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Relevancy of pricing European put option based on truncated Gumbel distribution in actual market

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Abstract
In this paper, we develop a solution for pricing a European put option under the assumption that the distribution returns (losses) are Gumbel distributed at maturity. Further, we use the derived solution to check its relevancy with the data of actual market. We compare the Black- Scholes model which is based on assumption that distribution returns follow log-normal distribution with our derived solution which is based on the assumption that returns follow truncated Gumbel distribution numerically and observe some interesting underlying phenomenon.

Keywords: *European put, Truncated Gumbel distribution, Fat tail, Black- Scholes model*

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I. Introduction

The shape of the tail of the distribution of returns has important factors in both practical as well as academic point of view. The fundamental view for economic and financial theories often rely on particular distributions whose parameters represents the 'macroscopic' variables the agents are sensitive to. For practitioners, and specifically for practical market risk management purposes, one typically needs to assess tail risks associated with the distributions of returns or profit and losses. In literature, Markose and Alentorn (2005) [2], it is assumed that negative asset returns or losses are Gumbel distribution (i.e. the fattest tail associated with the Gumbel distribution occurs on the negative axis). In this paper, we examine that, by assuming that asset price returns at maturity are truncated Gumbel distributed, the solution to the standard European option pricing problem which produced a significantly improved pricing performance over the standard Black- Scholes approach.

Madan and Seneta (1990) [3] in their theses derived The Variance Gamma (VG) model for share market returns which is based on levy process, i.e. Variance Gamma process. Geske & Torous (1991) [4] in their article named" Skewness, kurtosis and Black -Scholes option mispricing" showed that the distribution of stock returns exhibits deviations from normality; in particular skewness and kurtosis. Cont and Tankov (2004) [5] in their book titled "Financial Modeling with jump processes" represents the properties and construction of levy processes and discussed multivariate modeling via Brownian subordination using levy process.Sato, Ken-Iti (1999) [6] in their book titled "Levy Processes and Infinitely Divisible Distributions" represents processes with the properties of independent and stationary increments and made the link between such processes and their infinite divisible laws.

Singh and Gor (2020) [7], shows the relevancy of pricing European put option based on Gumbel distribution in actual market. Permana ,Lesmono & Chendra (2014) [8] in their paper titled " Valuation of European and American options under variance gamma process" conclude that the variance gamma model performs very well compared to the GBM model in Indonesian market. The VG model can match the first four moments including skewness and excess kurtosis.

The outline of the paper is as follows. Section 1; derive the solution formula for pricing call and put for European option where the returns follow truncated Gumbel distribution. Section 2; compare the real market data numerically with the formula which we derived with the basic formula of Black-Scholes model.

BLACK-SCHOLES (BS) MODEL

Black and Scholes (1973) [1] derived the theoretical valuation formula for options.
Let S_0 be the current price of stock, V be the value of European call option on this stock with, X = Strike price, T= time to expiration, σ = volatility of stock (constant), r = risk-free interest rate

Then the value, V of call today is given by,

$$V = S_0 N(d_1) - Xe^{-rT} N(d_2)$$

In this formula, $N(x)$ denotes the standard normal distribution function. That is,

$$N(x) = P[Z \leq x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

And, $d_1 = \frac{ln\left(\frac{S_0}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sqrt{T}\sigma}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

THE MODEL BASED ON TRUNCATED GUMBEL DISTRIBUTION

The Gumbel probability density function with parameters α and σ is given by

$$F(x; \alpha, \sigma) = \frac{1}{\sigma} e^{-\frac{(x-\alpha)}{\sigma}} \exp\left(-e^{-\frac{(x-\alpha)}{\sigma}}\right)$$

where α and $\sigma > 0$ are location (real) and Scale (real) respectively.

The truncated Gumbel probability density function with parameters α and σ is given by

$$F(x; \alpha, \sigma, a, b) = \frac{1}{\sigma} \frac{e^{-\left(\frac{x-\alpha}{\sigma} + e^{-\left(\frac{x-\alpha}{\sigma}\right)}\right)}}{e^{-e^{-\frac{b-\alpha}{\sigma}}} - e^{-e^{-\frac{a-\alpha}{\sigma}}}}, -\infty < a < b < \infty$$

The basic model requirements suggest that the asset price process, $S = \{S_t, t \in [0, T]\}$, may be approximately written as

$$S_t = S_0 e^{X_t}$$

where the process $X = \{X_t, t \in [0, T]\}$ represents the evolution of asset price returns.

The asset return process defined as

$S_t = S_0 e^{X_t}$ which implies that $X_t = ln\left(\frac{S_t}{S_0}\right)$ where $S_t \leq S_0$

That is, $Y_t = -X_t$ is Gumbel distributed.

The distribution of returns X_t, can be derived as follows:

$$\begin{aligned} P(X_t \leq x) &= P(-Y_t \leq x) \\ &= P(Y_t > -x) \\ &= \int_{-x}^{\infty} \frac{1}{\sigma} e^{\frac{\alpha-y}{\sigma}} \exp\left(-e^{\frac{\alpha-y}{\sigma}}\right) dy \\ &= \int_{\infty}^{-x} -\frac{1}{\sigma} e^{\frac{\alpha-y}{\sigma}} \exp\left(-e^{\frac{\alpha-y}{\sigma}}\right) dy \end{aligned}$$

Let us define $u = -y, as\ y \to \infty\ implies\ u \to -\infty\ and\ y \to -x\ implies\ u \to x$

$$= \int_{\infty}^{-x} \frac{1}{\sigma} e^{\frac{\alpha+u}{\sigma}} \exp\left(-e^{\frac{\alpha+u}{\sigma}}\right) du$$

The density function of returns, X_t, is given by

$$f_{X_t}(x) = \frac{1}{\sigma} e^{\frac{\alpha+x}{\sigma}} exp\left(-e^{\frac{\alpha+x}{\sigma}}\right) \quad for\ x \in R$$

Standard no arbitrage argument allows the price of a European call option under the stated assumptions to be written as follows:

$$\begin{aligned} C &= e^{-(r-q)T} E_Q[(S_t - k)] \\ &= e^{-(r-q)T} E_Q[(S_0 e^{X_T} - k)] \\ &= e^{-(r-q)T} \int_{-\infty}^{\infty} (S_0 e^x - k) f_{X_T}^Q(x)\, dx \\ &= e^{-(r-q)T} \int_{k}^{\infty} (S_0 e^x - k) f_{X_T}^Q(x)\, dx \end{aligned}$$

where $k = ln\left(\frac{k}{S_0}\right)$ and $f_{X_T}^Q$ is the risk neutral probability density function associated with X_T

The price of a call option can be written as

$$C = e^{-(r-q)T}\left(S_0 \int_k^{\infty} e^x f_{X_T}^Q(x)\, dx - k \int_k^{\infty} f_{X_T}^Q(x)\, dx\right) \text{\_\_\_\_\_\_\_\_ (1)}$$

These integrals can be evaluated under the assumption that X_T is Gumbel distributed.

A. First Integral

$$\int_k^{\infty} e^x f_{X_T}^Q(x)\, dx = \frac{1}{\sigma}\int_k^{\infty} e^x e^{\frac{\alpha+x}{\sigma}} exp\left(-e^{\frac{\alpha+x}{\sigma}}\right) dx$$

$$= \frac{1}{\sigma}\int_{ln\left(\frac{k}{s_o}\right)}^{\infty} e^x e^{\frac{\alpha+x}{\sigma}} exp\left(-e^{\frac{\alpha+x}{\sigma}}\right) dx$$

$$= \frac{1}{\sigma}\int_{e^{\frac{\alpha+k}{\sigma}}}^{\infty} e^x u\, e^{-u}\left(\frac{\sigma}{u}\right) du \qquad \text{Take } u = e^{\frac{\alpha+x}{\sigma}}$$

$$= \int_{e^{\frac{\alpha+k}{\sigma}}}^{\infty} e^x\, e^{-u}\, du$$

$$= e^{-\alpha}\int_{e^{\frac{\alpha+k}{\sigma}}}^{\infty} u^{\sigma}\, e^{-u}\, du$$

B. Second Integral

$$\int_k^{\infty} f_{X_T}(x)\, dx = \frac{1}{\sigma}\int_k^{\infty} e^{\frac{\alpha+x}{\sigma}} exp\left(-e^{\frac{\alpha+x}{\sigma}}\right) dx = \frac{1}{\sigma}\int_{e^{\frac{\alpha+k}{\sigma}}}^{\infty} u\, e^{-u}\left(\frac{\sigma}{u}\right) du$$

Take $u = e^{\frac{\alpha+x}{\sigma}}$

$$= \int_{e^{\frac{\alpha+k}{\sigma}}}^{\infty} u\, e^{-u} du$$

Now, by putting both the integrals in equation (1) we have price of a European call option under the assumption that losses are Gumbel distributed at maturity is given by

$$C = e^{-(r-q)T}\left(S_0 \frac{e^{-\alpha}}{e^{-e^{-\frac{b-\alpha}{\sigma}}} - e^{-e^{-\frac{a-\alpha}{\sigma}}}}\int_{e^{\frac{\alpha+k}{\sigma}}}^{\infty} u^{\sigma} e^{-u} du + K\frac{1}{e^{-e^{-\frac{b-\alpha}{\sigma}}} - e^{-e^{-\frac{a-\alpha}{\sigma}}}}\int_{e^{\frac{\alpha+k}{\sigma}}}^{\infty} e^{-u} du\right)$$

By using Put-Call Parity relation formula,

$$C - P = S_0 - Ke^{-(r-q)T}$$

The price of a European put option under the assumption that losses are Gumbel distributed at maturity is given by

$$P = S_0\left[e^{-(r-q)T}\frac{e^{-\alpha}}{e^{-e^{-\frac{b-\alpha}{\sigma}}} - e^{-e^{-\frac{a-\alpha}{\sigma}}}}\int_{e^{\frac{\alpha+k}{\sigma}}}^{\infty} u^{\sigma} e^{-u} du - 1\right] + K\frac{1}{e^{-e^{-\frac{b-\alpha}{\sigma}}} - e^{-e^{-\frac{a-\alpha}{\sigma}}}} e^{-(r-q)T}\left[\int_{e^{\frac{\alpha+k}{\sigma}}}^{\infty} u e^{-u} du + 1\right] \quad (2)$$

Further, we compare numerically both our model with classical Black-Scholes model on actual market data and conclude some interesting phenomenon.

Numerical Examples

1. Consider the Stock Index: NSE, Company Name: TATA STEEL LIMITED (TATASTEEL), Current price: Rs.358.20, Strike price: Rs.370, Expiration time: 6 days, Interest rate: 10%, Implied Volatility: 45.12%, Premium: Rs.15.90 (which was traded in the market), Expiration date: 31/10/2019, Current date: 25/10/2019.
Clearly, $T - t = 0.0247$
By using Black-Scholes put formula equation, we have $P_{BS} = Rs. 14.60$
Now, by using equation (2), we get put premium as $P = Rs. 14.95$

2. Consider the Stock Index: NSE, Company Name: INFOSYS LIMITED (INFY), Current price: Rs.637.10, Strike price: Rs.650, Expiration time: 6 days, Interest rate: 10%, Implied Volatility: 41.59%, Premium: Rs.20.10 (which was traded in the market), Expiration date: 31/10/2019, Current date: 25/10/2019.
Clearly, $T - t = 0.0247$. By using Black-Scholes put formula equation, we have $P_{BS} = Rs. 19.40$
Now, by using equation (2), we get put premium as $P = Rs. 19.75$

II. Conclusion

The basic Black-Scholes formula is formulated on the basis of the assumption that the returns follow log-normal distribution and it follows Geometric Brownian motion. But, our formula follow Gumbel distribution returns which follows Levy process. In this paper, we derived and compare our formula which follows Gumbel distribution with the classical Black-Scholes formula. This formula is relevant with the actual market data. Our derived formula helps the traders to predict the best premium price for the options traded in the market.

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