3\cdot(2N-3)} =$$

$$= \binom{2N}{3} - \binom{2N}{2}\cdot\varphi_1 + \binom{2N}{1}\cdot\varphi_2 - \binom{2N}{0}\cdot\varphi_3$$

Then, with $\varphi_1 = \lambda_1 = 4\cdot k\cdot N\cdot(2N-1)$ we have:

$$\binom{2N}{3}\cdot(1-6\cdot k\cdot(2N-3)) = \binom{2N}{3} - \binom{2N}{2}\cdot\varphi_2 + \binom{2N}{1}\cdot\varphi_1 - \varphi_3 +$$

$-6k\cdot 2N\cdot(2N-1)\cdot(2N-2)\cdot(2N-3)\cdot(1/6)\cdot\lambda_1 = -N\cdot(2N-1)\cdot\lambda_1 + 2N\cdot(2\cdot\varphi_1) - \varphi_3 +$
$-(N-1)\cdot(2N-3)\cdot\lambda_1 = -2\cdot N^2 + 5\cdot N\cdot\lambda_1 - \varphi_3$
$\varphi_3 = 3\cdot\varphi_1 + 0\cdot N = 3\cdot\lambda_1$ for all N.

APPENDIX 3

As a comment, concerning the use of some Polynomials to the study of the zeros on the critical line, we recall here that, on the R.H. and assuming simplicity of the zeros, the Equation for the n-ten zeros has been known since 2012 and may be found in more ways(see for example ref(4) in
Ref(5) of this work).

$$\left(\frac{t}{2\pi}\right)\cdot\left(\log\frac{t}{2\pi}-1\right)+\frac{7}{8}+\frac{1}{\pi}arc\left(\zeta'\left(\frac{1}{2}+i\cdot t\right)\right) = N(t)-\frac{1}{2}$$

The Mehta-Dyson Polynomials (with classical creation a* and annihilation a operators [4] may (it may be) be used for further developments in the quantum approach of to the R.H. For a Gauss-Lucas treatment of the Xi function inside the critical strip (Re(s)> 0.9), see a recent work of ours in another approach (Ref(6)) by means of the “Primitive Riemann wave”.

Danilo Merlini, et. al. "Spin system on a hexagon and Riemann Hypothesis." *IOSR Journal of Mathematics (IOSR-JM)*, 16(5), (2020): pp. 14-24.