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A Reviewon the TheoryofContinued Fractions

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Abstract: *In this article, the theory of continued fractions is presented. There aretwo types of continued fraction, one is the finite continued fraction and the other is the infinite continued fraction. A rational number can be expressed as a finite continued fraction. The value of an infinite continued fraction is an irrational number. The ratio of two successive Fibonacci numbers, which is a rational number, can be written as a simple finite continued fraction. The golden ratio can be expressed as an infinite continued fraction. The concept of golden ratio finds application in architecture. Using the convergents of finite continued fraction, the relation between Fibonacci numbers can be calculated and Linear Diophantine equations will be solved.*

Keywords*:Continued fraction, Convergent, Fibonacci numbers, Golden ratio.*

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I. Introduction

The credit for introducing continued fraction goes toFibonacci. The nickname of famous Italian mathematician Leonardo Pisano(1170-1250) is Fibonacci. In his book *LiberAbaci,* while dealing with the resolution of fractions into unit fractions, Fibonacci introduced a kind of "continued function". For example, he used the symbol $\frac{1\,1\,1}{3\,4\,5}$ as an abbreviation for

$$\frac{1+\frac{1+\frac{1}{5}}{4}}{3} = \frac{1}{3} + \frac{1}{3.4} + \frac{1}{3.4.5} \quad (1)$$

Continued fraction is two types, (a) Finite continued fraction and (b) Infinite continued fraction.A fraction of the form given below is known as finite continued fraction.

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots \, \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}} \quad (2)$$

Where, the numbers $a_1, a_2, \dots, a_n$ are the *partial denominators* of the finite continued fraction and they all are real numbers. The number a_0 may be zero or positive or negative. This fraction is denoted by the symbol $[a_0; a_1, a_2, \dots., a_n]$ and it is called *simple* if all of the a_i are integers. The value of any finite simple continued fraction will always be a rational number.

If $a_n > 1$ in the finite continued fraction (2), then $a_n = (a_n - 1) + 1 = (a_n - 1) + \frac{1}{1}$, where $(a_n - 1)$ is a positive integer. Hence, every rational number has two representations $[a_0; a_1, a_2, \dots., a_n]$ and $[a_0; a_1, a_2, \dots., a_{n-1}, 1]$ as a simple finite continued fraction if $a_n > 1$.

If $s = [a_0; a_1, a_2, \dots, a_n]$, where $s > 1$, then

$$\frac{1}{s} = [0; a_0, a_1, a_2, \dots, a_n] \quad (3)$$

Although due credit is given to Fibonacci for introducing continued fractions, most authorities agree that the theory of continued fractions began with Rafael Bombelli, the great algebraist of Italy. In his book *L'Algebra Opera* (1572), Bombelliattempted to find the value of square roots of integers by means of infinite continued fractions.He expressed $\sqrt{13}$ as an infinite continued fraction.

$$\sqrt{13} = 3 + \cfrac{4}{6 + \cfrac{4}{6 + \cfrac{4}{6 + \ddots}}} \quad (4)$$

In general, an infinite continued fraction is written as

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots}}} \quad (5)$$

Where $a_0, a_1, a_2, ...$ and $b_1, b_2, b_3, ...$ are real numbers. The expansion of an infinite continued fraction continues for ever. The infinite continued fraction in which there is 1 in all the numerators is called *simple* infinite continued fraction. Putting $b_1 = b_2 = b_3 = \cdots = 1$ in (5), the simple infinite continued fraction is written as

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}} \quad (6)$$

The notation $[a_0; a_1, a_2, ...]$indicates a simple infinite continued fraction. The value of any infinite continued fraction is an irrational number.

II. Finite Continued Fractions

Theorem2.1: Any rational number can be written as a simple finite continued fraction.

Proof: Let a/b, where $b > 0$, is an arbitrary rational number. Now, let us write the following equations.

$$\begin{aligned} a &= ba_0 + r_1 & 0 < r_1 < b \\ b &= r_1 a_1 + r_2 & 0 < r_2 < r_1 \\ r_1 &= r_2 a_2 + r_3 & 0 < r_3 < r_2 \quad (7) \\ &\ldots\ldots & \ldots\ldots \\ r_{n-2} &= r_{n-1} a_{n-1} + r_n & 0 < r_n < r_{n-1} \\ r_{n-1} &= r_n a_n + 0 & \end{aligned}$$

The above equations are rewritten as follows.

$$\begin{aligned} \frac{a}{b} &= a_0 + \frac{r_1}{b} = a_0 + \frac{1}{\frac{b}{r_1}} \\ \frac{b}{r_1} &= a_1 + \frac{r_2}{r_1} = a_1 + \frac{1}{\frac{r_1}{r_2}} \\ \frac{r_1}{r_2} &= a_2 + \frac{r_3}{r_2} = a_2 + \frac{1}{\frac{r_2}{r_3}} \quad (8) \\ &\ldots\ldots \\ \frac{r_{n-1}}{r_n} &= a_n \end{aligned}$$

Eliminating b/r_1 in the above first equation using the second equation, we get

$$\frac{a}{b} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\frac{r_1}{r_2}}} \quad (9)$$

Eliminating r_1/r_2 in (9) using third equation of (8), we obtain

$$\frac{a}{b} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \frac{r_2}{r_3}}}$$

Continuing in this way we get the following expression.

$$\frac{a}{b} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}}$$

Thus, the rational number $a/b$is expressed as a simple finite continued fraction. Hence, the Theorem2.1 is proved. As an example, let us apply this Theorem 2.1 to the rational number $\frac{71}{55}$.

$$\begin{aligned} 71 &= 1 \times 55 + 16 & \frac{71}{55} &= 1 + \frac{16}{55} \\ 55 &= 3 \times 16 + 7 & \frac{55}{16} &= 3 + \frac{7}{16} \\ 16 &= 2 \times 7 + 2 & \frac{16}{7} &= 2 + \frac{2}{7} \\ 7 &= 2 \times 3 + 1 & \frac{7}{2} &= 3 + \frac{1}{2} \end{aligned}$$

From the above equations, we obtain

$$\frac{71}{55} = 1 + \frac{1}{\frac{55}{16}} = 1 + \cfrac{1}{3 + \frac{7}{16}} = 1 + \cfrac{1}{3 + \cfrac{1}{2 + \frac{2}{7}}}$$

$$\Rightarrow \frac{71}{55} = 1 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{1}{2}}}} \quad (10)$$

This is the simple finite continued fraction of the rational number 71/55. Since, in general, the finite continued fraction (2) is denoted by the symbol $[a_0; a_1, a_2, \dots, a_n]$, the above continued fraction is denoted by the symbol $[1; 3,2,3,2]$. As, 2 =1 + 1/1, this continued fraction can also be denoted by the symbol $[1; 3,2,3,1,1]$. That is,

$$\frac{71}{55} = [1; 3,2,3,2] = [1; 3,2,3,1,1]$$

Now, using (10), let us represent 55/71 as continued fraction.

$$\frac{55}{71} = \frac{1}{\frac{71}{55}} = \frac{1}{1+\frac{1}{3+\frac{1}{2+\frac{1}{3+\frac{1}{2}}}}} \quad (11)$$

Hence, the rational number 55/71 is represented as [0;1,3,2,3,2] or [0;1,3,2,3,1,1].The examples (10) & (11) prove the statement (3).Similarly, the rational numbers $\frac{19}{51}$, $\frac{172}{51}$, $\left(-\frac{710}{457}\right)$ and $\left(-\frac{15}{23}\right)$ are denoted as given below.

$$\frac{19}{51} = [0; 2,1,2,6] = [0; 2,1,2,5,1]$$
$$\frac{172}{51} = [3; 2,1,2,6] = [3; 2,1,2,5,1]$$
$$-\frac{710}{457} = [-2; 2,4,6,8] = [-2; 2,4,6,7,1]$$
$$-\frac{15}{23} = [-1; 2,1,6,1]$$

The sequence of numbers introduced by Italian mathematician Leonardo Pisano is known as Fibonacci sequence, given by

1,1,2,3,5,8,13,21,34,55,89,144,233, ... (12)

Each term in the sequence after the first two is the sum of the immediately preceding two terms. The n^{th}term, denoted by F_n , is called n^{th} Fibonacci number. Note that

$F_3 = F_2 + F_1 = 1 + 1 = 2$ and $F_6 = F_5 + F_4 = 5 + 3 = 8$

In general, we can write

$F_n = F_{n-1} + F_{n-2}, \ (n \geq 3)$(13)

Let us write the following equations for Fibonacci numbers using (13).

$$F_{n+1} = 1 \times F_n + F_{n-1} \qquad \frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}$$
$$F_n = 1 \times F_{n-1} + F_{n-2} \qquad \frac{F_n}{F_{n-1}} = 1 + \frac{F_{n-2}}{F_{n-1}}$$

.... (14)

$$F_4 = 1 \times F_3 + F_2 \qquad \frac{F_4}{F_3} = 1 + \frac{F_2}{F_3}$$
$$F_3 = 2 \times F_2 + 0 \qquad \frac{F_3}{F_2} = 2$$

Using the above equations, we obtain

$$\frac{F_{n+1}}{F_n} = 1 + \frac{1}{\frac{F_n}{F_{n-1}}} = 1 + \frac{1}{1+\frac{1}{\frac{F_{n-1}}{F_{n-2}}}}$$

Continuing in this manner, one will get

$$\frac{F_{n+1}}{F_n} = 1 + \frac{1}{\frac{F_n}{F_{n-1}}} = 1 + \frac{1}{1+\frac{1}{\frac{F_{n-1}}{F_{n-2}}\ddots_{1+\frac{1}{2}}}} (15)$$

$$\Rightarrow \frac{F_{n+1}}{F_n} = [1; 1,1, \dots ,1,2] = [1; 1,1, \dots ,1,1,1] (16)$$

The above expression (16) shows that the ratio of two successive Fibonacci numbers F_{n+1}/F_n , which is a rational number, can be written as a simple finite continued fraction.

III. Convergentsof Finite Continued Fraction

Definition: The continued fraction made from $[a_0; a_1, a_2, \dots, a_n]$ by cutting off the expansion after the k^{th} partial denominator a_k is called the k^{th}*convergent* of the given continued fraction and denoted by C_k.

$C_k = [a_0; a_1, a_2, \dots, a_k]$(17)

$C_0 = a_0$(18)

The convergent C_0 is called the zeroth convergent. Going back to the example (10), let us write the successive convergents of$\frac{71}{55} = [1; 3,2,3,2]$.

$$C_0 = 1$$
$$C_1 = [1;3] = 1 + \frac{1}{3} = \frac{4}{3} = 1.3333\ldots$$
$C_2 = [1;3,2] = 1 + \frac{1}{3+\frac{1}{2}} = \frac{9}{7} = 1.2857$ ...(19)
$$C_3 = [1;3,2,3] = 1 + \frac{1}{3+\frac{1}{2+\frac{1}{3}}} = \frac{31}{24} = 1.2916\ldots$$
$$C_4 = [1;3,2,3,2] = \frac{71}{55} = 1.2909\ldots$$

The above values of convergents show that, except for the last convergent C_4, these are alternately less than or greater than 71/55, each convergent being closer in value to 71/55 than the previous one.

The following are the three important properties of convergents of a continued fraction.

(a) The convergents with even subscripts form an increasing sequence, that is,
$$C_0 < C_2 < C_4 < \cdots$$
(b) The convergents with odd subscripts form a decreasing sequence, that is,
$$C_1 > C_3 > C_5 > \cdots$$
(c) Every convergent with an odd subscript is greater than every convergent with an even subscript. That is,
$$C_0 < C_2 < C_4 < \cdots < C_{2n} < \cdots < C_{2n+1} < \cdots < C_5 < C_3 < C_1$$

Theorem3.1: The value of k^{th} convergent C_k of the finite simple continued fraction $[a_0; a_1, a_2, \ldots, a_n]$ is p_k/q_k.

$C_k = \frac{p_k}{q_k}$(20)

The numbers p_k and q_k $(k = 0,1,2,\ldots,n)$are defined as

$p_0 = a_0$ $\quad q_0 = 1$(21)

$p_1 = a_1 a_0 + 1$ $\quad q_1 = a_1$(22)

$p_k = a_k p_{k-1} + p_{k-2}$ $\quad q_k = a_k q_{k-1} + q_{k-2} fork = 2,3,\ldots,n$(23)

Proof: As per (17) and (18), the convergents of $[a_0; a_1, a_2, \ldots, a_n]$ are given by

$C_0 = a_0 = \frac{a_0}{1} = \frac{p_0}{q_0}$ [Using (21)] (24)

$C_1 = [a_0; a_1] = a_0 + \frac{1}{a_1} = \frac{a_1 a_0 + 1}{a_1} = \frac{p_1}{q_1}$ [Using (22)](25)

$C_2 = [a_0; a_1, a_2] = a_0 + \frac{1}{a_1 + \frac{1}{a_2}} = \frac{a_2(a_1 a_0 + 1) + a_0}{a_2 a_1 + 1}$(26)

For $k = 2$, we have from(21),(22) & (23),

$p_2 = a_2 p_1 + p_0 = a_2(a_1 a_0 + 1) + a_0$(27)

$q_2 = a_2 q_1 + q_0 = a_2 a_1 + 1$(28)

Substituting (27) & (28) in (26), we obtain

$C_2 = \frac{p_2}{q_2}$(29)

Noting (24), (25) and (29), we can write in general that
$$C_k = \frac{p_k}{q_k}$$

Hence,the Theorem3.1 is proved.

Let us see how this theorem works in case of the example $\frac{71}{55} = [1;3,2,3,2]$. In this case $a_0 = 1, a_1 = 3, a_2 = 2, a_3 = 3$ &$a_4 = 2$.Using (21), (22) and (23), we calculate p_k and q_k for $k = 0,1,2,3$ &4.

$$p_0 = a_0 = 1 \qquad q_0 = 1$$
$$p_1 = a_1 a_0 + 1 = 3 \times 1 + 1 = 4 \qquad q_1 = a_1 = 3$$
$$p_2 = a_2 p_1 + p_0 = 2 \times 4 + 1 = 9 \qquad q_2 = a_2 q_1 + q_0 = 2 \times 3 + 1 = 7$$
$$p_3 = a_3 p_2 + p_1 = 3 \times 9 + 4 = 31 \qquad q_3 = a_3 q_2 + q_1 = 3 \times 7 + 3 = 24$$
$$p_4 = a_4 p_3 + p_2 = 2 \times 31 + 9 = 71 \qquad q_4 = a_4 q_3 + q_2 = 2 \times 24 + 7 = 55$$

Using the above values, the convergents of$\frac{71}{55} = [1;3,2,3,2]$ aregiven by

$C_0 = \frac{p_0}{q_0} = \frac{1}{1} = 1$, $C_1 = \frac{p_1}{q_1} = \frac{4}{3}$, $C_2 = \frac{p_2}{q_2} = \frac{9}{7}$, $C_3 = \frac{p_3}{q_3} = \frac{31}{24}$and $C_4 = \frac{p_4}{q_4} = \frac{71}{55}$

The above values for convergents are same as those given in (19) for the continued fraction [1;3,2,3,2]. So, the Theorem3.1 is proved for the continued fraction [1;3,2,3,2].

Theorem3.2: If $C_k = \frac{p_k}{q_k}$ is the k^{th}convergent of the finite simple continued fraction $[a_0; a_1. a_2, \ldots, a_n]$, then

$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1} 1 \leq k \leq n$(30)

Proof:
(a) For $k = 1$, the LHS of (30) = $p_1 q_0 - q_1 p_0$
$= (a_1 a_0 + 1) \times 1 - a_1 \times a_0$ [Using (21) & (22)]
$= 1 = (-1)^{1-1}$ = RHS of (30) for $k = 1$
So, the given theorem is proved for $k = 1$.
(b) Let us assume that the formula (30) is true for $k = m$, where $1 < m < n$.
For $k = m + 1$, LHS of (30) = $p_{m+1} q_m - q_{m+1} p_m$
$= (a_{m+1} p_m + p_{m-1}) q_m - (a_{m+1} q_m + q_{m-1}) p_m$
[Using (23) for $k = m + 1$]
$= -(p_m q_{m-1} - q_m p_{m-1})$
$= -(-1)^{m-1}$ [Using (30) for $k = m$]
$= (-1)^m$
= RHS of (30) for $k = m + 1$
Hence, the theorem is true for $k = m + 1$, whenever it holds good for $k = m$.It follows by induction that the theorem is valid for all k with $1 \le k \le n$.
Corollary 3.1: If $C_k = p_k / q_k$ is the k^{th} convergent of the simple finite continued fraction $[a_0; a_1, a_2, \ldots, a_n]$ and $a_0 > 0$, then

$$\frac{p_k}{p_{k-1}} = [a_k; a_{k-1}, \ldots \ldots, a_1, a_0] \quad (31)$$

and

$$\frac{q_k}{q_{k-1}} = [a_k; a_{k-1}, \ldots \ldots, a_2, a_1] (32)$$

Proof: Using (21),(22) and (23), we obtain

$$p_k = a_k p_{k-1} + p_{k-2} \; \frac{p_k}{p_{k-1}} = a_k + \frac{p_{k-2}}{p_{k-1}}$$

$$p_{k-1} = a_{k-1} p_{k-2} + p_{k-3} \; \frac{p_{k-1}}{p_{k-2}} = a_{k-1} + \frac{p_{k-3}}{p_{k-2}}$$

$$p_{k-2} = a_{k-2} p_{k-3} + p_{k-4} \; \frac{p_{k-2}}{p_{k-3}} = a_{k-2} + \frac{p_{k-4}}{p_{k-3}}$$

....

$$p_1 = a_1 a_0 + 1 = a_1 p_0 + 1 \qquad \frac{p_1}{p_0} = a_1 + \frac{1}{p_0} = a_1 + \frac{1}{a_0}$$

$$\frac{p_k}{p_{k-1}} = a_k + \frac{1}{\frac{p_{k-1}}{p_{k-2}}} = a_k + \frac{1}{a_{k-1} + \frac{p_{k-3}}{p_{k-2}}} = a_k + \frac{1}{a_{k-1} + \frac{1}{a_{k-2} + \frac{p_{k-4}}{p_{k-3}}}}$$

$$\ddots$$

$$\frac{1}{a_1 + \frac{1}{a_0}}$$

$$\Rightarrow \frac{p_k}{p_{k-1}} = [a_k; a_{k-1}, \ldots, a_1, a_0]$$

Hence, the relation (31) is proved. Similarly, using (21),(22) & (23), we can write

$$q_k = a_k q_{k-1} + q_{k-2} \; \frac{q_k}{q_{k-1}} = a_k + \frac{q_{k-2}}{q_{k-1}}$$

$$q_{k-1} = a_{k-1} q_{k-2} + q_{k-3} \; \frac{q_{k-1}}{q_{k-2}} = a_{k-1} + \frac{q_{k-3}}{q_{k-2}}$$

$$q_{k-2} = a_{k-2} q_{k-3} + q_{k-4} \; \frac{q_{k-2}}{q_{k-3}} = a_{k-2} + \frac{q_{k-4}}{q_{k-3}}$$

....

$$q_2 = a_2 q_1 + q_0 \; \frac{q_2}{q_1} = a_2 + \frac{q_0}{q_1} = a_2 + \frac{1}{a_1}$$

$$\frac{q_k}{q_{k-1}} = a_k + \frac{1}{\frac{q_{k-1}}{q_{k-2}}} = a_k + \frac{1}{a_{k-1} + \frac{q_{k-3}}{q_{k-2}}} = a_k + \frac{1}{a_{k-1} + \frac{1}{a_{k-2} + \frac{q_{k-4}}{q_{k-3}}}} = [a_k; a_{k-1}, \ldots, a_2, a_1]$$

$$\ddots$$

$$\frac{1}{a_2 + \frac{1}{a_1}}$$

Thus, the expression (32) is proved.

*Corollary 3.2:*If $C_k = p_k/q_k$ is the k^{th} convergent of the simple finite continued fraction $[a_0; a_1, a_2, \dots, a_n]$, then

$C_k - C_{k-1} = \frac{(-1)^{k-1}}{q_k q_{k-1}}$ (33)

Proof: $C_k - C_{k-1} = \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{p_k q_{k-1} - q_k p_{k-1}}{q_k q_{k-1}}$

Using (30) in the numerator of RHS of the above expression, we get

$$C_k - C_{k-1} = \frac{(-1)^{k-1}}{q_k q_{k-1}}$$

Hence, the relation (33) is proved.

IV. Relation Between Fibonacci Numbers UsingConvergents

In modern usage, the Fibonacci sequence begins with 0. The Fibonacci sequence is given by

0,1,1,2,3,5,8,13,21,34,55,89,.....(34)

If F_k with $k \geq 0$ denotes k^{th}Fibonacci number, then

$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, \dots$..(35)

The Fibonacci sequence (34) satisfies the relation

$F_k = F_{k-1} + F_{k-2}$for $k \geq 2$(36)

Consider the continued fraction $[0; 1,1,1, \dots, 1]$ in which all the patial denominators are equal to 1. The first few convergents of this continued fraction are written in terms of Fibonacci numbers as given below.

$C_0 = 0 = \frac{0}{1} = \frac{F_0}{F_1}$, $C_1 = [0; 1] = 0 + \frac{1}{1} = \frac{1}{1} = \frac{F_1}{F_2}$, $C_2 = [0; 1,1] = 0 + \frac{1}{1+\frac{1}{1}} = \frac{1}{2} = \frac{F_2}{F_3}$,

$$C_3 = [0; 1,1,1] = \frac{2}{3} = \frac{F_3}{F_4}, \qquad C_4 = [0; 1,1,1,1] = \frac{3}{5} = \frac{F_4}{F_5}$$

Looking at the above relations, we can write in general

$C_k = \frac{F_k}{F_{k+1}}$ for $k \geq 2$(37)

Using the above expression in (20), we obtain

$C_k = \frac{p_k}{q_k} = \frac{F_k}{F_{k+1}}$(38)

Hence, we can take

$p_k = F_k$ and $q_k = F_{k+1}$(39)

Taking the relation (30), we have

$$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}$$

Using (39) in the above expression,we obtain

$$F_k F_k - F_{k+1} F_{k-1} = (-1)^{k-1}$$

$\Rightarrow F_k^2 - F_{k+1} F_{k-1} = (-1)^{k-1}$ (40)

Thus, the relation between Fibonacci numbersis derived from convergents of continued fraction. Forexample, if $k = 4$, LHS of (40) is$F_4^2 - F_5 F_3 = 3^2 - 5 \times 2 = -1 = (-1)^{4-1}$, which is the RHS of (40).

V. Solutionof Linear Diophantine Equation UsingConvergents

The linear Diophantine equation is

$ax + by = c$(41)

Where a, b &c are given integers. A solution of the above equation is obtained by solving the Diophantine equation

$ax + by = 1$ (42)

Putting $x = x_0$ and $y = y_0$ in (42), we get

$ax_0 + by_0 = 1$ (43)

Multiplication of both sides of above equation with c gives

$a(cx_0) + b(cy_0) = c$(44)

The solution of Diophantine equation (41) is given by comparing it with (44). So, the desired solution is

$x = cx_0$ and $y = cy_0$(45)

Let the rational number $\frac{a}{b}$ is expanded into simple finite continued fraction as given below

$$\frac{a}{b} = [a_0; a_1, a_2, \dots, a_n]$$

According to (17) and (20), the last two convergents of the above continued fraction are
$C_{n-1} = \frac{p_{n-1}}{q_{n-1}}$ and $C_n = \frac{p_n}{q_n} = \frac{a}{b}$(46)
From the above equation, we can write
$p_n = a$ and $q_n = b$(47)
By changing the index$k$ to n, the Eqn.(30) can be written a

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}$$

Using (47) in the above expression, we have
$aq_{n-1} - bp_{n-1} = (-1)^{n-1}$(48)
Let us now consider two cases for finding the solution of linear Diophantine equation.
(a) If n is odd, the Eqn. (48) is$aq_{n-1} - bp_{n-1} = 1$.Then,the Eqn.$ax + by = 1$ has a particular solution $x_0 = q_{n-1}$and $y_0 = -p_{n-1}$. Hence, according to (45), the solution of linear Diophantine equation is
$x = cq_{n-1}$ and $y = -cp_{n-1}$(49)
(b) If $n$is even, the Eqn. (48) is $(-aq_{n-1} + bp_{n-1}) = 1$. So, the Eqn. $ax + by = 1$ has a particular solution $x_0 = -q_{n-1}$ and $y_0 = p_{n-1}$. Then, the solution of linear Diophantine equation is
$x = -cq_{n-1}$ and $y = cp_{n-1}$(50)
The general solution of Linear Diophantine equation (41) is given by the equations
$x = cx_0 + bt$, $y = cy_0 - at$, for $t = 0, \pm1, \pm2, \dots$.(51)
Example: Let us solve the linear Diophantine equation $18x + 5y = 24$. In this example $a = 18$, $b = 5$ and $c = 24$.

$$18 = 3 \times 5 + 3 \qquad \frac{18}{5} = 3 + \frac{3}{5}$$

$$5 = 1 \times 3 + 2 \qquad \frac{5}{3} = 1 + \frac{2}{3}$$

$$3 = 1 \times 2 + 1 \qquad \frac{3}{2} = 1 + \frac{1}{2}$$

$$\frac{18}{5} = 3 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}} = [3; 1,1,2]$$

In this continued fraction $n = 3$ and the convergent C_2 is given by

$$C_2 = [3; 1,1] = 3 + \cfrac{1}{1 + \frac{1}{1}} = \frac{7}{2} = \frac{p_2}{q_2}$$

Therefore, $p_2 = 7$ and $q_2 = 2$. For $n = 3$, the Eqn. (48) is

$$aq_2 - bp_2 = 1 \ \Rightarrow 18 \times 2 - 5 \times 7 = 1$$

So, as per the above expression, the particular solution of $18x + 5y = 1$ is given by
$x_0 = 2$ and $y_0 = -7$
According to (45), the particular solution of the Diophantine equation $18x + 5y = 24$ is

$$x = cx_0 = 24 \times 2 = 48$$

$$y = cy_0 = 24 \times (-7) = -168$$

Using (51), the general solution of the givenDiophantine equation is
$x = 48 + 5t$, $y = -168 - 18t$, $t = 0, \pm1, \pm2, \dots\dots$

VI. Infinite Continued Fraction

An irrational number can be expressed as an infinite continued fraction.The infinite continued fraction in which there is 1 in all the numerators is known as simple infinite continued fraction. Two distinct infinite continued fractions represent two distinct irrational numbers.
An early example of an infinite continued fraction is found in the work of William Brouncker. In 1655, hehad converted Wallis's famous infinite product

$$\frac{4}{\pi} = \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times \dots\dots}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times \dots\dots}$$

into an infinite continued fraction

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ddots}}}}$$

Srinivasa Ramanujan (22 December 1887 – 26 April 1920) had contributed many problems on continued fractions to the *Journal of the Indian Mathematical Society* and his note books contain about 200 results on such fractions. The following are twoinfinite continued fractions given by Ramanujan.

- $e^{2\pi/5}\left\{\sqrt{\frac{5+\sqrt{5}}{2}}-\left(\frac{1+\sqrt{5}}{2}\right)\right\}=\cfrac{1}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\cfrac{e^{-6\pi}}{1+\ddots}}}}$
- $\pi=\cfrac{4}{1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{\ddots}}}}$

(1) Algorithm for expanding an irrational number into a simple infinite continued fraction:

Let x_0 is an arbitrary irrational number and the sequence of integers $a_0, a_1 . a_2, ...$ is defined as given below.

$x_1=\frac{1}{x_0-a_0}, x_2=\frac{1}{x_1-a_1}, x_3=\frac{1}{x_2-a_2}, \ldots ....$ (52)

In general, the above sequence can be written as

$x_{k+1}=\frac{1}{x_k-a_k} \Rightarrow x_k=a_k+\frac{1}{x_{k+1}} \quad (k \geq 0)$ (53)

Let us now expand x_0 into an infinite continued fraction using the above expression.

$$x_0=a_0+\frac{1}{x_1}$$

Then, by successive substitutionsof (53), we obtain

$$x_0=a_0+\cfrac{1}{a_1+\cfrac{1}{x_2}}=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{x_3}}}$$

$$\vdots$$

$\Rightarrow x_0=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\ddots \cfrac{1}{a_n+\cfrac{1}{x_{n+1}}}}}}=[a_0; a_1, a_2, \ldots, a_n, x_{n+1}]$(54)

Applying the definition (17) and the theorem (20) for the convergent to the above expression, we can write

$$x_0=[a_0; a_1, a_2, \ldots, a_n, x_{n+1}]=C_{n+1}$$

$\Rightarrow x_0=\frac{p_{n+1}}{q_{n+1}}$(55)

Using (23) in the above expression, we have

$$x_0=\frac{x_{n+1} p_n+p_{n-1}}{x_{n+1} q_n+q_{n-1}} [\because a_{n+1}=x_{n+1} in\ (54)]$$

Subtracting the convergent C_n from x_0, we get

$$x_0-C_n=\frac{x_{n+1} p_n+p_{n-1}}{x_{n+1} q_n+q_{n-1}}-\frac{p_n}{q_n}$$

$$=\frac{q_n p_{n-1}-p_n q_{n-1}}{(x_{n+1} q_n+q_{n-1}) q_n}$$

$=\frac{-(-1)^{n-1}}{q_{n+1} q_n}$ [Using (23) & (30)]

$$\Rightarrow x_0-C_n=\frac{(-1)^n}{q_{n+1} q_n}$$

Since, $C_n=p_n/q_n$,the values (19) for convergents of $\frac{71}{55}=[1; 3,2,3,2]$ show that, as n increases, the integers q_n are increasing. Hence, the above expression gives that

$$\lim_{n \to \infty}(x_0-C_n) \approx 0$$

$\Rightarrow x_0=\lim_{n \to \infty} C_n=[a_0; a_1, a_2, \ldots ....]$ (56)

Thus, the irrational numberx_0 is expanded into an infinite continued fraction $[a_0; a_1, a_2, a_3 ....]$.The following are four examples for expressing a given irrational number as an infinite continued fraction using this algorithm.

Example 1: Let $x_0=\sqrt{5} \approx 2.236$. The calculations for finding the simple infinite continued fraction expansion of $\sqrt{5}$ using (53) are given below.

$x_0=\sqrt{5}=2+(\sqrt{5}-2)=a_0+\frac{1}{x_1}$. i.e., $a_0=2$

$x_1=\frac{1}{\sqrt{5}-2}=\frac{\sqrt{5}+2}{(\sqrt{5}-2)(\sqrt{5}+2)}=\sqrt{5}+2=4+(\sqrt{5}-2)=a_1+\frac{1}{x_2}$. i.e., $a_1=4$

So, $x_2 = \frac{1}{\sqrt{5}-2}$. According to the above expression, $a_2 = 4$.similarly, $a_3 = a_5 = \cdots = 4$. Hence, the simple infinite continued fraction representation of $\sqrt{5}$ is

$\sqrt{5} = [2; 4,4,4,4, \ldots \ldots]$ (57)

Example 2: Let us now expand $\sqrt{7} \approx 2.6457$ as a simple infinite continued fraction. Taking $x_0 = \sqrt{7}$, let us make the following calculations using (53).

$x_0 = \sqrt{7} = 2 + (\sqrt{7} - 2) = a_0 + \frac{1}{x_1}$. i.e., $a_0 = 2$

$x_1 = \frac{1}{\sqrt{7}-2} = \frac{\sqrt{7}+2}{(\sqrt{7}-2)(\sqrt{7}+2)} = \frac{\sqrt{7}+2}{3} = \frac{3+(\sqrt{7}-1)}{3} = 1 + \frac{\sqrt{7}-1}{3} = a_1 + \frac{1}{x_2}$. i.e., $a_1 = 1$

$x_2 = \frac{3}{\sqrt{7}-1} = \frac{3(\sqrt{7}+1)}{(\sqrt{7}-1)(\sqrt{7}+1)} = \frac{3\sqrt{7}+3}{6} = \frac{6+(3\sqrt{7}-3)}{6} = 1 + \frac{\sqrt{7}-1}{2} = a_2 + \frac{1}{x_3}$. i.e., $a_2 = 1$

$x_3 = \frac{2}{\sqrt{7}-1} = \frac{2(\sqrt{7}+1)}{(\sqrt{7}-1)(\sqrt{7}+1)} = \frac{2\sqrt{7}+2}{6} = \frac{6+(2\sqrt{7}-4)}{6} = 1 + \frac{\sqrt{7}-2}{3} = a_3 + \frac{1}{x_4}$. i.e., $a_3 = 1$

$x_4 = \frac{3}{\sqrt{7}-2} = \frac{3(\sqrt{7}+2)}{3} = \frac{12+(3\sqrt{7}-6)}{3} = 4 + (\sqrt{7} - 2) = a_4 + \frac{1}{x_5}$. i.e., $a_4 = 4$

$x_5 = \frac{1}{\sqrt{7}-2} = x_1$. Then, we get $x_6 = x_2, x_7 = x_3 \& x_8 = x_4$. So, $a_5 = a_1 = 1$, $a_6 = a_2 = 1$, $a_7 = a_3 = 1$and $a_8 = a_4 = 4$. Then, we obtain $x_9 = x_5$,$x_{10} = x_6, x_{11} = x_7 \& x_{12} = x_8$. So, $a_9 = 1$, $a_{10} = 1$, $a_{11} = 1$and $a_{12} = 4$. This shows that the block of integers 1,1,1,4 repeat indefinitely. Thus, the simple infinite continued fraction expansion of $\sqrt{7}$ is given by

$\sqrt{7} = [2; 1,1,1,4,1,1,1,4,1,1,1,4, \ldots \ldots]$(58)

Example 3: Let us find out the infinite continued fraction expansion of $\pi = 3.14159265$ ....

$x_0 = \pi = 3 + (\pi - 3)$ i.e., $a_0 = 3$

$x_1 = \frac{1}{\pi-3} = \frac{1}{0.14159265 \ldots} = 7.06251330 \ldots = 7 + 0.06251330 \ldots$ i.e., $a_1 = 7$

$x_2 = \frac{1}{0.06251330 \ldots} = 15.99659440 \ldots = 15 + 0.99659440 \ldots$ i.e., $a_2 = 15$

$x_3 = \frac{1}{0.99659440 \ldots} = 1.00341723 \ldots = 1 + 0.00341723 \ldots$ i.e., $a_3 = 1$

$x_4 = \frac{1}{0.00341723 \ldots} = 292.63467 \ldots = 292 + 0.63467 \ldots$ i.e., $a_4 = 292$

....

Thus, the simple infinite continued fraction for π is written as

$\pi = [3; 7,15,1,292,1,1,1,2,1,3,1,14,2,1,1,2,2,2,2 \ldots \ldots]$(59)

Example 4: Let us express e, the base of natural logarithms, as an infinite continued fraction.

$$e = 2.718281828 \ldots.$$

$$x_0 = e = 2 + (e - 2) a_0 = 2$$

$$x_1 = \frac{1}{e-2} = \frac{1}{0.718281828} = 1.39221119 \qquad a_1 = 1$$

$$x_2 = \frac{1}{0.39221119} = 2.549646785 \qquad a_2 = 2$$

$$x_3 = \frac{1}{0.549646785} = 1.819350221 \qquad a_3 = 1$$

$$x_4 = \frac{1}{0.819350221} = 1.220479319 \qquad a_4 = 1$$

$$x_5 = \frac{1}{0.220479319} = 4.535572789 \qquad a_5 = 4$$

....

Thus, the pattern of infinite continued fraction expansion of e is given by

$e = [2; 1,2,1,1,4,1,1,6,1,1,8,1,1, \ldots \ldots]$ (60)

The continued fraction representation of e was found by Euler. In 1737, Euler showed that

$\frac{e-1}{e+1} = [0; 2,6,10,14,18, \ldots.]$(61)

and $\frac{e^2-1}{e^2+1} = [0; 1,3,5,7,9, \ldots \ldots]$(62)

In the above two infinitecontinued fractions, the partial denominators form an arithmetic progression.Following the given procedure, the value of x in equation $3/2 = 2^x$can be represented by an infinite continued fraction as shown below.

$$x = \frac{\log 3/2}{\log 2} = 0.584962500721 \ldots$$

$\Rightarrow x = [0; 1,1,2,2,3,1,5,2,, \ldots \ldots]$(63)

Corollary 6.1: We have shown that an irrational number is represented by an infinite continued fraction. The converse is that an infinite continued fraction represents an irrational number. This statement is proved by taking the infinite continued fraction

$$x_0 = [1; 2,3,1,3,1,3,1, \ldots.]$$

$\Rightarrow x_0 = 1 + \frac{1}{2+\frac{1}{y}}$(64)

Where, $y = [3; 1,3,1,3,1,3,1, \ldots.]$

$$\Rightarrow y = 3 + \frac{1}{1+\frac{1}{y}}$$

Solving the above equation, we obtain

$$y^2 - 3y - 3 = 0$$

The solutions of this quadratic equation are $y = \frac{3\pm\sqrt{21}}{2}$. Since $y > 3$, we have to take $y = (3+\sqrt{21})/2$. Substituting this value of y in (64), we get

$x_0 = 1 + \frac{1}{2+\frac{2}{3+\sqrt{21}}} = \frac{19-\sqrt{21}}{10}$, which is an irrational number.

Thus, the value of an infinite continued fraction x_0 is an irrational number.

(2) simple periodic infinite continued fraction:

If a simple infinite continued fraction contains a block of partial denominators $b_1, b_2, \ldots, b_3$that repeats indefinitely, then the continued fraction is known as *periodic.* A simple periodic infinite continued fraction is denoted as

$$[a_0; a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n, b_1, \ldots, b_n, \ldots\ldots]$$

In short, it is represented as $[a_0; a_1, a_2, \ldots, a_m, \overline{b_1, b_2, \ldots., b_n}]$, where the over bar indicates that this block repeats over and over. The block $b_1, b_2, \ldots, b_3$ is known as the *period* of the infinite continued fraction expansion and n gives the *length* of the period. The following are few examples of simple periodic infinite continued fraction.

(a) $\sqrt{23} = [4; 1,3,1,8,1,3,1,8, \ldots\ldots] = [4; \overline{1,3,1,8}]$

In this example, the period is 1,3,1,8 and the length of the period is 4.

(b) $\sqrt{2} = [1; 1,2,2,2,2, \ldots\ldots] = [1; 1, \overline{2}]$

(c) $\sqrt{3} = [1; 1,2,1,2,1,2, \ldots\ldots] = [1; \overline{1,2}]$

(d) $\sqrt{5} = [2; 4,4,4,4, \ldots\ldots.] = [2; \overline{4}]$

(e) $\sqrt{6} = [2; 2,4,2,4,2,4, \ldots\ldots] = [2; \overline{2,4}]$

(f) $\sqrt{7} = [2; 1,1,1,4,1,1,1,4, \ldots\ldots] = [2; \overline{1,1,1,4}]$

(g) $\sqrt{8} = [2; \overline{1,4}]$

(h) $\sqrt{10} = [3; \overline{6}]$

The following are the three general expressions for finding simple periodic infinite continued fractions of irrational numbers. For any positive integer n,

(i) $\sqrt{n^2+1} = [n; \overline{2n}]$(65)

(ii) $\sqrt{n^2+2} = [n; \overline{n, 2n}]$(66)

(ii) $\sqrt{n^2+2n} = [n; \overline{1,2n}]$(67)

Proof of (65):

$$n + \sqrt{n^2+1} = 2n + \left(\sqrt{n^2+1} - n\right)$$

$$\Rightarrow n + \sqrt{n^2+1} = 2n + \frac{1}{n+\sqrt{n^2+1}}$$

Substituting the above expression successively in RHS, we obtain

$$n + \sqrt{n^2+1} = 2n + \frac{1}{2n+\frac{1}{n+\sqrt{n^2+1}}} = \cdots = 2n + \frac{1}{2n+\frac{1}{2n+\frac{1}{2n+\ddots}}}$$

$$\Rightarrow \sqrt{n^2+1} = [n; 2n, 2n, \ldots] = [n; \overline{2n}]$$

Thus, the expression (65) is proved. For $n = 1,2,3, \ldots etc.$,this expression gives the infinite continued fractions of rational numbers $\sqrt{2}, \sqrt{5}, \sqrt{10}, \sqrt{17}, \ldots, etc.$

Proof of (66):

$$n + \sqrt{n^2+2} = 2n + \left(\sqrt{n^2+2} - n\right)(68)$$

$$\Rightarrow n + \sqrt{n^2+2} = 2n + \frac{2}{n+\sqrt{n^2+2}}$$

$$= 2n + \frac{1}{\left(n+\sqrt{n^2+2}\right)/2}$$

Applying (68)successively in RHS ofabove expression, we get

$$n+\sqrt{n^2+2}=2n+\frac{1}{\left[2n+\left(\sqrt{n^2+2}-n\right)\right]/2}=2n+\frac{1}{n+\frac{\left(\sqrt{n^2+2}-n\right)}{2}}=2n+\frac{1}{n+\frac{1}{\left(n+\sqrt{n^2+2}\right)}}$$

$$=2n+\frac{1}{n+\frac{1}{2n+\left(\sqrt{n^2+2}-n\right)}}=2n+\frac{1}{n+\frac{1}{2n+\frac{1}{\left(n+\sqrt{n^2+2}\right)/2}}}=2n+\frac{1}{n+\frac{1}{2n+\frac{1}{n+\frac{\left(\sqrt{n^2+2}-n\right)}{2}}}}=\ldots..$$

$$=2n+\frac{1}{n+\frac{1}{2n+\frac{1}{n+\frac{1}{2n+\frac{1}{n+\ddots}}}}}$$

$\Rightarrow\sqrt{n^2+2}=[n;n,2n,n,2n,\ldots]=[n;\overline{n,2n}]$

Hence, the expression (66) is proved. Using this expression, we can write the infinite continued fractions of rational numbers $\sqrt{3},\sqrt{6},\sqrt{11},\sqrt{18},\ldots,etc.$, taking $n=1,2,3,\ldots etc.$

Proof of (67):

$$n+\sqrt{n^2+2n}=2n+\left(\sqrt{n^2+2n}-n\right)(69)$$

$$\Rightarrow n+\sqrt{n^2+2n}=2n+\frac{2n}{n+\sqrt{n^2+2n}}$$

$$=2n+\frac{1}{\left[2n+\left(\sqrt{n^2+2n}-n\right)\right]/2n}$$

$$=2n+\frac{1}{1+\frac{1}{n+\sqrt{n^2+2n}}}$$

By successive substitutions of (69) in RHS of above expression, we obtain

$$n+\sqrt{n^2+2n}=2n+\frac{1}{1+\frac{1}{2n+\left(\sqrt{n^2+2n}-n\right)}}=2n+\frac{1}{1+\frac{1}{2n+\frac{2n}{n+\sqrt{n^2+2n}}}}$$

$$=2n+\frac{1}{1+\frac{1}{2n+\frac{1}{1+\frac{1}{n+\sqrt{n^2+2n}}}}}=\cdots=2n+\frac{1}{1+\frac{1}{2n+\frac{1}{1+\frac{1}{2n+\frac{1}{1+\ddots}}}}}$$

$\Rightarrow\sqrt{n^2+2n}=[n;1,2n,1,2n,\ldots]=[n;\overline{1,2n}]$

Thus, the expression (67) is proved. This expression gives the infinite continued fractions of rational numbers $\sqrt{3},\sqrt{8},\sqrt{15},\sqrt{24},\ldots,etc.$, for $n=1,2,3,\ldots etc.$

(3) Golden ratio as infinite continued fraction:

Consider the infinite continued fraction

$x=\ 1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ddots}}}}=[1;1,1,1,\ldots..](70)$

The above fraction shows that

$x=1+\frac{1}{x}$ or $x^2-x-1=0$

The solutions of this quadratic equation are $x=\frac{1\pm\sqrt{5}}{2}$. Since, $x>1$, we must take the solution having positive sign in the numerator of its RHS. Therefore,$x=\frac{1+\sqrt{5}}{2}=1.61803$ ..., which is the golden ratio ø. Hence, the infinite continued fraction (70) represents the golden ratio. That is,

$ø=\frac{1+\sqrt{5}}{2}=[1;1,1,1,\ldots]=[1;\bar{1}](71)$

Importance of golden ratio: A rectangle, whose length and breadth are in the ratio,$\emptyset:1$, is Known as the golden rectangle. The ratio of the diagonal of a regular pentagon to its side is equal to the golden ratio $\emptyset$.Golden ratio finds application in architectural designs.It has been observed that the Greeks have used the concept of golden ratio in the construction of temples. The Parthenon in Athens is the classic example of the golden ratio being used in architecture. It was constructed between 448-432 BC as a temple for the Goddess Athena. The great pyramid in Giza, Egypt, is another example of an ancient structure where golden ratio is used in its design. The ratio of the height of its triangular face to half of the side of its square base approximates to golden ratio. Some ratios in the human body, like the ratio of the height of a person to the distance between the naval point and the foot, are very close to golden ratio. If the ratios between two different parts of human body are close to golden ratio, the body appears beautiful.

VII .Conclusion

The theory of finite and infinite continued fractions was discussed in this article. A rational number can be expressed as a finite continued traction and an irrational number can be expressed as an infinite continued fraction. Linear Diophantine equations were solved using convergents of finite continued fraction. The golden ratio was represented as an infinite continued fraction.

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