

Formulae for the Sums of Squares of Generalized Tribonacci

Numbers: Closed Form Formulas of $\sum_{k=0}^n kW_k^2$

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Abstract. In this paper, closed forms of the sum formulas $\sum_{k=0}^n kW_k^2$, $\sum_{k=0}^n kW_{k+2}W_k$ and $\sum_{k=0}^n kW_{k+1}W_k$ for the squares of generalized Tribonacci numbers are presented. As special cases, we give summation formulas of Tribonacci, Tribonacci-Lucas, Padovan, Perrin numbers and the other third order recurrence relations.

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1. Introduction

The generalized Tribonacci sequence $\{W_n(W_0, W_1, W_2; r, s, t)\}_{n \geq 0}$ (or shortly $\{W_n\}_{n \geq 0}$) is defined as follows:

$$(1.1) \quad W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \quad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3$$

where W_0, W_1, W_2 are arbitrary complex numbers and r, s, t are real numbers. The generalized Tribonacci sequence has been studied by many authors, see for example [1,2,6,7,13,16,19,20,21,22,29,30,31,32,34].

The sequence $\{W_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, \dots$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer n .

In literature, for example, the following names and notations (see Table 1) are used for the special case of r, s, t and initial values.

Table 1 A few special case of generalized Tribonacci sequences.

| Sequences (Numbers) | Notation | OEIS [28] |
|------------------------------|---|------------------|
| Tribonacci | $\{T_n\} = \{W_n(0, 1, 1; 1, 1, 1)\}$ | A000073, A057597 |
| Tribonacci-Lucas | $\{K_n\} = \{W_n(3, 1, 3; 1, 1, 1)\}$ | A001644, A073145 |
| third order Pell | $\{P_n^{(3)}\} = \{W_n(0, 1, 2; 2, 1, 1)\}$ | A077939, A077978 |
| third order Pell-Lucas | $\{Q_n^{(3)}\} = \{W_n(3, 2, 6; 2, 1, 1)\}$ | A276225, A276228 |
| third order modified Pell | $\{E_n^{(3)}\} = \{W_n(0, 1, 1; 2, 1, 1)\}$ | A077997, A078049 |
| Padovan (Cordonnier) | $\{P_n\} = \{W_n(1, 1, 1; 0, 1, 1)\}$ | A000931 |
| Perrin (Padovan-Lucas) | $\{E_n\} = \{W_n(3, 0, 2; 0, 1, 1)\}$ | A001608, A078712 |
| Padovan-Perrin | $\{S_n\} = \{W_n(0, 0, 1; 0, 1, 1)\}$ | A000931, A176971 |
| Pell-Padovan | $\{R_n\} = \{W_n(1, 1, 1; 0, 2, 1)\}$ | A066983, A128587 |
| Pell-Perrin | $\{C_n\} = \{W_n(3, 0, 2; 0, 2, 1)\}$ | |
| Jacobsthal-Padovan | $\{Q_n\} = \{W_n(1, 1, 1; 0, 1, 2)\}$ | A159284 |
| Jacobsthal-Perrin (-Lucas) | $\{L_n\} = \{W_n(3, 0, 2; 0, 1, 2)\}$ | A072328 |
| Narayana | $\{N_n\} = \{W_n(0, 1, 1; 1, 0, 1)\}$ | A078012 |
| Narayana-Lucas | $\{U_n\} = \{W_n(3, 1, 1; 1, 0, 1)\}$ | A001609 |
| Narayana-Perrin | $\{H_n\} = \{W_n(3, 0, 2; 1, 0, 1)\}$ | |
| third order Jacobsthal | $\{J_n^{(3)}\} = \{W_n(0, 1, 1; 1, 1, 2)\}$ | A077947 |
| third order Jacobsthal-Lucas | $\{j_n^{(3)}\} = \{W_n(2, 1, 5; 1, 1, 2)\}$ | A226308 |
| 3-primes | $\{G_n\} = \{W_n(0, 1, 2; 2, 3, 5)\}$ | |
| Lucas 3-primes | $\{H_n\} = \{W_n(3, 2, 10; 2, 3, 5)\}$ | |
| modified 3-primes | $\{E_n\} = \{W_n(0, 1, 1; 2, 3, 5)\}$ | |

The evaluation of sums of powers of these sequences is a challenging issue. Two pretty examples are

$$\sum_{k=0}^n kT_k^2 = \frac{1}{4}(- (n + 4) T_{n+3}^2 - (4n + 11) T_{n+2}^2 - (5n + 9) T_{n+1}^2 + 2(2n + 6) T_{n+3}T_{n+2} + 2(n + 3) T_{n+3}T_{n+1} - 2T_{n+2}T_{n+1} + 2)$$

and

$$\sum_{k=0}^n kP_k^2 = - (n + 6) P_{n+3}^2 - (n + 5) P_{n+2}^2 - (2n + 8) P_{n+1}^2 + 2(n + 4) P_{n+3}P_{n+2} + 2(n + 5) P_{n+3}P_{n+1} - 2P_{n+2}P_{n+1} + 3.$$

In this work, we derive expressions for sums of second powers of generalized Tribonacci numbers. We present some works on sum formulas of powers of the numbers in the following Table 2.

Table 2. A few special study on sum formulas of second, third and arbitrary powers.

| Name of sequence | sums of second powers | sums of third powers | sums of powers |
|------------------------|-----------------------|----------------------|----------------|
| Generalized Fibonacci | [3,4,10,11,12,23] | [9,24,26,27,33] | [5,8,14] |
| Generalized Tribonacci | [17,25] | | |
| Generalized Tetranacci | [15,18] | | |

Let

$$\Delta = (s + rt - t^2 + 1)(r + s + t - 1)(r - s + t + 1).$$

The following theorem gives closed forms of the sum formulas for the squares of generalized Tribonacci numbers.

THEOREM 1.1. *If $\Delta \neq 0$ then*

(a):

$$\sum_{k=1}^n W_k^2 = \frac{\Delta_1}{\Delta},$$

(b):

$$\sum_{k=1}^n W_{k+1}W_k = \frac{\Delta_2}{\Delta},$$

(c):

$$\sum_{k=1}^n W_{k+2}W_k = \frac{\Delta_3}{\Delta},$$

where

$$\begin{aligned} \Delta_1 = & -(t^2 + rt + s - 1)W_{n+3}^2 - (r^3t + r^2t^2 + r^2s + r^2 + t^2 + 2rst + rt + s - 1)W_{n+2}^2 \\ & - (r^3t + r^2t^2 + s^2t^2 - rs^2t - s^3 + r^2s + 4rst + r^2 + s^2 + t^2 + rt + s - 1)W_{n+1}^2 \\ & + 2(r + t)(s + rt)W_{n+3}W_{n+2} + 2t(r + st)W_{n+3}W_{n+1} - 2t(s - 1)(s + rt)W_{n+2}W_{n+1} \\ & + (2rst + 2r^2 + t^2 + rt + s - 1)W_3^2 + (r^3t + r^2t^2 + r^2s + 2rst + r^2 + t^2 + rt + s - 1)W_2^2 \\ & + (r^3t + r^2t^2 + s^2t^2 - rs^2t - s^3 + r^2s + 4rst + r^2 + s^2 + t^2 + rt + s - 1)W_1^2 \\ & - 2(r + st)W_4W_3 - 2t(r^2 - s^2 + rt + s)W_3W_2 + 2t(s - 1)(s + rt)W_2W_1 \end{aligned}$$

and

$$\begin{aligned} \Delta_2 = & (r + st)W_{n+3}^2 + (s + rt)(t + rs)W_{n+2}^2 + t^2(r + st)W_{n+1}^2 \\ & - (2rst + r^2 + s^2 + t^2 - 1)W_{n+3}W_{n+2} + t(r^2 - s^2 - t^2 + 1)W_{n+3}W_{n+1} \\ & - (r^3t - rt^3 - rs^2t + r^2s - s^3 - st^2 + 2rst + r^2 + s^2 + t^2 + rt + s - 1)W_{n+2}W_{n+1} \\ & + (r^3 - rs^2 - rt^2 - st)W_3^2 - (t + rs)(s + rt)W_2^2 - t^2(r + st)W_1^2 \\ & - (r^2 - s^2 - t^2 + 1)W_4W_3 + (r^2s - st^2 - s^3 + 2rst + r^2 + s^2 + t^2 + s - 1)W_3W_2 \\ & + (-rt^3 + r^3t - rs^2t + r^2s - st^2 - s^3 + r^2 + s^2 + t^2 + 2rst + rt + s - 1)W_2W_1 \end{aligned}$$

and

$$\begin{aligned} \Delta_3 = & (r^2 - s^2 + rt + s)W_{n+3}^2 - (rs^2t - rt^3 - r^2t^2 + r^2s + s^2 - s)W_{n+2}^2 + t^2(r^2 - s^2 + rt + s)W_{n+1}^2 \\ & - (r + t)(r^2 - s^2 + t^2 - 1)W_{n+3}W_{n+2} - (r^2s - st^2 - s^3 + 2rst + r^2 + s^2 + t^2 + s - 1)W_{n+3}W_{n+1} \\ & + t(s - 1)(r^2 - s^2 + t^2 - 1)W_{n+2}W_{n+1} + (rs^2t + r^4 - r^2s^2 - r^2t^2 + 2r^2s - rt^3 + r^3t + s^2 - s)W_3^2 \\ & + (rs^2t - rt^3 - r^2t^2 + r^2s + s^2 - s)W_2^2 - t^2(r^2 - s^2 + rt + s)W_1^2 \\ & - (r^3 - t^3 - rs^2 - rt^2 + r^2t + s^2t + 2rs + r + t)W_4W_3 \\ & + (r^3s - st^3 + s^3t - rst^2 - rs^3 + r^2st + rs^2 + rt^2 + r^2t - s^2t + r^3 + t^3 + rs + st - r - t)W_3W_2 \\ & + (s + rt - t^2 + 1)(r - s + t + 1)(r + s + t - 1)W_3W_1 - t(s - 1)(r^2 - s^2 + t^2 - 1)W_2W_1 \end{aligned}$$

Proof. The proof of the Theorem is given in [25].

2. Main Result

Let

$$\Omega = (r - s + t + 1)^2(r + s + t - 1)^2(s + rt - t^2 + 1)^2.$$

THEOREM 2.1. *If $\Omega \neq 0$ then*

(a):

$$\sum_{k=0}^n kW_k^2 = \frac{\Omega_1}{\Omega}$$

(b):

$$\sum_{k=0}^n kW_{k+1}W_k = \frac{\Omega_2}{\Omega}$$

(c):

$$\sum_{k=0}^n kW_{k+2}W_k = \frac{\Omega_3}{\Omega}$$

where

$$\Omega_1 = \sum_{k=1}^{12} \Gamma_k, \quad \Omega_2 = \sum_{k=1}^{12} \Phi_k, \quad \Omega_3 = \sum_{k=1}^{12} \Psi_k,$$

with

$$\begin{aligned} \Gamma_1 = & (n(s + rt + t^2 - 1)(-s - rt + t^2 - 1)(r - s + t + 1)(r + s + t - 1) + 6s - 2r^2s - 6rt^3 - 4r^3t - \\ & 6st^2 - 2r^2s^2 - 9r^2t^2 - 2r^2t^4 - 4r^3t^3 - 2r^4t^2 - 6s^2t^2 + 4s^3t^2 - s^2t^4 + 6rt + 2r^2 - 2s^2 - 2s^3 + s^4 + 6t^2 - 3 \\ & t^4 + r^2s^2t^2 - 6rs^2t - 12rst^3 + 2rs^3t - 4r^3st - 12r^2st^2 + 2rs^2t^3 - 6rst - 3)W_{n+3}^2, \end{aligned}$$

$$\begin{aligned} \Gamma_2 = & (n(-s - rt + t^2 - 1)(s + r^2s + r^3t + r^2t^2 + rt + r^2 + t^2 + 2rst - 1)(r + s + t - 1)(r - \\ & s + t + 1) + 4s - 4r^4s - 2rt^3 - 2st^2 - 2rt^5 - 4r^5t - 2st^4 - 5r^2s^2 - 2r^2s^3 + r^2s^4 - 2r^2t^2 - 2r^4 \\ & s^2 - 11r^2t^4 - 12r^3t^3 - 8r^4t^2 - 7s^2t^2 + 4s^3t^2 - 2r^4t^4 - 4r^5t^3 - 2r^6t^2 + 4rt + 4r^2 - 2r^4 - 2s^2 + 3t^2 - \\ & t^6 - 18r^2s^2t^2 + 6r^2s^3t^2 - r^2s^2t^4 + 2r^3s^2t^3 + r^4s^2t^2 - 8rs^2t - 18rst^3 - 6rs^3t - 16r^3st + 4rs^4t - 2rs \\ & t^5 - 4r^5st - 20r^2st^2 + 4rs^2t^3 - 14r^3s^2t - 14r^3st^3 + 2r^3s^3t - 16r^4st^2 + 2rst - 2)W_{n+2}^2, \end{aligned}$$

$$\Gamma_3 = (n(-s - rt + t^2 - 1)(r + s + t - 1)(r - s + t + 1)(s + r^2s + r^3t + r^2t^2 + s^2t^2 + rt + r^2 + s^2 - s^3 + t^2 - rs^2t + 4rst - 1) + 2s - 2r^4s + 2rt^3 + 2st^2 - 4rt^5 - 2r^5t - 4st^4 - 4r^2s^2 + 2r^2s^4 - r^4s^2 - 10r^2t^4 - 8r^3t^3 - 4r^4t^2 - 2s^2t^2 - r^2t^6 - 2r^3t^5 - 2r^4t^4 - 8s^2t^4 - 2r^5t^3 - r^6t^2 - 2s^4t^2 + 4s^3t^4 + 2s^5t^2 - s^4t^4 + 2rt + 2r^2 - r^4 + s^2 - 4s^3 + s^4 + 2s^5 - s^6 + 3t^4 - 2t^6 - 19r^2s^2t^2 + 8r^2s^3t^2 - 2r^2s^2t^4 - r^2s^4t^2 + 2r^4s^2t^2 - 12r^2st - 16rst^3 - 4rs^3t - 12r^3st + 10rs^4t - 8rst^5 - 2rs^5t - 2r^5st - 14r^2st^2 - 8r^3s^2t - 8rs^3t^3 - 6r^2st^4 - 10r^3st^3 + 4r^3s^3t - 10r^4st^2 + 2rs^2t^5 + 2rs^4t^3 + 6rst - 1)W_{n+1}^2,$$

$$\Gamma_4 = 2(n(r + t)(r + s + t - 1)(r - s + t + 1)(s + rt)(s + rt - t^2 + 1) + 3rs^2 + 2rs^3 + 3r^3s - rs^4 - 6rt^2 - 5r^2t + 6rt^4 + 4r^4t + 4s^2t + 4st^3 + 3s^3t - 2s^4t + st^5 + 2r^3s^2 + 8r^2t^3 + 8r^3t^2 + 2r^3t^4 + 4r^4t^3 - s^2t^3 + 2r^5t^2 - 4rs - 5st - 2r^2s^2t^3 - r^3s^2t^2 + 10rst^2 + 10r^2st + 4r^4st + 12rs^2t^2 + 10r^2s^2t - 5rs^3t^2 + 13r^2st^3 - 2r^2s^3t + 14r^3st^2 + rs^2t^4)W_{n+3}W_{n+2},$$

$$\Gamma_5 = 2t(n(r - s + t + 1)(r + st)(s + rt - t^2 + 1)(r + s + t - 1) - 4r + 2rs^2 - rs^3 + 2r^3s + 2rt^2 + 2r^2t + 2rt^4 + r^4t + 5s^2t + 6st^3 + 4s^3t - 3s^4t + r^2t^3 - 2s^2t^3 + s^3t^3 + 3rs - 6st + 3r^3 + 4rs^2t + 9r^2st - rst^4 + 11rs^2t^2 + 4r^2s^2t - 2rs^3t^2 + 2r^2st^3 + 3r^3st^2)W_{n+3}W_{n+1},$$

$$\Gamma_6 = -2t(n(s - 1)(s + rt)(r - s + t + 1)(r + s + t - 1)(s + rt - t^2 + 1) + 4s - 3r^2s - 4rt^3 - 4r^3t - 2st^2 - rt^5 - 2st^4 + 2r^2s^2 + 3r^2s^3 - 3r^2t^2 - r^3t^3 - 2r^4t^2 + 4s^2t^2 + s^2t^4 + 5rt - 8s^2 + 2s^3 + 4s^4 - 2s^5 + 14r^2s^2t^2 - 2r^2s^3t^2 + rs^2t + 6rst^3 + 14rs^3t + r^3st - 4rs^4t - 4r^2st^2 - 2rs^2t^3 + 6r^3s^2t + rs^3t^3 - r^2st^4 + 2r^3st^3 + 3r^4st^2 - 12rst)W_{n+2}W_{n+1},$$

$$\Gamma_7 = (-4s + 2r^2s + 2rt^3 + 4r^3t + 2st^2 + 2rt^5 + 2st^4 + r^2s^2 + 8r^2t^2 + 2r^2t^4 + 2r^3t^3 + r^4t^2 + 7s^2t^2 - 4s^3t^2 - 4rt - r^2 + 2s^2 - 3t^2 + t^6 + 10rst^3 + 2r^3st + 6r^2st^2 - 2rs^2t^3 + 8rst + 2)W_2^2,$$

$$\Gamma_8 = (-2s + 2r^4s - 2rt^3 - 2st^2 + 4rt^5 + 2r^5t + 4st^4 + 2r^2s^2 + 2r^2s^3 + r^4s^2 + 10r^2t^4 + 8r^3t^3 + 4r^4t^2 + 8s^2t^2 - 4s^3t^2 + r^2t^6 + 2r^3t^5 + 2r^4t^4 - s^2t^4 + 2r^5t^3 + r^6t^2 - 2rt - 2r^2 + r^4 + 2s^2 - 2s^3 + s^4 - 3t^4 + 2t^6 + 10r^2s^2t^2 - 4r^2s^3t^2 - 2r^3s^2t^3 + 12rst^3 + 6rs^3t + 6r^3st - 2rs^4t + 4rst^5 + 2r^5st + 10r^2st^2 - 2rs^2t^3 + 8r^3s^2t - 2rs^3t^3 + 4r^2st^4 + 10r^3st^3 + 8r^4st^2 + 2rst + 1)W_1^2,$$

$$\Gamma_9 = t^2(-6s + 2r^2s + 6rt^3 + 4r^3t + 6st^2 + 2r^2s^2 + 9r^2t^2 + 2r^2t^4 + 4r^3t^3 + 2r^4t^2 + 6s^2t^2 - 4s^3t^2 + s^2t^4 - 6rt - 2r^2 + 2s^2 + 2s^3 - s^4 - 6t^2 + 3t^4 - r^2s^2t^2 + 6rs^2t + 12rst^3 - 2rs^3t + 4r^3st + 12r^2st^2 - 2rs^2t^3 + 6rst + 3)W_0^2,$$

$$\Gamma_{10} = -2(2rs^2 + rs^3 + 2r^3s - 5rt^2 - 4r^2t + 4rt^4 + 3r^4t + 3s^2t + 2st^3 + 2s^3t + rt^6 - s^4t + 2st^5 + r^3s^2 + 5r^2t^3 + 6r^3t^2 + 2r^2t^5 + 2r^3t^4 + 2r^4t^3 + r^5t^2 - s^3t^3 - 3rs - 4st - 2r^2s^2t^3 + 6rst^2 + 7r^2st + 3rst^4 + 2r^4st + 8rs^2t^2 + 4r^2s^2t - 4rs^3t^2 + 10r^2st^3 + 7r^3st^2)W_2W_1,$$

$$\Gamma_{11} = -2t(-3r + rs^2 + r^3s + r^2t + 3rt^4 + 4s^2t + 4st^3 + 3s^3t - 2s^4t + st^5 + 2r^2t^3 - r^3t^2 - s^2t^3 + 2rs - 5st + 2r^3 + 4rst^2 + 4r^2st + 6rs^2t^2 + 4r^2s^2t - rs^3t^2 + r^2st^3 + 2r^3st^2)W_2W_0,$$

$$\Gamma_{12} = 2t(3s - 2r^2s - 2rt^3 - 3r^3t - 2rt^5 - 3st^4 + 2r^2s^2 + 2r^2s^3 - 2r^2t^2 - r^2t^4 - r^4t^2 + s^2t^2 + 2s^3t^2 + 2s^2t^4 - s^4t^2 + 4rt - 6s^2 + 2s^3 + 2s^4 - s^5 + 8r^2s^2t^2 - r^2s^3t^2 + 4rs^2t + 2rst^3 + 7rs^3t + 2r^3st - 2rs^4t + rst^5 + rs^2t^3 + 4r^3s^2t + r^3st^3 + 2r^4st^2 - 9rst)W_1W_0$$

and

$$\Phi_1 = (n(r + st)(r + s + t - 1)(r - s + t + 1)(s + rt - t^2 + 1) - 3r + rs^2 + r^3s + r^2t + 3rt^4 + 4s^2t + 4st^3 + 3s^3t - 2s^4t + st^5 + 2r^2t^3 - r^3t^2 - s^2t^3 + 2rs - 5st + 2r^3 + 4rst^2 + 4r^2st + 6rs^2t^2 + 4r^2s^2t - rs^3t^2 + r^2st^3 + 2r^3st^2)W_{n+3}^2,$$

$$\Phi_2 = (t + rs)(n(r + s + t - 1)(s + rt - t^2 + 1)(r - s + t + 1)(s + rt) - 4s + 3r^2s + 4rt^3 + 4r^3t + 2st^2 + rt^5 + 2st^4 + 2r^2s^2 + 3r^2t^2 + r^3t^3 + 2r^4t^2 - s^3t^2 - 5rt + 3s^2 + 2s^3 - s^4 - r^2s^2t^2 + 7rs^2t - 2rs^3t + 4r^3st + 8r^2st^2 + 6rst)W_{n+2}^2$$

$$\Phi_3 = t^2(n(r - s + t + 1)(s + rt - t^2 + 1)(r + s + t - 1)(r + st) - 4r + 2rs^2 - rs^3 + 2r^3s + 2rt^2 + 2r^2t + 2rt^4 + r^4t + 5s^2t + 6st^3 + 4s^3t - 3s^4t + r^2t^3 - 2s^2t^3 + s^3t^3 + 3rs - 6st + 3r^3 + 4rs^2t + 9r^2st - rst^4 + 11rs^2t^2 + 4r^2s^2t - 2rs^3t^2 + 2r^2st^3 + 3r^3st^2)W_{n+1}^2,$$

$$\Phi_4 = (n(r+s+t-1)(r-s+t+1)(-s-rt+t^2-1)(r^2+s^2+t^2+2rst-1)+s-2r^2s-r^4s-2r^3t-2st^2+st^4-4r^2s^2-2r^2s^3-6r^2t^2-4r^2t^4-4r^3t^3+r^4t^2-8s^2t^2+2s^3t^2-2s^2t^4+s^4t^2+4r^2-2r^4+4s^2-2s^3-2s^4+3t^2+s^5-t^6-14r^2s^2t^2+2r^2s^3t^2-8rs^2t-16rst^3-10rs^3t-8r^3st+4rs^4t-2rst^5-10r^2st^2+2rs^2t^3-8r^3s^2t-2r^3st^3-4r^4st^2-2+6rst)W_{n+3}W_{n+2},$$

$$\Phi_5 = t(n(s + rt - t^2 + 1)(r - s + t + 1)(r + s + t - 1)(r^2 - s^2 - t^2 + 1) + 2s + 4r^2s + 2r^4s - 2rt^3 + 2r^3t - 4st^2 + rt^5 + r^5t + 2st^4 - 2r^2s^2 - 4r^2s^3 - 4r^2t^2 - 2r^3t^3 - 10s^2t^2 + 4s^3t^2 - 2s^2t^4 + rt - 2r^2 + 3r^4 + 6s^2 - 4s^3 - 3s^4 + 6t^2 + 2s^5 - 3t^4 - 2r^2s^2t^2 - 2rs^2t - 12rst^3 - 8rs^3t + 4r^3st + rs^4t - 4r^2st^2 + 2rs^2t^3 - 2r^3s^2t - 3)W_{n+3}W_{n+1},$$

$$\Phi_6 = (n(-s-rt+t^2-1)(r+s+t-1)(r-s+t+1)(s+r^2s-rt^3+r^3t-st^2+rt+r^2+s^2-s^3+t^2-rs^2t+2rst-1)+2s-2r^4s-4rt^3-3st^2+2rt^5-2r^5t+st^6-4r^2s^2+2r^2s^4-5r^2t^2-r^4s^2-2r^2t^4-4s^2t^2+8s^3t^2-r^2t^6+2r^4t^4-3s^2t^4-r^6t^2+2s^3t^4-s^5t^2+2rt+2r^2-r^4+s^2-4s^3+s^4+2s^5-s^6+3t^4-2t^6+10r^2s^3t^2-2r^2s^2t^4-r^2s^4t^2+2r^3s^2t^3+2r^4s^2t^2-6rs^2t-4rst^3-6r^3st+8rs^4t-6rst^5-2rs^5t-2r^5st-2r^2st^2+16rs^2t^3-4r^3s^2t-2rs^3t^3+8r^2st^4+4r^3st^3+4r^3s^3t-5r^4st^2+2rs^2t^5-2rst-1)W_{n+2}W_{n+1},$$

$$\Phi_7 = (2r - rs^3 + 2rt^2 - 4rt^4 + r^4t - 3s^2t - 2st^3 - 2s^3t + s^4t - 2st^5 - 3r^2t^3 + 2r^3t^2 + s^3t^3 - rs + 4st - r^3 - 4rst^2 + r^2st - rst^4 - rs^2t^2 - 4r^2s^2t - r^3st^2)W_2^2,$$

$$\Phi_8 = -(t + rs)(-3s + 2r^2s + 2rt^3 + 3r^3t + 2rt^5 + 3st^4 + r^2s^2 + 2r^2t^2 + r^2t^4 + r^4t^2 + s^2t^2 - 2s^3t^2 - 4rt + 2s^2 + s^3 + 2rs^2t + 2rst^3 + 2r^3st + 3r^2st^2 - rs^2t^3 + 4rst)W_1^2,$$

$$\Phi_9 = -t^2(-3r + rs^2 + r^3s + r^2t + 3rt^4 + 4s^2t + 4st^3 + 3s^3t - 2s^4t + st^5 + 2r^2t^3 - r^3t^2 - s^2t^3 + 2rs - 5st + 2r^3 + 4rst^2 + 4r^2st + 6rs^2t^2 + 4r^2s^2t - rs^3t^2 + r^2st^3 + 2r^3st^2)W_0^2,$$

$$\Phi_{10} = -(-2r^2s + 2rt^3 - 2r^3t - rt^5 + r^5t - 2r^2s^2 - 2r^2s^3 - 4r^2t^2 - 4r^2t^4 - 4r^3t^3 + 2r^4t^2 - 6s^2t^2 - 2s^2t^4 + 2s^4t^2 - rt + 2r^2 - r^4 + 2s^2 - s^4 + 3t^4 - 2t^6 - 4r^2s^2t^2 - 4rs^2t - 8rst^3 - 4rs^3t - 2r^3st + rs^4t - 4rst^5 - 8r^2st^2 - 2rs^2t^3 - 6r^3s^2t + 2rs^3t^3 - 2r^2st^4 - 2r^4st^2 - 1)W_2W_1,$$

$$\Phi_{11} = +t(-s-2r^2s-r^4s+2st^2-st^4+2r^2s^2+2r^2s^3+6r^2t^2-2r^2t^4+r^4t^2+8s^2t^2-2s^3t^2+2s^2t^4-s^4t^2+2r^2-2r^4-4s^2+2s^3+2s^4-3t^2-s^5+t^6+2r^2s^2t^2+8rst^3+4rs^3t+2r^2st^2+4rst+2)W_2W_0,$$

$$\Phi_{12} = +t(3t - 4rs^2 - 2rs^3 - 4r^3s - rt^2 + 2r^2t + 2rt^4 - 3r^4t + 4s^2t + 4st^3 - 4s^3t - rt^6 - 3s^4t - 2st^5 + 2s^5t - 2r^3s^2 + 4r^2t^3 - 2r^3t^2 + 2r^3t^4 + 2s^2t^3 - r^5t^2 - 2s^3t^3 + 6rs - 2st - 6t^3 + 3t^5 - 2r^3s^2t^2 - 6r^2st + 6rst^4 - 2r^4st - 6rs^2t^2 - 6r^2s^2t - 4rs^3t^2 - 4r^2s^3t - 2r^3st^2 - 2rs^2t^4 + rs^4t^2)W_1W_0$$

and

$$\Psi_1 = (n(s+rt-t^2+1)(r-s+t+1)(r+s+t-1)(s+rt+r^2-s^2) - 3s+4r^2s+r^4s+2rt^3+4r^3t+2rt^5+3st^4-r^2s^2-2r^2s^3+2r^2t^2+4r^2t^4+2r^3t^3-s^2t^2-2s^3t^2-2s^2t^4+s^4t^2-4rt-3r^2+2r^4+6s^2-2s^3-2s^4+s^5-2r^2s^2t^2+2rst^3-4rs^3t+2r^3st+4r^2st^2-2rs^2t^3+4rst)W_{n+3}^2,$$

$$\Psi_2 = -(n(r+s+t-1)(r-s+t+1)(s+rt-t^2+1)(-s+r^2s-rt^3-r^2t^2+s^2+rs^2t)+2s-4r^2s+2r^4s+6rt^3+2st^2-6rt^5-4st^4+4r^2s^2+2r^2s^3+5r^2t^2+r^4s^2-8r^2t^4-8r^3t^3-4r^4t^2-2s^2t^2+4s^3t^2-2r^3t^5-4r^4t^4+3s^2t^4-2r^5t^3-2s^4t^2-4s^2+2s^3+8r^2s^3t^2+2r^2s^2t^4-r^2s^4t^2+2r^3s^2t^3+2r^4s^2t^2-8rst^3+2rs^3t+2r^3st+4rs^4t+2rst^5-2rs^5t-2r^2st^2+2rs^2t^3+4r^3s^2t+2rs^3t^3-8r^2st^4-10r^3st^3+4r^3s^3t-4r^4st^2)W_{n+2}^2,$$

$$\Psi_3 = t^2(n(r+s+t-1)(r-s+t+1)(s+rt-t^2+1)(s+rt+r^2-s^2)-4s+6r^2s+2r^4s+4rt^3+6r^3t+2st^2+rt^5+r^5t+2st^4-4r^2s^3+5r^2t^2+2r^2t^4+2r^3t^3+2r^4t^2-4s^2t^2-s^2t^4-5rt-4r^2+3r^4+8s^2-2s^3-4s^4+2s^5-3r^2s^2t^2+4rs^2t-10rs^3t+8r^3st+rs^4t+8r^2st^2-2r^3s^2t+6rst)W_{n+1}^2$$

$$\Psi_4 = (n(r+t)(r+s+t-1)(-s-rt+t^2-1)(r-s+t+1)(r^2-s^2+t^2-1)-2r-3t-4rs^2+4rs^3-2r^3s+2rs^4+4rt^2+6r^2t-rs^5-r^5s-2rt^4-5r^4t-4s^2t-4st^3+4s^3t+3s^4t+2st^5-2s^5t+2r^3s^2+2r^3s^3-8r^2t^3-8r^3t^2-4r^2t^5-8r^3t^4-4r^4t^3-2s^2t^3+2s^3t^3+rs+2st+4r^3-2r^5+6t^3-3t^5+4r^2s^2t^3+4r^3s^2t^2-2rst^2-6r^2st-11rst^4-2r^4st+6r^2s^2t+10rs^3t^2-12r^2st^3+8r^2s^3t-8r^3st^2+4rs^2t^4-2rs^4t^2)W_{n+3}W_{n+2}$$

$$\Psi_5 = (n(r+s+t-1)(-s-rt+t^2-1)(r-s+t+1)(s+r^2s-st^2+r^2+s^2-s^3+t^2+2rst-1)+2s-2r^4s+2rt^3-2r^3t-3st^2-rt^5+r^5t+st^6-4r^2s^2+2r^2s^4-4r^2t^2-r^4s^2-4r^2t^4-4r^3t^3+2r^4t^2-4s^2t^2+8s^3t^2-3s^2t^4+2s^3t^4-s^5t^2-rt+2r^2-r^4+s^2-4s^3+s^4+2s^5-s^6+3t^4-2t^6-2r^2s^2t^2+2r^2s^3t^2-8rst^3-4rs^3t-2r^3st+5rs^4t-4rst^5-2r^2st^2+6rs^2t^3-6r^3s^2t+2rs^3t^3-4r^2st^4-r^4st^2-1)W_{n+3}W_{n+1},$$

$$\Psi_6 = t(n(s-1)(s+rt-t^2+1)(r+s+t-1)(r-s+t+1)(r^2-s^2+t^2-1)+4s-8r^2s+2r^4s-2r^3t-8st^2+4st^4+4r^2s^2-4r^2s^4-6r^2t^2+2r^4s^2-4r^2t^4-4r^3t^3+r^4t^2+4s^3t^2-3s^4t^2+4r^2-2r^4-6s^2+8s^3-2s^4+3t^2-4s^5+2s^6-t^6+6r^2s^2t^2-2r^2s^3t^2+4rs^2t-6rst^3+2r^3st-8rs^4t-rst^5+r^5t+r^5st+4r^2st^2+14rs^2t^3+4r^3s^2t-2rs^3t^3+4r^2st^4+4r^3st^3-2r^3s^3t-5rst-2)W_{n+2}W_{n+1},$$

$$\Psi_7 = (2s-2r^2s-2r^3t+2st^2-3rt^5+r^5t-4st^4+2r^2s^2+r^2t^2-6r^2t^4-2r^3t^3+2r^4t^2-2s^2t^2+4s^3t^2+3s^2t^4-2s^4t^2+3rt+2r^2-r^4-4s^2+2s^3+r^2s^2t^2+4rs^2t-4rst^3-2rs^3t+4r^3st+rs^4t+4rs^2t^3-2r^3s^2t-2rst)W_2^2,$$

$$\Psi_8 = -(-s+2r^2s-r^4s-5rt^3-4st^2+4rt^5+5st^4+rt^7-3r^2s^2-r^2s^4-4r^2t^2+5r^2t^4+6r^3t^3+3r^4t^2+5s^2t^2-6s^3t^2+2r^2t^6+2r^3t^5+2r^4t^4-4s^2t^4+r^5t^3+3s^4t^2+2s^2-2s^3+2s^4-s^5-2r^2s^3t^2-3r^2s^2t^4-r^4s^2t^2-4rs^2t+8rst^3+4rs^3t-2r^3st-4rs^4t-rst^5+rs^5t+r^5st+2r^2st^2-2rs^2t^3+2r^3s^2t-2rs^3t^3+4r^2st^4+4r^3st^3-4r^3s^3t+4r^4st^2-2rs^2t^5+rs^4t^3-rst)W_1^2,$$

$$\Psi_9 = t^2(3s-4r^2s-r^4s-2rt^3-4r^3t-2rt^5-3st^4+r^2s^2+2r^2s^3-2r^2t^2-4r^2t^4-2r^3t^3+s^2t^2+2s^3t^2+2s^2t^4-s^4t^2+4rt+3r^2-2r^4-6s^2+2s^3+2s^4-s^5+2r^2s^2t^2-2rst^3+4rs^3t-2r^3st-4r^2st^2+2rs^2t^3-4rst)W_0^2,$$

$$\Psi_{10} = -(-r-2t-4rs^2+4rs^3-2r^3s+rs^4+3r^2t+3rt^4-4r^4t-4s^2t-2st^3+4s^3t-2rt^6+r^6t+2s^4t+st^5-s^5t+2r^3s^2-4r^2t^3-6r^3t^2-5r^2t^5-8r^3t^4-3r^4t^3-4s^2t^3+2r^5t^2+2s^3t^3+2s^2t^5-$$

$$s^4t^3 + st + 2r^3 - r^5 + 3t^3 - t^7 + 4r^2s^2t^3 + 2r^3s^2t^2 - 4rst^2 - 10r^2st - 8rst^4 + 3r^4st - 2rs^2t^2 + 6r^2s^2t + 6rs^3t^2 - 8r^2st^3 + 2r^2s^3t - 4r^3st^2 + 6rs^2t^4 - 2rs^4t^2 + r^2s^4t - 2r^4s^2t)W_2W_1,$$

$$\Psi_{11} = t(2r + 3t - 8rs^2 - 6r^3s + 2rs^4 - 4rt^2 + 2r^2t - rs^5 - r^5s + 2rt^4 - 3r^4t + 4s^2t + 4st^3 - 4s^3t - 3s^4t - 2st^5 + 2s^5t + 2r^3s^3 + 4r^2t^3 + 4r^3t^2 + 2s^2t^3 - 2s^3t^3 + 5rs - 2st + 2r^3 - 2r^5 - 6t^3 + 3t^5 + 2rst^2 - 2r^2st + 5rst^4 - 2r^4st + 2rs^2t^2 - 6r^2s^2t - 6rs^3t^2 + 8r^2st^3)W_2W_0,$$

$$\Psi_{12} = -t(2s - 6r^2s + 2r^4s + 2rt^3 - 2r^3t - 3st^2 - rt^5 + r^5t + st^6 + 4r^2s^2 - 2r^2s^3 - 2r^2s^4 - 4r^2t^2 + r^4s^2 - 4r^2t^4 - 4r^3t^3 + 2r^4t^2 - 4s^2t^2 + 6s^3t^2 + 3s^2t^4 - 4s^4t^2 - 2s^3t^4 + s^5t^2 - rt + 2r^2 - r^4 - 5s^2 + 8s^3 - 3s^4 - 2s^5 + s^6 + 3t^4 - 2t^6 + 6r^2s^2t^2 - 2r^2s^3t^2 + 8rs^2t - 4rst^3 - 4rs^3t + 6r^3st - 3rs^4t + 2r^2st^2 + 10rs^2t^3 - 2r^3s^2t - 2rs^3t^3 + 4r^2st^4 + 4r^3st^3 - r^4st^2 - 8rst - 1)W_1W_0$$

Proof. First, we obtain $\sum_{k=1}^n kW_k^2$. Using the recurrence relation

$$W_{n+3} = rW_{n+2} + sW_{n+1} + tW_n$$

or

$$tW_n = W_{n+3} - rW_{n+2} - sW_{n+1}$$

i.e.

$$t^2W_n^2 = (W_{n+3} - rW_{n+2} - sW_{n+1})^2 = W_{n+3}^2 + r^2W_{n+2}^2 + s^2W_{n+1}^2 - 2rW_{n+3}W_{n+2} - 2sW_{n+3}W_{n+1} + 2rsW_{n+2}W_{n+1}$$

we obtain

$$\begin{aligned} t^2 \times n \times W_n^2 &= n \times W_{n+3}^2 + r^2 \times n \times W_{n+2}^2 + s^2 \times n \times W_{n+1}^2 \\ &\quad - 2r \times n \times W_{n+3}W_{n+2} - 2s \times n \times W_{n+3}W_{n+1} + 2rs \times n \times W_{n+2}W_{n+1} \\ t^2(n-1)W_{n-1}^2 &= (n-1)W_{n+2}^2 + r^2(n-1)W_{n+1}^2 + s^2(n-1)W_n^2 \\ &\quad - 2r(n-1)W_{n+2}W_{n+1} - 2s(n-1)W_{n+2}W_n + 2rs(n-1)W_{n+1}W_n \\ t^2(n-2)W_{n-2}^2 &= (n-2)W_{n+1}^2 + r^2(n-2)W_n^2 + s^2(n-2)W_{n-1}^2 \\ &\quad - 2r(n-2)W_{n+1}W_n - 2s(n-2)W_{n+1}W_{n-1} + 2rs(n-2)W_nW_{n-1} \\ &\quad \vdots \\ t^2 \times 2 \times W_2^2 &= 2 \times W_5^2 + r^2 \times 2 \times W_4^2 + s^2 \times 2 \times W_3^2 \\ &\quad - 2r \times 2 \times W_5W_4 - 2s \times 2 \times W_5W_3 + 2rs \times 2 \times W_4W_3 \\ t^2 \times 1 \times W_1^2 &= 1 \times W_4^2 + r^2 \times 1 \times W_3^2 + s^2 \times 1 \times W_2^2 \\ &\quad - 2r \times 1 \times W_4W_3 - 2s \times 1 \times W_4W_2 + 2rs \times 1 \times W_3W_2 \\ t^2 \times 0 \times W_0^2 &= 0 \times W_3^2 + r^2 \times 0 \times W_2^2 + s^2 \times 0 \times W_1^2 \\ &\quad - 2r \times 0 \times W_3W_2 - 2s \times 0 \times W_3W_1 + 2rs \times 0 \times W_2W_1 \end{aligned}$$

If we add the equations by side by, we get

$$\begin{aligned}
 t^2 \sum_{k=0}^n kW_k^2 &= (nW_{n+3}^2 + (n-1)W_{n+2}^2 + (n-2)W_{n+1}^2 + 1 \times W_2^2 + 2 \times W_1^2 + 3 \times W_0^2 + \sum_{k=0}^n (k-3)W_k^2) \\
 &+ r^2(nW_{n+2}^2 + (n-1)W_{n+1}^2 + 1 \times W_1^2 + 2 \times W_0^2 + \sum_{k=0}^n (k-2)W_k^2) \\
 &+ s^2(nW_{n+1}^2 + 1 \times W_0^2 + \sum_{k=0}^n (k-1)W_k^2) \\
 &- 2r(nW_{n+3}W_{n+2} + (n-1)W_{n+2}W_{n+1} + 1 \times W_2W_1 + 2 \times W_1W_0 + \sum_{k=0}^n (k-2)W_{k+1}W_k) \\
 &- 2s(nW_{n+3}W_{n+1} + 1 \times W_2W_0 + \sum_{k=0}^n (k-1)W_{k+2}W_k) \\
 &+ 2rs(nW_{n+2}W_{n+1} + 1 \times W_1W_0 + \sum_{k=0}^n (k-1)W_{k+1}W_k)
 \end{aligned}$$

and so

$$\begin{aligned}
 (2.1) \quad t^2 \sum_{k=0}^n kW_k^2 &= (r^2 + s^2 + 1) \sum_{k=0}^n kW_k^2 - 2s \sum_{k=0}^n kW_{k+2}W_k + 2r(s-1) \sum_{k=0}^n kW_{k+1}W_k \\
 &- (2r^2 + s^2 + 3) \sum_{k=0}^n W_k^2 + 2s \sum_{k=0}^n W_{k+2}W_k - 2r(s-2) \sum_{k=0}^n W_{k+1}W_k + nW_{n+3}^2 \\
 &+ (n + nr^2 - 1)W_{n+2}^2 + (n + nr^2 + ns^2 - r^2 - 2)W_{n+1}^2 - 2nrW_{n+3}W_{n+2} \\
 &- 2nsW_{n+3}W_{n+1} + 2r(-n + ns + 1)W_{n+2}W_{n+1} + W_2^2 + (r^2 + 2)W_1^2 \\
 &+ (2r^2 + s^2 + 3)W_0^2 - 2rW_2W_1 - 2sW_2W_0 + 2r(s-2)W_1W_0.
 \end{aligned}$$

Next we obtain $\sum_{k=0}^n kW_{k+1}W_k$. Multiplying the both side of the recurrence relation

$$tW_n = W_{n+3} - rW_{n+2} - sW_{n+1}$$

by W_{n+1} we get

$$tW_{n+1}W_n = W_{n+3}W_{n+1} - rW_{n+2}W_{n+1} - sW_{n+1}^2.$$

Then using last recurrence relation, we obtain

$$\begin{aligned}
 t \times n \times W_{n+1}W_n &= nW_{n+3}W_{n+1} - r \times n \times W_{n+2}W_{n+1} - s \times n \times W_{n+1}^2 \\
 t(n-1)W_nW_{n-1} &= (n-1)W_{n+2}W_n - r(n-1)W_{n+1}W_n - s(n-1)W_n^2 \\
 t(n-2)W_{n-1}W_{n-2} &= (n-2)W_{n+1}W_{n-1} - r(n-2)W_nW_{n-1} - s(n-2)W_{n-1}^2 \\
 t(n-3)W_{n-2}W_{n-3} &= (n-3)W_nW_{n-2} - r(n-3)W_{n-1}W_{n-2} - s(n-3)W_{n-2}^2 \\
 &\vdots \\
 t \times 3 \times W_4W_3 &= 3 \times W_6W_4 - r \times 3 \times W_5W_4 - s \times 3 \times W_4^2 \\
 t \times 2 \times W_3W_2 &= 2 \times W_5W_3 - r \times 2 \times W_4W_3 - s \times 2 \times W_3^2 \\
 t \times 1 \times W_2W_1 &= 1 \times W_4W_2 - r \times 1 \times W_3W_2 - s \times 1 \times W_2^2 \\
 t \times 0 \times W_1W_0 &= 0 \times W_3W_1 - r \times 0 \times W_2W_1 - s \times 0 \times W_1^2
 \end{aligned}$$

If we add the equations by side by, we get

$$\begin{aligned}
 t \sum_{k=0}^n kW_{k+1}W_k &= (nW_{n+3}W_{n+1} + 1 \times W_2W_0 + \sum_{k=0}^n (k-1)W_{k+2}W_k) \\
 &\quad - r(nW_{n+2}W_{n+1} + 1 \times W_1W_0 + \sum_{k=0}^n (k-1)W_{k+1}W_k) \\
 &\quad - s(nW_{n+1}^2 + 1 \times W_0^2 + \sum_{k=0}^n (k-1)W_k^2)
 \end{aligned}$$

and so

$$\begin{aligned}
 (2.2) \quad t \sum_{k=0}^n kW_{k+1}W_k &= -s \sum_{k=0}^n kW_k^2 + \sum_{k=0}^n kW_{k+2}W_k - r \sum_{k=0}^n kW_{k+1}W_k + s \sum_{k=0}^n W_k^2 \\
 &\quad - \sum_{k=0}^n W_{k+2}W_k + r \sum_{k=0}^n W_{k+1}W_k - nW_{n+3}W_{n+1} + nW_{n+2}W_{n+1} \\
 &\quad - nrW_{n+2}W_{n+1} - sW_0^2 + W_2W_0 - rW_1W_0
 \end{aligned}$$

Next we obtain $\sum_{k=0}^n kW_{k+2}W_k$. Multiplying the both side of the recurrence relation

$$tW_n = W_{n+3} - rW_{n+2} - sW_{n+1}$$

by W_{n+2} we get

$$tW_{n+2}W_n = W_{n+3}W_{n+2} - rW_{n+2}^2 - sW_{n+2}W_{n+1}.$$

Then using last recurrence relation, we obtain

$$\begin{aligned}
 t \times n \times W_{n+2}W_n &= nW_{n+3}W_{n+2} - r \times n \times W_{n+2}^2 - s \times n \times W_{n+2}W_{n+1} \\
 t(n-1)W_{n+1}W_{n-1} &= (n-1)W_{n+2}W_{n+1} - r(n-1)W_{n+1}^2 - s(n-1)W_{n+1}W_n \\
 t(n-2)W_nW_{n-2} &= (n-2)W_{n+1}W_n - r(n-2)W_n^2 - s(n-2)W_nW_{n-1} \\
 t(n-3)W_{n-1}W_{n-3} &= (n-3)W_nW_{n-1} - r(n-3)W_{n-1}^2 - s(n-3)W_{n-1}W_{n-2} \\
 t(n-4)W_{n-2}W_{n-4} &= (n-4)W_{n-1}W_{n-2} - r(n-4)W_{n-2}^2 - s(n-4)W_{n-2}W_{n-3} \\
 &\vdots \\
 t \times 3 \times W_5W_3 &= 3 \times W_6W_5 - r \times 3 \times W_5^2 - s \times 3 \times W_5W_4 \\
 t \times 2 \times W_4W_2 &= 2 \times W_5W_4 - r \times 2 \times W_4^2 - s \times 2 \times W_4W_3 \\
 t \times 1 \times W_3W_1 &= 1 \times W_4W_3 - r \times 1 \times W_3^2 - s \times 1 \times W_3W_2 \\
 t \times 0 \times W_2W_0 &= 0 \times W_3W_2 - r \times 0 \times W_2^2 - s \times 0 \times W_2W_1
 \end{aligned}$$

If we add the equations by side by, we get

$$\begin{aligned}
 t \sum_{k=0}^n kW_{k+2}W_k &= (nW_{n+3}W_{n+2} + (n-1)W_{n+2}W_{n+1} + 1 \times W_2W_1 + 2 \times W_1W_0 + \sum_{k=0}^n (k-2)W_{k+1}W_k) \\
 &\quad -r(nW_{n+2}^2 + (n-1)W_{n+1}^2 + 1 \times W_1^2 + 2 \times W_0^2 + \sum_{k=0}^n (k-2)W_k^2) \\
 &\quad -s(nW_{n+2}W_{n+1} + 1 \times W_1W_0 + \sum_{k=0}^n (k-1)W_{k+1}W_k)
 \end{aligned}$$

and so

$$\begin{aligned}
 (2.3) \quad t \sum_{k=0}^n kW_{k+2}W_k &= -r \sum_{k=0}^n kW_k^2 - (s-1) \sum_{k=0}^n kW_{k+1}W_k + 2r \sum_{k=0}^n W_k^2 \\
 &\quad + (s-2) \sum_{k=0}^n W_{k+1}W_k - nrW_{n+2}^2 - r(n-1)W_{n+1}^2 + nW_{n+3}W_{n+2} \\
 &\quad + (n-n_s-1)W_{n+2}W_{n+1} - rW_1^2 - 2rW_0^2 + W_1W_2 - (s-2)W_1W_0
 \end{aligned}$$

Using Theorem 1.1 and solving the system (2.1)-(2.2)-(2.3), the results in (a), (b) and (c) follow.

3. Specific Cases

In this section, we present the closed form solutions (identities) of the sums $\sum_{k=0}^n kW_k^2$, $\sum_{k=0}^n kW_{k+1}W_k$ and $\sum_{k=0}^n kW_{k+2}W_k$ for the specific case of sequence $\{W_n\}$.

Taking $r = s = t = 1$ in Theorem 2.1, we obtain the following Proposition.

PROPOSITION 3.1. *If $r = s = t = 1$ then for $n \geq 0$ we have the following formulas:*

- (a): $\sum_{k=0}^n kW_k^2 = \frac{1}{4}(- (n + 4) W_{n+3}^2 - (4n + 11) W_{n+2}^2 - (5n + 9) W_{n+1}^2 + 2(2n + 6) W_{n+3}W_{n+2} + 2(n + 3) W_{n+3}W_{n+1} - 2W_{n+2}W_{n+1} + 3W_2^2 + 7W_1^2 + 4W_0^2 - 8W_2W_1 - 4W_2W_0 + 2W_1W_0)$.
- (b): $\sum_{k=0}^n kW_{k+1}W_k = \frac{1}{4}((n + 2) W_{n+3}^2 + (2n + 5) W_{n+2}^2 + (n + 3) W_{n+1}^2 - (2n + 6) W_{n+3}W_{n+2} - 2W_{n+3}W_{n+1} - 2nW_{n+2}W_{n+1} - W_2^2 - 3W_1^2 - 2W_0^2 + 4W_2W_1 + 2W_2W_0 - 2W_1W_0)$.
- (c): $\sum_{k=0}^n kW_{k+2}W_k = \frac{1}{4}((n + 1) W_{n+3}^2 + W_{n+2}^2 + (n + 2) W_{n+1}^2 - 2W_{n+3}W_{n+2} - 2(n + 1) W_{n+3}W_{n+1} - W_1^2 - W_0^2 + 2W_2W_1)$.

From the above proposition, we have the following Corollary which gives sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

COROLLARY 3.2. For $n \geq 0$, Tribonacci numbers have the following properties:

- (a): $\sum_{k=0}^n kT_k^2 = \frac{1}{4}(- (n + 4) T_{n+3}^2 - (4n + 11) T_{n+2}^2 - (5n + 9) T_{n+1}^2 + 2(2n + 6) T_{n+3}T_{n+2} + 2(n + 3) T_{n+3}T_{n+1} - 2T_{n+2}T_{n+1} + 2)$.
- (b): $\sum_{k=0}^n kT_{k+1}T_k = \frac{1}{4}((n + 2) T_{n+3}^2 + (2n + 5) T_{n+2}^2 + (n + 3) T_{n+1}^2 - (2n + 6) T_{n+3}T_{n+2} - 2T_{n+3}T_{n+1} - 2nT_{n+2}T_{n+1})$.
- (c): $\sum_{k=0}^n kT_{k+2}T_k = \frac{1}{4}((n + 1) T_{n+3}^2 + T_{n+2}^2 + (n + 2) T_{n+1}^2 - 2T_{n+3}T_{n+2} - 2(n + 1) T_{n+3}T_{n+1} + 1)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the above Proposition, we have the following Corollary which presents sum formulas of Tribonacci-Lucas numbers.

COROLLARY 3.3. For $n \geq 0$, Tribonacci-Lucas numbers have the following properties:

- (a): $\sum_{k=0}^n kK_k^2 = \frac{1}{4}(- (n + 4) K_{n+3}^2 - (4n + 11) K_{n+2}^2 - (5n + 9) K_{n+1}^2 + 2(2n + 6) K_{n+3}K_{n+2} + 2(n + 3) K_{n+3}K_{n+1} - 2K_{n+2}K_{n+1} + 16)$.
- (b): $\sum_{k=0}^n kK_{k+1}K_k = \frac{1}{4}((n + 2) K_{n+3}^2 + (2n + 5) K_{n+2}^2 + (n + 3) K_{n+1}^2 - (2n + 6) K_{n+3}K_{n+2} - 2K_{n+3}K_{n+1} - 2nK_{n+2}K_{n+1} - 6)$.
- (c): $\sum_{k=0}^n kK_{k+2}K_k = \frac{1}{4}((n + 1) K_{n+3}^2 + K_{n+2}^2 + (n + 2) K_{n+1}^2 - 2K_{n+3}K_{n+2} - 2(n + 1) K_{n+3}K_{n+1} - 4)$.

Taking $r = 2, s = 1, t = 1$ in Theorem 2.1, we obtain the following Proposition.

PROPOSITION 3.4. If $r = 2, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:

- (a): $\sum_{k=0}^n kW_k^2 = \frac{1}{27}(- (3n + 10) W_{n+3}^2 - (27n + 63) W_{n+2}^2 - (30n + 40) W_{n+1}^2 + 2(9n + 24) W_{n+3}W_{n+2} + 2(3n + 7) W_{n+3}W_{n+1} - 6W_{n+2}W_{n+1} + 7W_2^2 + 36W_1^2 + 10W_0^2 - 30W_2W_1 - 8W_2W_0 + 6W_0W_1)$.
- (b): $\sum_{k=0}^n kW_{k+1}W_k = \frac{1}{27}((3n + 4) W_{n+3}^2 + 3(3n + 7) W_{n+2}^2 + (3n + 7) W_{n+1}^2 - (9n + 18) W_{n+3}W_{n+2} + (3n + 1) W_{n+3}W_{n+1} - (18n + 9) W_{n+2}W_{n+1} - W_2^2 - 12W_1^2 - 4W_0^2 + 9W_2W_1 + 2W_2W_0 - 9W_1W_0)$.
- (c): $\sum_{k=0}^n kW_{k+2}W_k = \frac{1}{27}((6n + 5) W_{n+3}^2 + 9W_{n+2}^2 + (6n + 11) W_{n+1}^2 + (-9n - 15) W_{n+3}W_{n+2} + (-12n - 7) W_{n+3}W_{n+1} + 3W_{n+2}W_{n+1} + W_2^2 - 9W_1^2 - 5W_0^2 + 6W_2W_1 - 5W_2W_0 - 3W_1W_0)$.

From the last Proposition, we have the following Corollary which gives sum formulas of Third-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 1$).

COROLLARY 3.5. For $n \geq 0$, third-order Pell numbers have the following properties:

- (a): $\sum_{k=0}^n kP_k^2 = \frac{1}{27}(- (3n + 10) P_{n+3}^2 - (27n + 63) P_{n+2}^2 - (30n + 40) P_{n+1}^2 + 2(9n + 24) P_{n+3}P_{n+2} + 2(3n + 7) P_{n+3}P_{n+1} - 6P_{n+2}P_{n+1} + 4)$.
- (b): $\sum_{k=0}^n kP_{k+1}P_k = \frac{1}{27}((3n + 4) P_{n+3}^2 + 3(3n + 7) P_{n+2}^2 + (3n + 7) P_{n+1}^2 - (9n + 18) P_{n+3}P_{n+2} + (3n + 1) P_{n+3}P_{n+1} - (18n + 9) P_{n+2}P_{n+1} + 2)$.
- (c): $\sum_{k=0}^n kP_{k+2}P_k = \frac{1}{27}((6n + 5) P_{n+3}^2 + 9P_{n+2}^2 + (6n + 11) P_{n+1}^2 + (-9n - 15) P_{n+3}P_{n+2} + (-12n - 7) P_{n+3}P_{n+1} + 3P_{n+2}P_{n+1} + 7)$.

Taking $W_n = Q_n$ with $Q_0 = 3, Q_1 = 2, Q_2 = 6$ in the last Proposition, we have the following Corollary which presents sum formulas of third-order Pell-Lucas numbers.

COROLLARY 3.6. For $n \geq 0$, third-order Pell-Lucas numbers have the following properties:

- (a): $\sum_{k=0}^n kQ_k^2 = \frac{1}{27}(- (3n + 10) Q_{n+3}^2 - (27n + 63) Q_{n+2}^2 - (30n + 40) Q_{n+1}^2 + 2(9n + 24) Q_{n+3}Q_{n+2} + 2(3n + 7) Q_{n+3}Q_{n+1} - 6Q_{n+2}Q_{n+1} + 18)$.
- (b): $\sum_{k=0}^n kQ_{k+1}Q_k = \frac{1}{27}((3n + 4) Q_{n+3}^2 + 3(3n + 7) Q_{n+2}^2 + (3n + 7) Q_{n+1}^2 - (9n + 18) Q_{n+3}Q_{n+2} + (3n + 1) Q_{n+3}Q_{n+1} - (18n + 9) Q_{n+2}Q_{n+1} - 30)$.
- (c): $\sum_{k=0}^n kQ_{k+2}Q_k = \frac{1}{27}((6n + 5) Q_{n+3}^2 + 9Q_{n+2}^2 + (6n + 11) Q_{n+1}^2 + (-9n - 15) Q_{n+3}Q_{n+2} + (-12n - 7) Q_{n+3}Q_{n+1} + 3Q_{n+2}Q_{n+1} - 81)$.

From the last Proposition, we have the following Corollary which gives sum formulas of third-order modified Pell numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

COROLLARY 3.7. For $n \geq 0$, third-order modified Pell numbers have the following properties:

- (a): $\sum_{k=0}^n kE_k^2 = \frac{1}{27}(- (3n + 10) E_{n+3}^2 - (27n + 63) E_{n+2}^2 - (30n + 40) E_{n+1}^2 + 2(9n + 24) E_{n+3}E_{n+2} + 2(3n + 7) E_{n+3}E_{n+1} - 6E_{n+2}E_{n+1} + 13)$.
- (b): $\sum_{k=0}^n kE_{k+1}E_k = \frac{1}{27}((3n + 4) E_{n+3}^2 + 3(3n + 7) E_{n+2}^2 + (3n + 7) E_{n+1}^2 - (9n + 18) E_{n+3}E_{n+2} + (3n + 1) E_{n+3}E_{n+1} - (18n + 9) E_{n+2}E_{n+1} - 4)$.
- (c): $\sum_{k=0}^n kE_{k+2}E_k = \frac{1}{27}((6n + 5) E_{n+3}^2 + 9E_{n+2}^2 + (6n + 11) E_{n+1}^2 + (-9n - 15) E_{n+3}E_{n+2} + (-12n - 7) E_{n+3}E_{n+1} + 3E_{n+2}E_{n+1} - 2)$.

Taking $r = 0, s = 1, t = 1$ in Theorem 2.1, we obtain the following Proposition.

PROPOSITION 3.8. If $r = 0, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:

- (a): $\sum_{k=0}^n kW_k^2 = - (n + 6) W_{n+3}^2 - (n + 5) W_{n+2}^2 - (2n + 8) W_{n+1}^2 + 2(n + 4) W_{n+3}W_{n+2} + 2(n + 5) W_{n+3}W_{n+1} - 2W_{n+2}W_{n+1} + 5W_2^2 + 4W_1^2 + 6W_0^2 - 6W_2W_1 - 8W_2W_0 + 2W_1W_0$.
- (b): $\sum_{k=0}^n kW_{k+1}W_k = (n + 4) W_{n+3}^2 + (n + 3) W_{n+2}^2 + (n + 5) W_{n+1}^2 + (-n - 6) W_{n+3}W_{n+2} + (-n - 7) W_{n+3}W_{n+1} + W_{n+2}W_{n+1} - 3W_2^2 - 2W_1^2 - 4W_0^2 + 5W_2W_1 + 6W_2W_0 - W_1W_0$.
- (c): $\sum_{k=0}^n kW_{k+2}W_k = -W_{n+3}^2 - W_{n+2}^2 - W_{n+1}^2 + (n + 1) W_{n+3}W_{n+2} + W_{n+3}W_{n+1} - W_{n+2}W_{n+1} + W_2^2 + W_1^2 + W_0^2 - W_0W_2 + W_0W_1$.

From the last Proposition, we have the following Corollary which gives sum formulas of Padovan numbers (take $W_n = P_n$ with $P_0 = 1, P_1 = 1, P_2 = 1$).

COROLLARY 3.9. For $n \geq 0$, Padovan numbers have the following properties:

- (a): $\sum_{k=0}^n kP_k^2 = - (n + 6) P_{n+3}^2 - (n + 5) P_{n+2}^2 - (2n + 8) P_{n+1}^2 + 2(n + 4) P_{n+3}P_{n+2} + 2(n + 5) P_{n+3}P_{n+1} - 2P_{n+2}P_{n+1} + 3$.
- (b): $\sum_{k=0}^n kP_{k+1}P_k = (n + 4) P_{n+3}^2 + (n + 3) P_{n+2}^2 + (n + 5) P_{n+1}^2 + (-n - 6) P_{n+3}P_{n+2} + (-n - 7) P_{n+3}P_{n+1} + P_{n+2}P_{n+1} + 1$.
- (c): $\sum_{k=0}^n kP_{k+2}P_k = -P_{n+3}^2 - P_{n+2}^2 - P_{n+1}^2 + (n + 1) P_{n+3}P_{n+2} + P_{n+3}P_{n+1} - P_{n+2}P_{n+1} + 3$.

Taking $W_n = E_n$ with $E_0 = 3, E_1 = 0, E_2 = 2$ in the last Proposition, we have the following Corollary which presents sum formulas of Perrin numbers.

COROLLARY 3.10. *For $n \geq 0$, Perrin numbers have the following properties:*

- (a): $\sum_{k=0}^n kE_k^2 = -(n+6)E_{n+3}^2 - (n+5)E_{n+2}^2 - (2n+8)E_{n+1}^2 + 2(n+4)E_{n+3}E_{n+2} + 2(n+5)E_{n+3}E_{n+1} - 2E_{n+2}E_{n+1} + 5E_2^2 + 6.$
- (b): $\sum_{k=0}^n kE_{k+1}E_k = (n+4)E_{n+3}^2 + (n+3)E_{n+2}^2 + (n+5)E_{n+1}^2 + (-n-6)E_{n+3}E_{n+2} + (-n-7)E_{n+3}E_{n+1} + E_{n+2}E_{n+1} - 12.$
- (c): $\sum_{k=0}^n kE_{k+2}E_k = -E_{n+3}^2 - E_{n+2}^2 - E_{n+1}^2 + (n+1)E_{n+3}E_{n+2} + E_{n+3}E_{n+1} - E_{n+2}E_{n+1} + 7.$

From the last Proposition, we have the following Corollary which gives sum formulas of Padovan-Perrin numbers (take $W_n = S_n$ with $S_0 = 0, S_1 = 0, S_2 = 1$).

COROLLARY 3.11. *For $n \geq 0$, Padovan-Perrin numbers have the following properties:*

- (a): $\sum_{k=0}^n kS_k^2 = -(n+6)S_{n+3}^2 - (n+5)S_{n+2}^2 - (2n+8)S_{n+1}^2 + 2(n+4)S_{n+3}S_{n+2} + 2(n+5)S_{n+3}S_{n+1} - 2S_{n+2}S_{n+1} + 5.$
- (b): $\sum_{k=0}^n kS_{k+1}S_k = (n+4)S_{n+3}^2 + (n+3)S_{n+2}^2 + (n+5)S_{n+1}^2 + (-n-6)S_{n+3}S_{n+2} + (-n-7)S_{n+3}S_{n+1} + S_{n+2}S_{n+1} - 3.$
- (c): $\sum_{k=0}^n kS_{k+2}S_k = -S_{n+3}^2 - S_{n+2}^2 - S_{n+1}^2 + (n+1)S_{n+3}S_{n+2} + S_{n+3}S_{n+1} - S_{n+2}S_{n+1} + 1.$

Taking $r = 0, s = 1, t = 2$ in Theorem 2.1, we obtain the following proposition.

PROPOSITION 3.12. *If $r = 0, s = 1, t = 2$ then for $n \geq 0$ we have the following formulas:*

- (a): $\sum_{k=0}^n kW_k^2 = \frac{1}{8}((4n-9)W_{n+3}^2 + (4n-13)W_{n+2}^2 + (8n-28)W_{n+1}^2 - 2(2n-7)W_{n+3}W_{n+2} - 4(2n-5)W_{n+3}W_{n+1} + 4W_{n+2}W_{n+1} + 13W_2^2 + 17W_1^2 + 36W_0^2 - 18W_2W_1 - 28W_2W_0 - 4W_1W_0).$
- (b): $\sum_{k=0}^n kW_{k+1}W_k = \frac{1}{8}(-(2n-7)W_{n+3}^2 - (2n-9)W_{n+2}^2 - 4(2n-5)W_{n+1}^2 + 4(n-3)W_{n+3}W_{n+2} + 8(n-2)W_{n+3}W_{n+1} - 4W_{n+2}W_{n+1} - 9W_2^2 - 11W_1^2 - 28W_0^2 + 16W_2W_1 + 24W_2W_0 + 4W_1W_0).$
- (c): $\sum_{k=0}^n kW_{k+2}W_k = \frac{1}{8}(W_{n+3}^2 + W_{n+2}^2 + 4W_{n+1}^2 + 2(2n-1)W_{n+3}W_{n+2} - 4W_{n+3}W_{n+1} - 4W_{n+2}W_{n+1} - W_2^2 - W_1^2 - 4W_0^2 + 6W_2W_1 + 4W_2W_0 + 4W_1W_0).$

From the last Proposition, we have the following Corollary which gives sum formulas of Jacobsthal-Padovan numbers (take $W_n = Q_n$ with $Q_0 = 1, Q_1 = 1, Q_2 = 1$).

COROLLARY 3.13. *For $n \geq 0$, Jacobsthal-Padovan numbers have the following properties:*

- (a): $\sum_{k=0}^n kQ_k^2 = \frac{1}{8}((4n-9)Q_{n+3}^2 + (4n-13)Q_{n+2}^2 + (8n-28)Q_{n+1}^2 - 2(2n-7)Q_{n+3}Q_{n+2} - 4(2n-5)Q_{n+3}Q_{n+1} + 4Q_{n+2}Q_{n+1} + 16).$
- (b): $\sum_{k=0}^n kQ_{k+1}Q_k = \frac{1}{8}(-(2n-7)Q_{n+3}^2 - (2n-9)Q_{n+2}^2 - 4(2n-5)Q_{n+1}^2 + 4(n-3)Q_{n+3}Q_{n+2} + 8(n-2)Q_{n+3}Q_{n+1} - 4Q_{n+2}Q_{n+1} - 4).$
- (c): $\sum_{k=0}^n kQ_{k+2}Q_k = \frac{1}{8}(Q_{n+3}^2 + Q_{n+2}^2 + 4Q_{n+1}^2 + 2(2n-1)Q_{n+3}Q_{n+2} - 4Q_{n+3}Q_{n+1} - 4Q_{n+2}Q_{n+1} + 8).$

COROLLARY 3.14. For $n \geq 0$, Jacobsthal-Perrin numbers have the following properties:

- (a): $\sum_{k=0}^n kL_k^2 = \frac{1}{8}((4n-9)L_{n+3}^2 + (4n-13)L_{n+2}^2 + (8n-28)L_{n+1}^2 - 2(2n-7)L_{n+3}L_{n+2} - 4(2n-5)L_{n+3}L_{n+1} + 4L_{n+2}L_{n+1} + 208)$.
- (b): $\sum_{k=0}^n kL_{k+1}L_k = \frac{1}{8}(- (2n-7)L_{n+3}^2 - (2n-9)L_{n+2}^2 - 4(2n-5)L_{n+1}^2 + 4(n-3)L_{n+3}L_{n+2} + 8(n-2)L_{n+3}L_{n+1} - 4L_{n+2}L_{n+1} - 144)$.
- (c): $\sum_{k=0}^n kL_{k+2}L_k = \frac{1}{8}(L_{n+3}^2 + L_{n+2}^2 + 4L_{n+1}^2 + 2(2n-1)L_{n+3}L_{n+2} - 4L_{n+3}L_{n+1} - 4L_{n+2}L_{n+1} - 16)$.

Taking $r = 1, s = 0, t = 1$ in Theorem 2.1, we obtain the following Proposition.

PROPOSITION 3.15. If $r = 1, s = 0, t = 1$ then for $n \geq 0$ we have the following formulas:

- (a): $\sum_{k=0}^n kW_k^2 = \frac{1}{9}(- (3n+19)W_{n+3}^2 - (12n+43)W_{n+2}^2 - (12n+31)W_{n+1}^2 + 2(6n+23)W_{n+3}W_{n+2} + 2(3n+7)W_{n+3}W_{n+1} + 2(3n+10)W_{n+2}W_{n+1} + 16W_2^2 + 31W_1^2 + 19W_0^2 - 34W_2W_1 - 8W_2W_0 - 14W_1W_0)$.
- (b): $\sum_{k=0}^n kW_{k+1}W_k = \frac{1}{9}((3n+4)W_{n+3}^2 + (3n+10)W_{n+2}^2 + (3n+7)W_{n+1}^2 + (-3n-13)W_{n+3}W_{n+2} + (3n-2)W_{n+3}W_{n+1} + (-6n-8)W_{n+2}W_{n+1} - W_2^2 - 7W_1^2 - 4W_0^2 + 10W_2W_1 + 5W_2W_0 + 2W_1W_0)$.
- (c): $\sum_{k=0}^n kW_{k+2}W_k = \frac{1}{9}((6n+11)W_{n+3}^2 + (6n+23)W_{n+2}^2 + (6n+17)W_{n+1}^2 + (-6n-29)W_{n+3}W_{n+2} + (-3n-10)W_{n+3}W_{n+1} + (-3n-13)W_{n+2}W_{n+1} - 5W_2^2 - 17W_1^2 - 11W_0^2 + 23W_2W_1 + 7W_2W_0 + 10W_1W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of Narayana numbers (take $W_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 1$).

COROLLARY 3.16. For $n \geq 0$, Narayana numbers have the following properties:

- (a): $\sum_{k=0}^n kN_k^2 = \frac{1}{9}(- (3n+19)N_{n+3}^2 - (12n+43)N_{n+2}^2 - (12n+31)N_{n+1}^2 + 2(6n+23)N_{n+3}N_{n+2} + 2(3n+7)N_{n+3}N_{n+1} + 2(3n+10)N_{n+2}N_{n+1} + 13)$.
- (b): $\sum_{k=0}^n kN_{k+1}N_k = \frac{1}{9}((3n+4)N_{n+3}^2 + (3n+10)N_{n+2}^2 + (3n+7)N_{n+1}^2 + (-3n-13)N_{n+3}N_{n+2} + (3n-2)N_{n+3}N_{n+1} + (-6n-8)N_{n+2}N_{n+1} + 2)$.
- (c): $\sum_{k=0}^n kN_{k+2}N_k = \frac{1}{9}((6n+11)N_{n+3}^2 + (6n+23)N_{n+2}^2 + (6n+17)N_{n+1}^2 + (-6n-29)N_{n+3}N_{n+2} + (-3n-10)N_{n+3}N_{n+1} + (-3n-13)N_{n+2}N_{n+1} + 1)$.

Taking $N_n = U_n$ with $U_0 = 3, U_1 = 1, U_2 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Narayana-Lucas numbers.

COROLLARY 3.17. For $n \geq 0$, Narayana-Lucas numbers have the following properties:

- (a): $\sum_{k=0}^n kU_k^2 = \frac{1}{9}(- (3n+19)U_{n+3}^2 - (12n+43)U_{n+2}^2 - (12n+31)U_{n+1}^2 + 2(6n+23)U_{n+3}U_{n+2} + 2(3n+7)U_{n+3}U_{n+1} + 2(3n+10)U_{n+2}U_{n+1} + 118)$.
- (b): $\sum_{k=0}^n kU_{k+1}U_k = \frac{1}{9}((3n+4)U_{n+3}^2 + (3n+10)U_{n+2}^2 + (3n+7)U_{n+1}^2 + (-3n-13)U_{n+3}U_{n+2} + (3n-2)U_{n+3}U_{n+1} + (-6n-8)U_{n+2}U_{n+1} - 13)$.
- (c): $\sum_{k=0}^n kU_{k+2}U_k = \frac{1}{9}((6n+11)U_{n+3}^2 + (6n+23)U_{n+2}^2 + (6n+17)U_{n+1}^2 + (-6n-29)U_{n+3}U_{n+2} + (-3n-10)U_{n+3}U_{n+1} + (-3n-13)U_{n+2}U_{n+1} - 47)$.

From the last proposition, we have the following corollary which gives sum formulas of Narayana-Perrin numbers (take $U_n = H_n$ with $H_0 = 3, H_1 = 0, H_2 = 2$).

COROLLARY 3.18. For $n \geq 0$, Narayana-Perrin numbers have the following properties:

- (a): $\sum_{k=0}^n kH_k^2 = \frac{1}{9}(- (3n + 19) H_{n+3}^2 - (12n + 43) H_{n+2}^2 - (12n + 31) H_{n+1}^2 + 2(6n + 23) H_{n+3}H_{n+2} + 2(3n + 7) H_{n+3}H_{n+1} + 2(3n + 10) H_{n+2}H_{n+1} + 187)$.
- (b): $\sum_{k=0}^n kH_{k+1}H_k = \frac{1}{9}((3n + 4) H_{n+3}^2 + (3n + 10) H_{n+2}^2 + (3n + 7) H_{n+1}^2 + (-3n - 13) H_{n+3}H_{n+2} + (3n - 2) H_{n+3}H_{n+1} + (-6n - 8) H_{n+2}H_{n+1} - 10)$.
- (c): $\sum_{k=0}^n kH_{k+2}H_k = \frac{1}{9}((6n + 11) H_{n+3}^2 + (6n + 23) H_{n+2}^2 + (6n + 17) H_{n+1}^2 + (-6n - 29) H_{n+3}H_{n+2} + (-3n - 10) H_{n+3}H_{n+1} + (-3n - 13) H_{n+2}H_{n+1} - 77)$.

Taking $r = 2, s = 3, t = 5$ in Theorem 2.1, we obtain the following Proposition.

PROPOSITION 3.19. If $r = 2, s = 3, t = 5$ then for $n \geq 0$ we have the following formulas:

- (a): $\sum_{k=0}^n kW_k^2 = \frac{1}{245025}((18315n - 26674) W_{n+3}^2 + (125235n - 230461) W_{n+2}^2 + (212850n - 454000) W_{n+1}^2 - 2(45045n - 73042) W_{n+3}W_{n+2} - 10(8415n - 11279) W_{n+3}W_{n+1} + 10(12870n - 10757) W_{n+2}W_{n+1} + 44989W_2^2 + 355696W_1^2 + 666850W_0^2 - 236174W_2W_1 - 196940W_2W_0 + 236270W_1W_0)$.
- (b): $\sum_{k=0}^n kW_{k+1}W_k = \frac{1}{245025}(- (8415n - 19694) W_{n+3}^2 - 11(6435n - 15031) W_{n+2}^2 - 25(8415n - 11279) W_{n+1}^2 + (48015n - 108004) W_{n+3}W_{n+2} + 5(14355n - 14873) W_{n+3}W_{n+1} - (138600n - 80395) W_{n+2}W_{n+1} - 28109W_2^2 - 236126W_1^2 - 492350W_0^2 + 156019W_2W_1 + 146140W_2W_0 - 218995W_1W_0)$.
- (c): $\sum_{k=0}^n kW_{k+2}W_k = \frac{1}{245025}(- (3960n - 15761) W_{n+3}^2 - (119790n - 112079) W_{n+2}^2 - 25(3960n - 11801) W_{n+1}^2 + (65835n - 77701) W_{n+3}W_{n+2} + (4950n - 68335) W_{n+3}W_{n+1} - 5(18810n - 7421) W_{n+2}W_{n+1} - 19721W_2^2 - 231869W_1^2 - 394025W_0^2 + 143536W_2W_1 + 73285W_2W_0 - 131155W_1W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of 3-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 2$).

COROLLARY 3.20. For $n \geq 0$, 3-primes numbers have the following properties:

- (a): $\sum_{k=0}^n kG_k^2 = \frac{1}{245025}((18315n - 26674) G_{n+3}^2 + (125235n - 230461) G_{n+2}^2 + (212850n - 454000) G_{n+1}^2 - 2(45045n - 73042) G_{n+3}G_{n+2} - 10(8415n - 11279) G_{n+3}G_{n+1} + 10(12870n - 10757) G_{n+2}G_{n+1} + 63304)$.
- (b): $\sum_{k=0}^n kG_{k+1}G_k = \frac{1}{245025}(- (8415n - 19694) G_{n+3}^2 - 11(6435n - 15031) G_{n+2}^2 - 25(8415n - 11279) G_{n+1}^2 + (48015n - 108004) G_{n+3}G_{n+2} + 5(14355n - 14873) G_{n+3}G_{n+1} - (138600n - 80395) G_{n+2}G_{n+1} - 36524)$.
- (c): $\sum_{k=0}^n kG_{k+2}G_k = \frac{1}{245025}(- (3960n - 15761) G_{n+3}^2 - (119790n - 112079) G_{n+2}^2 - 25(3960n - 11801) G_{n+1}^2 + (65835n - 77701) G_{n+3}G_{n+2} + (4950n - 68335) G_{n+3}G_{n+1} - 5(18810n - 7421) G_{n+2}G_{n+1} - 23681)$.

Taking $G_n = H_n$ with $H_0 = 3, H_1 = 2, H_2 = 10$ in the last proposition, we have the following corollary which presents sum formulas of Lucas 3-primes numbers.

COROLLARY 3.21. For $n \geq 0$, Lucas 3-primes numbers have the following properties:

- (a): $\sum_{k=0}^n kH_k^2 = \frac{1}{245025}((18315n - 26674) H_{n+3}^2 + (125235n - 230461) H_{n+2}^2 + (212850n - 454000) H_{n+1}^2 - 2(45045n - 73042) H_{n+3}H_{n+2} - 10(8415n - 11279) H_{n+3}H_{n+1} + 10(12870n - 10757) H_{n+2}H_{n+1} + 2709274)$.

(b): $\sum_{k=0}^n kH_{k+1}H_k = \frac{1}{245025}(- (8415n - 19694) H_{n+3}^2 - 11(6435n - 15031) H_{n+2}^2 - 25(8415n - 11279) H_{n+1}^2 + (48015n - 108004) H_{n+3}H_{n+2} + 5(14355n - 14873) H_{n+3}H_{n+1} - (138600n - 80395) H_{n+2}H_{n+1} - 1995944).$

(c): $\sum_{k=0}^n kH_{k+2}H_k = \frac{1}{245025}(- (3960n - 15761) H_{n+3}^2 - (119790n - 112079) H_{n+2}^2 - 25(3960n - 11801) H_{n+1}^2 + (65835n - 77701) H_{n+3}H_{n+2} + (4950n - 68335) H_{n+3}H_{n+1} - 5(18810n - 7421) H_{n+2}H_{n+1} - 2163461).$

From the last proposition, we have the following corollary which gives sum formulas of modified 3-primes numbers (take $H_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

COROLLARY 3.22. *For $n \geq 0$, modified 3-primes numbers have the following properties:*

(a): $\sum_{k=0}^n kE_k^2 = \frac{1}{245025}((18315n - 26674) E_{n+3}^2 + (125235n - 230461) E_{n+2}^2 + (212850n - 454000) E_{n+1}^2 - 2(45045n - 73042) E_{n+3}E_{n+2} - 10(8415n - 11279) E_{n+3}E_{n+1} + 10(12870n - 10757) E_{n+2}E_{n+1} + 164511).$

(b): $\sum_{k=0}^n kE_{k+1}E_k = \frac{1}{245025}(- (8415n - 19694) E_{n+3}^2 - 11(6435n - 15031) E_{n+2}^2 - 25(8415n - 11279) E_{n+1}^2 + (48015n - 108004) E_{n+3}E_{n+2} + 5(14355n - 14873) E_{n+3}E_{n+1} - (138600n - 80395) E_{n+2}E_{n+1} - 108216).$

(c): $\sum_{k=0}^n kE_{k+2}E_k = \frac{1}{245025}(- (3960n - 15761) E_{n+3}^2 - (119790n - 112079) E_{n+2}^2 - 25(3960n - 11801) E_{n+1}^2 + (65835n - 77701) E_{n+3}E_{n+2} + (4950n - 68335) E_{n+3}E_{n+1} - 5(18810n - 7421) E_{n+2}E_{n+1} - 108054).$

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