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Relation between Bernoulli Polynomial Matrix and *k*-Fibonacci Matrix

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Abstract: *The Bernoulli polynomial matrix is expressed by $\mathscr{B}_n(x)$, with each entry being a Bernoulli polynomial. The k-Fibonacci matrix is represented by $F_n(k)$, with each entry being a k-Fibonacci number, whose first term is 0, the second term is 1, and the next term depends on a positive integer k. In this paper, we discuss about relation between the Bernoulli polynomial matrix and k-Fibonacci matrix. The results are define two new matrix, $C_n(x)$ and $D_n(x)$ such that $\mathscr{B}_n(x) = F_n(k)\, C_n(x) = D_n(x) F_n(k)$.*

Keyword: *Bernoulli number, Bernoulli polynomial, Bernoulli matrix, Bernoulli polynomial matrix, k-Fibonacci number, k-Fibonacci matrix.*

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I. Introduction

Bernoulli numbers are defined by Jacob Bernoulli (1654-1705) [1]. They first appeared in *Ars Conjectandi*, page 97, a famous treatise published in 1713. The Bernoulli polynomials are polynomials B_n of the degree n which are convenient for integration by part as well as polynomials x^n. They also are orthogonal to constant function. Bernoulli numbers and Bernoulli polynomials have been represented in matrices. The Bernoulli matrix is a lower trianguler matrix with each entry being a Bernoulli number, and if each entry being a Bernoulli polynomial, then that is Bernoulli polynomial matrix.

Matrices and matrix theory are recently used in number theory and combinatorics. In particular, Bernoulli type lower triangular matrices are studied with Fibonacci, Pascal, and Stirling numbers, and other special numbers sequences. In 2008, Ernst [2] factorized (generalized) q-Bernoulli matrix by Pascal matrix and obtained some combinatorial identities. Can and Dagli [3] discussed generalized Bernoulli and Stirling matrices and obtained some of related combinatorial identities. Tuglu and Kus [4] studied q-Bernoulli and its properties. Earlier, in 2006, Zhang and Wang [5] disscussed relation between Bernoulli polynomial matrix and some of other matrices. They gave a product formula for the Bernoulli matrix and established several identities involving Fibonacci matrix, Stirling matrix, and Bernoulli polynomial matrix.

The k-Fibonacci number was introduced by Falcon [6]. That is a number whose first term is 0, the second term is 1, and the next term depends on a positive integer k, and it can be represent in a matrix. The k-Fibonacci matrix is a lower trianguler matrix with each entry being a k-Fibonacci number. Wahyuni et al. [7] was introduced some identities of k-Fibonacci sequences modulo ring Z_6 and Z_{10}. Mawaddaturrohmah and S. Gemawati [8] have discussed about relationship of Bell's polynomial matrix is expressed as B_n and k-Fibonacci matrix is expressed as $F_n(k)$, such that obtained two matrices, those are Y_n and Z_n, thus $B_n = F_n(k)\, Y_n = Z_n\, F_n(k)$, for any n, k are natural number.

In this paper, we discuss relation between Bernoulli polynomial matrix and k-Fibonacci matrix. Using two ideas of [2, 5] we define two matrices, those are $C_n(x)$ and $D_n(x)$ which states the relation between Bernoulli polynomial and k-Fibonacci matrices.

II. Preliminaries

In this section, we introduce some definitions which are essential for the subsequent sections. We begin with some definitions and theories of Bernoulli and k-Fibonacci numbers and matrices. Firstly, we mention that Bernoulli numbers. Then using these numbers, a matrix can be delivered. This matrix is called Bernoulli matrix. Extending this matrix some matrices are obtained.

Some definitions and theories related to Bernoulli and k-Fibonacci numbers and matrices that have been discussed by several authors [5, 9, 10, 11].

Definition 2.1. Bernoulli numbers are defined initial condition by $B_0 = 1$ and for any natural number n hold

$$B_n = -\frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k} B_k.$$

The first few Bernoulli numbers are:
$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, B_7 = 0, B_8 = -\frac{1}{30}$.

For any B_n is a Bernoulli number, then $B_n(x)$ is a Bernoulli polynomial, that given in following definition.

Definition 2.2. Let n be a natural number, the Bernoulli polynomials $B_n(x)$ are defined by

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k} B_k\, x^{n-k}.$$

The first few Bernoulli polynomials are:

$$B_0(x) = 1,$$
$$B_1(x) = x - \frac{1}{2},$$
$$B_2(x) = x^2 - x + \frac{1}{6},$$
$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,$$
$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30},$$
$$B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x,$$
$$B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}.$$

Zhang [17] defined Bernoulli matrices by using Bernoulli numbers and polynomials. Also, they obtained factorization and some properties of Bernoulli matrices.

Definition 2.3. Let B_n be n^{th} Bernoulli number and $B_n(x)$ be Bernoulli polynomial, $(n+1)\times(n+1)$ type Bernoulli matrix $\mathscr{B}_n = [b_{i,j}]$ and Bernoulli polynomial matrix $\mathscr{B}_n(x) = [b_{i,j}(x)]$ defined respectively as follows

$$b_{i,j} = \begin{cases} \binom{i}{j} B_{i-j} & if\ i \geq j, \\ 0 & otherwise, \end{cases}$$

and

$$b_{i,j}(x) = \begin{cases} \binom{i}{j} B_{i-j}(x) & if\ i \geq j, \\ 0 & otherwise. \end{cases}$$

Example 1. Bernoulli matrix $\mathscr{B}_4$ and Bernoulli polynomial matrix $\mathscr{B}_4(x)$ are

$$\mathscr{B}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{6} & -1 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{bmatrix} \quad \text{and} \quad \mathscr{B}_4(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x-\frac{1}{2} & 1 & 0 & 0 \\ x^2 - x + \frac{1}{6} & 2x-1 & 1 & 0 \\ x^3 - \frac{3}{2}x^2 + \frac{1}{2}x & 3x^2 - 3x + \frac{1}{2} & 3x - \frac{3}{2} & 1 \end{bmatrix}.$$

Definition 2.4. For any integer number $k \geq 1$, the k^{th} Fibonacci sequence, say $\{F_{k,n}\}_{n\in N}$ is defined recurrently by

$$F_{k,0} = 0,\ F_{k,1} = 1, \text{ and } F_{k,n+1} = k\, F_{k,n} + F_{k,n-1} \quad \text{for } n \geq 1.$$

The first k-Fibonacci numbers are:

$$F_{k,1} = 1,$$
$$F_{k,2} = k,$$

$F_{k,3} = k^2 + 1,$

$F_{k,4} = k^3 + 2k,$

$F_{k,5} = k^4 + 3k^2 + 1,$

$F_{k,6} = k^5 + 4k^3 + 3k,$

$F_{k,7} = k^6 + 5k^4 + 6k^2 + 1,$

$F_{k,8} = k^7 + 6k^5 + 10k^3 + 4k.$

Definition 2.5. Let $F_{k,n}$ be n^{th} *k*-Fibonacci number, the $n \times n$ *k*-Fibonacci matrix as the unipotent lower triangular matrix $F_n(k) = [f_{i,j}]_{i,j=1,\dots,n}$ defined with entries $f_{i,j} = F_{k,i-j+1}$ if $i \geq j$, 0 otherwise. That is

$$F_n(k) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ F_{k,2} & 1 & 0 & 0 & 0 & 0 \\ F_{k,3} & F_{k,2} & 1 & 0 & 0 & 0 \\ F_{k,4} & F_{k,3} & F_{k,2} & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ F_{k,n} & F_{k,n-1} & F_{k,n-2} & F_{k,n-3} & \cdots & 1 \end{bmatrix}.$$

Example 2. The 4×4 *k*-Fibonacci matrix is

$$F_4(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ k & 1 & 0 & 0 \\ k^2+1 & k & 1 & 0 \\ k^3+2k & k^2+1 & k & 1 \end{bmatrix}.$$

Definition 2.6. Let $F_n^{-1}(k)$ be inverse matrix of the *k*-Fibonacci matrix, the $n \times n$ inverse *k*-Fibonacci matrix as lower triangular matrix $F_n^{-1}(k) = [f'_{i,j}(k)]_{i,j=1,\dots,n}$ where

$$f'_{i,j}(k) = \begin{cases} 1 & if\ j = i, \\ -k & if\ j = i-1, \\ -1 & if\ j = i-2, \\ 0 & otherwise, \end{cases}$$

that is,

$$F_n^{-1}(k) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -k & 1 & 0 & 0 & 0 & 0 \\ -1 & -k & 1 & 0 & 0 & 0 \\ 0 & -1 & -k & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & -1 & -k & 1 \end{bmatrix}.$$

Example 3. The 4×4 inverse matrix of the *k*-Fibonacci matrix is

$$F_4^{-1}(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -k & 1 & 0 & 0 \\ -1 & -k & 1 & 0 \\ 0 & -1 & -k & 1 \end{bmatrix}.$$

III. Relation Between Bernoulli Polynomial Matrix and *k*-Fibonacci Matrix

In this section, we discuss relation between Bernoulli polynomial matrix and *k*-Fibonacci matrix by the similar way in [5]. We obtain definitions of new matrix $C_n(x)$ and $D_n(x)$. Furthermore, factorizations obtained on the Bernoulli polynomial matrix and *k*-Fibonacci matrix.

3.1 The First Factorization for Bernoulli Polynomial Matrix and *k*-Fibonacci Matrix

We obtain relation between Bernoulli polynomial matrix and *k*-Fibonacci matrix by doing multiplication between inverse of *k*-Fibonacci matrix $F_n^{-1}(k)$ and Bernoulli polynomial matrix $\mathscr{B}_n(x)$, for any n is natural number. The first step, for $n = 2$, by multiply $F_2^{-1}(k)$ and $\mathscr{B}_2(x)$ obtained $C_2(x)$, that is

$$F_2^{-1}(k)\ \mathscr{B}_2(x)=\begin{bmatrix}1 & 0\\ -k & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ x-\frac{1}{2} & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ x-k-\frac{1}{2} & 1\end{bmatrix}=C_2(x).$$

The second step, for $n=3$, by multiply $F_3^{-1}(k)$ and $\mathscr{B}_3(x)$, we have $C_3(x)$ as follows

$$\begin{aligned}F_3^{-1}(k)\ \mathscr{B}_3(x)&=\begin{bmatrix}1 & 0 & 0\\ -k & 1 & 0\\ -1 & -k & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ x-\frac{1}{2} & 1 & 0\\ x^2-x+\frac{1}{6} & 2x-1 & 1\end{bmatrix}\\ &=\begin{bmatrix}1 & 0 & 0\\ x-k-\dfrac{1}{2} & 1 & 0\\ x^2-(k+1)x+\dfrac{1}{2}k-\dfrac{5}{6} & 2x-k-1 & 1\end{bmatrix}\\ &=C_3(x).\end{aligned}$$

The next step, for $n>3$, to simplify calculations of multiplication between inverse of k-Fibonacci matrix and Bernoulli polynomial matrix used Maple, such that obtained $C_4(x), C_5(x), \dots, C_n(x)$. Then, by look at each entry of $C_4(x)$, we have that each entry of main diagonal is 1 (for $i=j$ entry) and for $i>j$ obtained as follows:

i. On the 1st row, then $c_{1,1}(x)=1$, $c_{1,j}(x)=0$ for $j\geq 2$.
ii. On the 2nd row, then $c_{2,1}(x)=x-k-\frac{1}{2}$, $c_{2,2}(x)=1$, $c_{2,j}(x)=0$ for $j\geq 3$.
iii. On the 3rd row, then $c_{3,1}(x)=x^2-(k+1)x+\frac{1}{2}k-\frac{5}{6}$, $c_{3,2}(x)=2x-k-1$, $c_{3,3}(x)=1$, $c_{3,j}(x)=0$ for $j\geq 3$.
iv. Entry $c_{i,j}(x)=1$ for any $i=j$ and $c_{i,j}(x)=0$ for $i<j$.

Thus, a list of all values of the $C_4(x)$ matrix entries are given in the following table.

Table 1: Element values for matrix $C_n(x)$

Matrix $C_n(x)$ Entry	Matrix $C_n(x)$ Entry Value
$c_{1,1}(x)$	$\binom{1}{1} B_0(x)= B_0(x)$
$c_{2,2}(x)$	$\binom{2}{2} B_0(x)= B_0(x)$
$c_{3,3}(x)$	$\binom{3}{3} B_0(x)= B_0(x)$
$c_{4,4}(x)$	$\binom{4}{4} B_0(x)= B_0(x)$
$c_{2,1}(x)$	$2B_1(x)-kB_0(x)=\binom{2}{1} B_1(x)+(-k)\binom{1}{1} B_0(x)$
$c_{3,2}(x)$	$3B_1(x)-kB_0(x)=\binom{3}{2} B_1(x)+(-k)\binom{2}{2} B_0(x)$
$c_{4,3}(x)$	$4B_1(x)-kB_0(x)=\binom{4}{3} B_1(x)+(-k)\binom{3}{3} B_0(x)$
$c_{3,1}(x)$	$3B_2(x)-2kB_1(x)-B_0(x)=\binom{3}{1} B_2(x)+(-k)\binom{2}{1} B_1(x)+(-1)\binom{1}{1} B_0(x)$
$c_{4,2}(x)$	$6B_2(x)-3kB_1(x)-B_0(x)=\binom{4}{2} B_2(x)+(-k)\binom{3}{2} B_1(x)+(-1)\binom{2}{2} B_0(x)$
$c_{4,1}(x)$	$4B_3(x)-3kB_2(x)-2B_1(x)=\binom{4}{1} B_3(x)+(-k)\binom{3}{1} B_2(x)+(-1)\binom{2}{1} B_1(x)$
$\vdots$	$\vdots$
$c_{i,j}(x)$	$\binom{i}{j} B_{i-j}(x)-k\binom{i-1}{j} B_{i-j-1}(x)-\binom{i-2}{j} B_{i-j-2}(x)$

Furthermore, by look at each matrix entry and the matrix value of Table 1 obtained following definition.

Definition 3.1. For every natural number n, it is defined an $(n+1)\times(n+1)$ matrix $C_n(x)=[c_{i,j}(x)]$ with $i,j=0,1,2,\dots,n$ as follows

$$c_{i,j}(x) = \binom{i}{j} B_{i-j}(x) - k\binom{i-1}{j} B_{i-j-1}(x) - \binom{i-2}{j} B_{i-j-2}(x).$$

From Definition 3.1 obtained

$c_{0,0}(x) = B_0(x) = 1,$

$c_{0,j}(x) = 0$ for $j \geq 1,$

$c_{1,0}(x) = B_1(x) - kB_0(x) = x - \dfrac{1}{2} - k,$

$c_{i,0}(x) = B_i(x) - kB_{i-1}(x) - B_{i-2}(x)$ for $i \geq 2,$

$c_{1,1}(x) = B_0(x) = 1,$

$c_{1,j}(x) = 0$ for $j \geq 2.$

From the definitions of Bernoulli polynomials matrix, k-Fibonacci matrix, and $C_n(x)$ matrix, the following theorem can be derived.

Theorem 3.2. If $\mathscr{B}_n(x)$ be a Bernoulli polynomial matrix, $F_n(k)$ be a k-Fibonacci matrix, and $C_n(x)$ be a new matrix in Definition 3.1, then $\mathscr{B}_n(x) = F_n(k)\, C_n(x)$.

Proof. Since $F_n(k)$ be a k-Fibonacci matrix, then it is a invertible matrix. So that, we will be proven that

$$C_n(x) = F_n^{-1}(k)\, \mathscr{B}_n(x).$$

Let $F_n^{-1}(k)$ be a inverse of k-Fibonacci matrix and $\mathscr{B}_n(x)$ be a Bernoulli polynomial matrix, then from Definition 2.6 and 2.3 we have the value of main diagonal of them is 1. The value of main diagonal $C_n(x)$ matrix is 1. From Definition 3.1 obtained $c_{i,j}(x) = 1$ for $i = j$ and $c_{i,j}(x) = 0$ for $i < j$, thus for $i > 2$, we get

$$\begin{aligned} c_{i,j}(x) &= f'_{i,t}(k)\, b_{t,j}(x) \\ &= f'_{i,i}(k)\, b_{i,j}(x) + f'_{i,i-1}(k)\, b_{i-1,j}(x) + f'_{i,i-2}(k)\, b_{i-2,j}(x) + f'_{i,i-3}(k)\, b_{i-3,j}(x) + \cdots + f'_{i,n}(k)\, b_{n,j}(x), \\ &= \sum_{t=0}^{n} f'_{i,t}(k)\; b_{t,j}(x). \end{aligned}$$

Hence, $F_n^{-1}(k)\mathscr{B}_n(x) = C_n(x)$ such that $\mathscr{B}_n(x) = F_n(k)C_n(x)$.

3.2 The Second Factorization for Bernoulli Polynomial Matrix and *k*-Fibonacci Matrix

We obtain relation between Bernoulli polynomial matrix and k-Fibonacci matrix by doing multiplication between Bernoulli polynomial matrix $\mathscr{B}_n(x)$ and inverse of k-Fibonacci matrix $F_n^{-1}(k)$, for any natural number n. The first step, for $n = 2$, by multiply $\mathscr{B}_2(x)$ and $F_2^{-1}(k)$ obtained $D_2(x)$, that is

$$\mathscr{B}_2(x)\, F_2^{-1}(k) = \begin{bmatrix} 1 & 0 \\ x - \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x - k - \frac{1}{2} & 1 \end{bmatrix} = D_2(x).$$

The second step, for $n = 3$, by multiply $\mathscr{B}_3(x)$ and $F_3^{-1}(k)$, we have $D_3(x)$ as follows:

$$\begin{aligned} \mathscr{B}_3(x)F_3^{-1}(k) &= \begin{bmatrix} 1 & 0 & 0 \\ x - \frac{1}{2} & 1 & 0 \\ x^2 - x + \frac{1}{6} & 2x - 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ -1 & -k & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ x - k - \dfrac{1}{2} & 1 & 0 \\ x^2 - (2k+1)x + k - \dfrac{5}{6} & 2x - k - 1 & 1 \end{bmatrix} \\ &= D_3(x). \end{aligned}$$

The next step, for $n > 3$, to simplify calculations of multiplication between Bernoulli polynomial matrix and inverse of k-Fibonacci matrix used Maple, such that obtained $D_4(x), D_5(x), \ldots, D_n(x)$. Then, by look at each entry of $D_4(x)$, we have that each entry of main diagonal is 1 (for $i = j$ entry) and for $i > j$ obtained as follows:

i. On the 1st row, then $d_{1,1}(x) = 1$, $d_{1,j}(x) = 0$ for $j \geq 2$.

ii. On the 2nd row, then $d_{2,1}(x) = x - k - \frac{1}{2}$, $d_{2,2}(x) = 1$, $d_{2,j}(x) = 0$ for $j \geq 3$.

iii. On the 3rd row, then $d_{3,1}(x) = x^2 - (2k+1)x + k - \frac{5}{6}$, $d_{3,2}(x) = 2x - k - 1$, $d_{3,3}(x) = 1$, $d_{3,j}(x) = 0$ for $j \geq 3$.

iv. Entry $d_{i,j}(x) = 1$ for any $i = j$ and $d_{i,j}(x) = 0$ for $i < j$.

Thus, a list of all values of the $D_4(x)$ matrix entries are given in the following table.

Table 2: Element values for matrix $D_n(x)$

Matrix $D_n(x)$ Entry	Matrix $D_n(x)$ Entry Value
$d_{1,1}(x)$	$\binom{1}{1} B_0(x) = B_0(x)$
$d_{2,2}(x)$	$\binom{2}{2} B_0(x) = B_0(x)$
$d_{3,3}(x)$	$\binom{3}{3} B_0(x) = B_0(x)$
$d_{4,4}(x)$	$\binom{4}{4} B_0(x) = B_0(x)$
$d_{2,1}(x)$	$2B_1(x) - kB_0(x) = \binom{2}{1} B_1(x) + (-k)\binom{2}{2} B_0(x)$
$d_{3,2}(x)$	$3B_1(x) - kB_0(x) = \binom{3}{2} B_1(x) + (-k)\binom{3}{3} B_0(x)$
$d_{4,3}(x)$	$4B_1(x) - kB_0(x) = \binom{4}{3} B_1(x) + (-k)\binom{4}{4} B_0(x)$
$d_{3,1}(x)$	$3B_2(x) - 3kB_1(x) - B_0(x) = \binom{3}{1} B_2(x) + (-k)\binom{3}{2} B_1(x) + (-1)\binom{3}{3} B_0(x)$
$d_{4,2}(x)$	$6B_2(x) - 4kB_1(x) - B_0(x) = \binom{4}{2} B_2(x) + (-k)\binom{4}{3} B_1(x) + (-1)\binom{4}{4} B_0(x)$
$d_{4,1}(x)$	$4B_3(x) - 6kB_2(x) - 4B_1(x)$ $= \binom{4}{1} B_3(x) + (-k)\binom{4}{2} B_2(x) + (-1)\binom{4}{3} B_1(x)$ $+ (0)\binom{4}{4} B_0(x)$
$\vdots$	$\vdots$
$d_{i,j}(x)$	$\binom{i}{j} B_{i-j}(x) - k\binom{i}{j+1} B_{i-j-1}(x) - \binom{i}{j+2} B_{i-j-2}(x)$

Furthermore, by look at each matrix entry and the matrix value of Table 2 obtained following definition.

Definition 3.3. For every natural number n, it is defined an $(n+1) \times (n+1)$ matrix $D_n(x) = [d_{i,j}(x)]$ with $i, j = 0,1,2, \ldots, n$ as follows

$$d_{i,j}(x) = \binom{i}{j} B_{i-j}(x) - k\binom{i}{j+1} B_{i-j-1}(x) - \binom{i}{j+2} B_{i-j-2}(x).$$

From Definition 3.3 obtained

$d_{0,0}(x) = B_0(x) = 1$,

$d_{0,j}(x) = 0$ for $j \geq 1$,

$d_{1,0}(x) = B_1(x) - kB_0(x) = x - \dfrac{1}{2} - k$,

$d_{i,0}(x) = B_i(x) - k\binom{i}{1}B_{i-1}(x) - \binom{i}{2}B_{i-2}(x)$ for $i \geq 2$,

$d_{1,1}(x) = B_0(x) = 1$,

$d_{1,j}(x) = 0$ for $j \geq 2$.

From the definitions of Bernoulli polynomials matrix, k-Fibonacci matrix, and $D_n(x)$ matrix, the following theorem can be derived.

Theorem 3.4. If $\mathscr{B}_n(x)$ be a Bernoulli polynomial matrix, $F_n(k)$ be a k-Fibonacci matrix, and $D_n(x)$ be a new matrix in Definition 3.3, then $\mathscr{B}_n(x) = D_n(x)F_n(k)$.
Proof. Since $F_n(k)$ be a k-Fibonacci matrix, then it is a invertible matrix. So that, we will be proven that

$$D_n(x) = \mathscr{B}_n(x)F_n^{-1}(k).$$

Let $F_n^{-1}(k)$ be a inverse of k-Fibonacci matrix and $\mathscr{B}_n(x)$ be a Bernoulli polynomial matrix, then from Definition 2.6 and 2.3 we have the value of main diagonal of them is 1. The value of main diagonal $D_n(x)$ matrix is 1. From Definition 3.3 obtained $d_{i,j}(x) = 1$ for $i = j$ and $d_{i,j}(x) = 0$ for $i < j$, thus for $i > 2$, we get

$$\begin{aligned} d_{i,j}(x) &= b_{i,t}(x)\, f'_{t,j}(k) \\ &= b_{i,j}(x)\, f'_{j,j}(k) + b_{i,j+1}(x)\, f'_{j+1,j}(k) + b_{i,j+2}(x)\, f'_{j+2,j}(k) + b_{i,j+3}(x)\, f'_{j+3,j}(k) + \cdots + b_{i,n}(x)\, f'_{n,j}(k), \\ &= \sum_{t=0}^{n} b_{i,t}(x)\ f'_{t,j}(k)\,. \end{aligned}$$

Hence, $\mathscr{B}_n(x)F_n^{-1}(k) = D_n(x)$such that $\mathscr{B}_n(x) = D_n(x)F_n(k)$.

IV. Conclusion

In this paper, the author discuss the relation between Bernoulli polynomial matrix and k-Fibonacci matrix, such that obtained two new matrices. Then, multiplication between the k-Fibonacci matrix and the first new matrix is the Bernoulli polynomial matrix. However, multiplication between the second new matrix and the k-Fibonacci matrix is the Bernoulli polynomial matrix too. Therefore, the first new matrix is not equal to the second new matrix and both of them are not commutative. For future research, it is necessary to think about the relation between q-Bernoulli polynomial matrix and k-Fibonacci matrix as well as the relation between Bernoulli polynomial and k-Fibonacci matrix.

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