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Riemannian Curvature Tensor on Trans -Sasakian Manifold

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Abstract:
Background: *Oubina, J.A.[1] defined and initiated the study of Trans-Sasakian manifolds. Blair [2] ,Prasad and Ojha [3], Hasan Shahid [4] and some other authors have studied different properties of C-R-Sub –manifolds of Trans-Sasakian manifolds. Golab, S. [5] studied the properties of semi-symmetric and Quarter symmetric connections in Riemannian manifold. Yano,K.[6] has defined contact conformal connection and studied some of its properties in a sasakian manifold. Mishra and Pandey [7] have studied the properties in Quarter symmetric metric F-connections in an almost Grayan manifold.*
***Result** :In this paper we have studied Riemannian curvature tensor on Trans-Sasakian manifold. Following the patterns of Yano [6], we have proved that a Trans –Sasakian manifold admitting a killing structure vector is an (α, 0) type Trans –Sasakian manifold. Further we have proved that a Trans –Sasakian manifold with structure 1-form A is closed,becomes (β,0) type Trans –Sasakian manifold.*
***Conclusion:** Trans –Sasakian manifold admitting a killing structure vector is an (α, 0) type Trans –Sasakian manifold. And a Trans –Sasakian manifold with structure 1-form A is closed ,becomes (β,0) type Trans –Sasakian manifold.*
***Key words:** Riemannian curvature tensor, Trans-Sasakian manifold,C-R-Sub –manifolds of Trans-Sasakian manifolds,semi-symmetric and Quarter symmetric connections in Riemannian manifold, almost Grayan manifold.*

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I. Introduction

Let M_n (n = 2m + 1) be an almost contact metric manifold endowed with a (1,1)-type structure tensor F, a contravariant vector field T, a -1 form A associated with T and a metric tensor 'g' satisfying :---

(1.1)(a) $F^2X = -X + A(X)T$

(1.1)(b) $FT = 0$

(1.1)(c) $A(FX) = 0$

(1.1)(d) $A(T) = 1$

and

(1.2)(a) $g(\underline{X},\underline{Y}) = g(X, Y) – A(X)A(Y)$

Where

(1.2)(b) $\underline{X} \stackrel{\text{def}}{=} FX$

And

(1.2)(c) $g(T, X) \stackrel{\text{def}}{=} A(X)$

For all C^∞- vector fields X,Y in M_n also , a fundamental 2-form 'F in M_n is defined as

(1.3) $'F(X,Y) = g(\underline{X},Y) = -g(X,\underline{Y}) = -'F(Y,X)$

Then, we call the structure bundle {F,T,A,g}an almost contact-metric structure [1]

An almost contact metric structure is called normal [1], if

(1.4)(a) $(dA)(X,Y)T + N(X,Y) = 0$

Where

(1.4)(b) $(dA)(X,Y) = (D_XA)(Y) - (D_YA)(X)$, D is the Riemannian connection in M_n .

And

(1.5) $N(X,Y) = (D_{\bar{X}}\ F)(\bar{Y}) - (D_{\bar{Y}}\ F)(\bar{X}) -\underline{(D_X\ F)(Y)}+\underline{(D_YF)(X)}$

Is Nijenhenus tensor in M_n .

An almost contact metric manifold M_n with structure bundle {F,T,A,g} is called a Trans-Sasakian manifold [3]&[1],if

(1.6) $(D_XF)(Y)=\alpha\{g(X,Y)T –A(Y)X\}+\beta\{'F(X,Y)T –A(Y)\underline{X}\}$

Where α , β are non -zero constants.

Itcan be easily seen that a Trans-Sasakian manifold is normal. In view of (1.6) one can easily obtain in M_n , the relations

(1.7) $N(X, Y) = 2\alpha' F(X,Y)T$
(1.8) $(dA)(X,Y) = -2\alpha' F(X,Y)$
(1.9) $(D_X A)(Y) + (D_Y A)(X) = 2\beta\{g(X,Y) - A(Y)A(X)\}$
(1.10) $(D_X\ 'F)(Y,Z) + (D_Y\ 'F)(Z,X) + (D_Z\ 'F)(X,Y)$
$=2\beta[A(Z)'F(X,Y) + A(X)\ 'F(Y,Z) + A(Y)\ 'F(Z,X)]$
(1.11)(a) $(D_X A)(Y) = -\alpha' F(X,Y) + \beta\{g(X,Y) - A(X)A(Y)\}$
(1.11)(b) $(D_X T) = -\alpha\underline{X} + \beta\{X - A(X)T\}$
REMARK (1.1): In the above and in what follows, the letters X,Y,Z …..etc. an C^∞- vector fields in M_n .

II. Riemannian Curvature Tensor On Trans-Sasakian Manifold:

From (1.11)(b) given by
$(D_Y T) = -\alpha\underline{Y} + \beta\{Y - A(Y)T\}$
we obtain, in view of (1.6)
(2.1) $K(X,Y,T) = D_X D_Y T - D_Y D_X T - D_{[X,Y]}T$
$= (\alpha^2 - \beta^2)\{A(Y)X - A(X)Y\} + 2\alpha\beta\{A(Y)\underline{X} - A(X)\underline{Y}\}$
Where K(X, Y, Z) is the Riemannian curvature tensor with respect to the Riemannian connection D. From (2.1),we have the following relations
(2.2)(a) $K(X,T,T) = -(\alpha^2 - \beta^2)\{X - A(X)T\} + 2\alpha\beta\underline{X}$
(2.2)(b) $K(T,T,T) = 0$
(2.2)(c) $'K(X,Y,T,T) \overset{def}{=} g(K(X,Y,T),T) = 0$
Also by contracting (2.1)with respect to X, we get
(2.3)(a) $Ric(Y,T) = (n-1)(\alpha^2 - \beta^2)A(Y)$
Further, putting T for Y in (2.3)(a), we get
(2.3)(b) $Ric(T,T) = (n-1)(\alpha^2 - \beta^2)$
Again, barring Y in (2.3)(a),we can get
(2.3)(c) $Ric(\underline{Y},T) = 0$
Also (2.3)(a) gives
(2.3)(d) $R(T) = (n-1)(\alpha^2 - \beta^2)T$
Thus, we have
THEOREM (2.1): In a Trans-Sasakian manifold M_n the equation (2.1),(2.2) and (2.3) hold good.
Now, differentiating covariantly the equation (2.1) with respect to a vector field Z, we obtain, in view of the equation (1.6), (1.11)(b)
(2.4) $(D_Z K)(X,Y,T) - \alpha K(X,Y,\underline{Z}) + \beta K(X.Y.Z) - \beta A(Z)K(X,Y,T)$
$= \alpha(\alpha^2 - \beta^2)['F(Z,X)Y - 'F(Z,Y)X] + \beta(\alpha^2 - \beta^2)[g(Z,Y)X - g(Z,X)Y - A(Z)A(Y)X + A(Z)A(X)Y] + 2\alpha^2\beta['F(Z,X)\underline{Y}$
$- 'F(Z,Y)\underline{X} + A(Y)g(Z,X)T - A(X)g(Z,Y)T] + 2\alpha\beta^2[g(Z,Y)\underline{X} - g(Z,X)\underline{Y} - A(Z)A(Y)\underline{X} + A(Z)A(X)\underline{Y}$
$+ A(Y)'F(Z,X)T - A(X)'F(Z,Y)T]$
Now, putting T for Z in (2.4), we get
(2.5) $(D_T K)(X,Y,T) = 0$
Also, contracting (2.4) with respect to Z, we obtain
(2.6) $(D_{iv}K)(X,Y,T) = -2\alpha(\alpha^2 - 2\beta^2)'F(X,Y)$
Thus,we have
THEOREM (2.2): In a Trans-Sasakian manifold M_n we have
$(D_T K)(X,Y,T) = 0$
$(D_{iv}K)(X,Y,T) = -2\alpha(\alpha^2 - 2\beta^2)'F(X,Y)$
Now,suppose T is a killing vector,i.e.
(2.7) $(D_X A)(Y) + (D_Y A)(X) = 0$
Then, in view of (1.8) and (2.7), we easily get
(2.8)(a) $(D_X A)(Y) = -\alpha' F(X,Y)$
(2.8)(b) $D_X T = -\alpha\underline{X}$
from which , we have
COROLLARY(2.1):A Trans-Sasakian manifold M_n admitting a killing structure vector T is an (α ,0) type Trans –Sasakian manifold.
COROLLARY(2.2): In a (α ,0) type Trans –Sasakian manifold, we have
(2.9)(a) $(D_Z K)(X,Y,T) - \alpha K(X,Y,\underline{Z}) = \alpha^3\{'F(Z,X)Y - 'F(Z,Y)X\}$
(2.9)(b) $(D_{IV}K)(X,Y,T) = -2\alpha^3\ 'F(X,Y)$
PROOF: Putting β = 0 in (2.4) and (2.6), we immediately obtain the above result in (2.9)

COROLLARY(2.3):A Trans-Sasakian manifold M_n with structure 1-form A is closed, becomes (0,β) type Trans-Sasakian manifold.
PROOF: The 1-form A is closed, i.e.
(2.10) $(dA)(X,Y) = (D_X A)(Y) - (D_Y A)(X) = 0$
Using this in (1.8), we easily get α = 0, so that M_n becomes (0,β) type Trans-Sasakian manifold.
COROLLARY(2.4): In a (0,β) Trans-Sasakian manifold, we have
(2.11)(a) $K(X,Y,T) = -\beta^2[A(Y)X - A(X)Y]$
(2.11)(b) $(D_Z K)(X,Y,T) + \beta K(X,Y,Z) - \beta A(Z)K(X,Y,T)$
$= -\beta^2[g(Z,Y)X - g(Z,X)Y - A(Z)A(Y)X + A(Z)A(X)Y]$
(2.11)(c) $(D_{IV}K)(X,Y,T) = 0$
PROOF: The above results are also immediate consequence of (2.1),(2.4) and (2.6) for α = 0. Now, we have
(2.12) $K(X,Y,\underline{Z}) = D_X D_Y \underline{Z} - D_Y D_X \underline{Z} - D_{[X,Y]}\underline{Z}$
$= D_X\{(D_Y F)(Z)\} + D_X\{\underline{D_Y Z}\} - D_Y\{(D_Y F)(Z)\} - D_Y\{\underline{D_X Z}\} - \{D_{[X,Y]}F\}(Z) - \underline{D_{[X,Y]}Z}$
$= D_X\{(D_Y F)(Z)\} + (D_X F)(D_Y Z) + \underline{D_X D_Y} Z - D_Y\{(D_X F)(Z)\} - (D_Y F)(D_X Z) - \underline{D_Y D_X} Z$
$-\{D_{[X,Y]}F\}(Z) - \underline{D_{[X,Y]}Z}$
Using (1.6) in the above equation, we get
$K(X,Y,\underline{Z}) = D_X[\alpha\{g(Y,Z)T - A(Z)Y\} + \beta\{'F(Y,Z)T - A(Z)\underline{Y}\}] + \alpha\{g(X,D_Y Z)T - A(D_Y Z)X\}$
$+ \beta\{'F(X,D_Y Z)T) - A(D_Y Z)\underline{X}\} - D_Y[\alpha\{g(X,Z)T - A(Z)X\} + \beta\{'F(X,Z)T - A(Z)\underline{X}\}]$
$- \alpha\{g(Y,D_X Z)T - A(D_X Z)Y\} - \beta\{'F(Y,D_X Z)T - A(D_X Z)\underline{Y}\} + K(\underline{X,Y,Z}) - \alpha\{g[(X,Y),Z]T$
$-A(Z)(X,Y)\} - \beta\{'F[(X,Y),Z]T - A(Z)\underline{(X,Y)}\}$
again using (1.6), (1.11)(b) in this result, we obtain

$K(X,Y,\underline{Z}) = K(\underline{X,Y,Z}) - (\alpha^2 - \beta^2)\{g(Y,Z)\underline{X} - g(X,Z)\underline{Y}\} + 2\alpha\beta\{g(Y,Z)X - g(X,Z)Y\}$
$+ \alpha^2\{'F(X,Z)Y - 'F(Y,Z)X\} - \beta^2\{A(Y)'F(X,Z)T - A(X)'F(Y,Z)T\} + \alpha\beta\{'F(X,Z)\underline{Y}$
$- 'F(Y,Z)\underline{X}\}$
From which, we easily obtain
(2.13) $'K(X,Y,\underline{Z},U) + 'K(X,Y,Z,\underline{U})$
$= (\alpha^2 - \beta^2)\{g(Y,Z)'F(X,U) - g(X,Z)'F(Y,U)\} + 2\alpha\beta\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\}$
$+ \alpha^2\{'F(X,Z)g(Y,U) - 'F(Y,Z)g(X,U)\} - \beta^2\{A(Y)A(U)'F(X,Z) - A(X)A(U)'F(Y,Z)T\}$
$+ \alpha\beta\{'F(X,Z)\,'F(Y,U) - 'F(Y,Z)\,'F(X,U)\}$
Putting T for U in the above and then barring X and Y, we easily get
(2.14)(a) $'K(X,Y,\underline{Z},T) = 2\alpha\beta\{A(X)g(Y,Z) - A(Y)g(X,Z)\} - (\alpha^2 - \beta^2)\{A(X)'F(Y,Z) - A(Y)\,'F(X,Z)\}$
And
(2.14)(b) $'K(\underline{X},\underline{Y},\underline{Z},T) = 0$
Thus,we have
THEOREM (2.3): In a Trans-Sasakian manifold M_n , we have
$'K(X,Y,\underline{Z},T) = 2\alpha\beta\{A(X)g(Y,Z) - A(Y)g(X,Z)\} - (\alpha^2 - \beta^2)\{A(X)'F(Y,Z) - A(Y)\,'F(X,Z)\}$
And
$'K(\underline{X},\underline{Y},\underline{Z},T) = 0$

III. Conclusion

Trans –Sasakian manifold admitting a killing structure vector is an (α, 0) type Trans –Sasakian manifold. And a Trans –Sasakian manifold with structure 1-form A is closed ,becomes (β,0) type Trans –Sasakian manifold.

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