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Shift Map and Cantor Set of Logistic Function

Hena Rani Biswas[1,*] and Md. Monirul Islam Sumon[2]

[1]*Department of Mathematics, University of Barishal, Barishal- 8200, Bangladesh.*
[2]*Department of Business Administration, Z. H. Sikder University of Science and Technology, Shariatpur, Bangladesh*

Abstract: *This paper presents some basic concepts of shift map and cantor set. We have proved some properties of shift map such as continuous, chaotic and homeomorphism. We have described the construction and the formula of the cantor ternary set, which is the most common modern construction. We selected a problem about cantor set of Logistic Function.*

Keywords: *Shift Map, Chaotic dynamical systems, Homeomorphism, Cantor Set.*

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I. Introduction

A dynamical systems consists of an abstract phase space or state space, whose coordinates describe the state at any instant, and a dynamical rule that specifies the immediate future of all state variables, given only the present values of those same state variables. The study of dynamical systems is the focus of dynamical systems theory, which has applications to a wide variety of fields such as mathematics, physics, biology, chemistry, engineering, economics and medicine.

Symbolic dynamics is part of dynamical systems theory. It is a powerful tool used in the study of dynamical systems. We all know that symbolic dynamical systems are a very interesting example of topological dynamical systems. Symbolic dynamics is also an example of chaotic dynamical systems. Robert L. Devaney [1] have given vivid description of the space $\mathbf{\Sigma_2}$. By symbolic dynamical systems we mean here the space of sequences $\mathbf{\Sigma_2} = \{\alpha : \alpha = (\alpha_0 \alpha_1 \dots \dots \dots), \alpha_i = 0 \text{ or } 1\}$ along with the shift map defined on it. It is known that $\mathbf{\Sigma_2}$ is a compact metric space by the metric $d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^{i+1}}$, where $s = (s_0 s_1 \dots \dots \dots)$ and $t = (t_0 t_1 \dots \dots \dots)$ are any two points of $\mathbf{\Sigma_2}$. Symbolic dynamics is concerned with maps on sets. We focus here on an example using the set Σ_2.We define Σ_2 as the set of infinite sequences of $0's$ and $1's$. An element or point in Σ_0 is something like $000000000000 \dots \dots$ or $010101010101 \dots \dots$ repeating. We can refer to the point s as $s_0 s_1 s_2 s_3 s_4 \dots \dots$ and the point t as $t_0 t_1 t_2 t_3 \dots \dots$.We can think of s and t as close to one another if their sequences are similar at the beginning of the sequence. This space has a nice metric that we can use to tell if two points in $\Sigma_2$are close to each other.

The Cantor set plays a very important role in many branches of mathematics, above all in set theory, chaotic dynamical systems and fractal theory. It is simply a subset of the interval $[0, 1]$, but it has a number of remarkable and deep properties. The Cantor set [12] is a famous set first introduced by German mathematician Georg Cantor in 1883. He was made famous by introducing the Cantor set in his works of mathematics. The ternary Cantor set is the most well- known of the Cantor sets, and can be best described by its construction.

This set starts with the closed interval zero to one, and is constructed in iterations. The first iteration requires removing the middle third of this interval. The second iteration will remove the middle third of each of these two remaining intervals.

These iterations continue in this fashion infinitely. Finally, the ternary Cantor set is described as the intersection of all of these intervals. This set is particularly interesting due to its unique properties being uncountable, closed, length of zero, and more. A more general Cantor set is created by taking the intersection of iterations that remove any middle portion during each iteration.

In section 2 of this paper we describe the mathematical preliminaries which are requirements for the subsequent chapters. In section 3 we discuss properties of the shift map. In section 4 we obtain chaotic properties of shift map $\sigma : \Sigma \to \Sigma$ and we see that it is homeomorphism. In this section we also solve a problem about cantor set of Logistic Function. The paper ends with a conclusion which is section 5.

*E-mail: biswas.hena@yahoo.com

II. Mathematical Preliminaries

In this section, we include some definitions, lemma and proposition which is important for the proof of main theorem. We discuss some properties of shift map, proof of different theorem such as shift map is continuous in $\sum_2$, shift map $\sigma: \sum \to \sum$ is a chaotic dynamical systems and the shift map $\sigma : \Sigma \to \Sigma$ is homeomorphism.

Definition 2.1 (**Sequence space):** The set of all infinite sequences of 0's and 1's is called the sequence space **[2]** of 0 and 1 or the symbol space of 0 and 1 and is denoted by Σ_2. More precisely, $\Sigma_2 = \{(s_0s_1s_2 \dots\dots\dots) \mid s_i = 0 \text{ or } s_i = 1 \text{ for all } i\}$.

We often refer to elements of Σ_2 as points in Σ_2.

Example 2.1: $(0000 \dots\dots\dots), (010101 \dots\dots\dots), (101010 \dots\dots\dots)$ and $(1111 \dots\dots\dots)$ are all distinct elements of Σ_2.

Definition 2.2 (Shift Map): The shift map **[2]** $\sigma : \Sigma_2 \to \Sigma_2$ is defined by $\sigma(s_0s_1s_2 \dots) = s_1s_2s_3 \dots$. In other words, the shift maps "forgets" the first digit of the sequence.

Example 2.2: $\sigma(010101 \dots\dots\dots) = (10101 \dots\dots\dots), \sigma(01110101 \dots\dots\dots) = (1110101 \dots\dots\dots)$

Now $\sigma^2(s_0s_1s_2 \dots\dots\dots) = \sigma(\sigma(s_0s_1s_2 \dots\dots\dots)) = \sigma(s_1s_2s_3 \dots\dots\dots) = s_2s_3s_4 \dots\dots\dots$

Continue in this way, we get $\sigma^n(s_0s_1s_2 \dots\dots\dots) = s_ns_{n+1}s_{n+2} \dots\dots\dots$

Definition 2.3: Suppose X is a set and Y is a subset of X. We say that Y is **dense** in X if , for any point $x \in X$. there is a point y in the subset Y arbitrarily close to .

Definition 2.4: Let (X, f) be a dynamical systems. Then f is **transitive** if for any two non-empty open subsets U, V of X there exists $n \geq 0$ such that $f^n(U) \cap V \neq \emptyset$.

Definition 2.5: Let (X, f) be a dynamical systems where X is a metric space with metric d. Then f is **sensitive** if there exists $\delta > 0$ with the property that $\forall\, x \in X$ and $\forall\, x > 0, \exists\, y \in B(x, \in)$ and $\exists\, n \in N$ such that

$$d\left(f^n(x), f^n(y)\right) > \delta.$$

These three properties of dense, transitivity and sensitivity are the basic ingredient of a chaotic system.

Definition 2.6: A dynamical system (X, f) where X is a metric space, is **chaotic [1]** if

i) f is topologically transitive,
ii) the set of periodic points of f is dense,
iii) f is sensitive dependence on initial condition.

Proposition 2.1: The distance d on Σ given by $d[s,t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$ is a metric on Σ_2.

Proof: Let $s = (s_0s_1s_2 \dots\dots)$, $t = (t_0t_1t_2 \dots\dots)$ and $u = (u_0u_1u_2 \dots\dots)$. Clearly, $d[s,t] \geq 0$ and $d[s,t] = 0$ if and only if $s = t$. Since $|s_i - t_i| = |t_i - s_i|$, it follows that $d[s,t] = d[t,s]$. Finally, for any three real numbers s_i, t_i, u_i, we have the triangle inequality

$$|s_i - t_i| + |t_i - u_i| \geq |s_i - u_i|$$

From which we deduce that

$$d[s,t] + d[t,u] \geq d[s,u].$$

Hence d is a metric on .

Definition 2.7: A nonempty set $C \subset \mathbb{R}$ is called a **cantor set [2]** if

a) C is closed and bounded. (Sets of real numbers with these characteristics are called compact sets)
b) C contains no intervals. (Sets of this nature are called totally disconnected sets)
c) Every point in C is an accumulation point of C. (When closed, such sets are called perfect sets)

Definition 2.8 (Cantor Ternary Set): The set obtained by repeatedly deleting the open-middle third from the closed interval $[0, 1]$ is termed as the Cantor Ternary Set **[17].**

Definition 2.9 (Totally disconnected Set): A set is said to totally disconnected in $\mathbb{R}$ if it contains no intervals.

Definition 2.10 (Perfect set): A set is said to be **perfect** if every point in it is an accumulation point or limit point of other points in the set.

Definition 2.11 (Cantor's Middle-Thirds Set) : From the unit interval $I = [0,1]$, delete the middle third part *i.e.* delete the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$. Again delete the middle thirds *i.e.* the pair of open intervals $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$ from the remaining part which is the closed interval$\left[0, \frac{1}{3}\right] \cup \left[\frac{1}{3}, 1\right]$. We continue removing middle thirds in this fashion infinitely many times. The remaining set is called as the Cantor's middle-thirds set. At the k^{th}step, the total length of the 2^kclosed intervals is $\left(\frac{2}{3}\right)^k$ which tends to zero as $k \to \infty$.

Lemma 2.1 (Compact Sets are Bounded) : Let $C \subseteq \mathbb{R}$ be a compact set. Then C is bounded. In other words, compact sets in $\mathbb{R}$ are bounded.

Lemma 2.2 (Compact Sets are Closed) : Let $C \subseteq \mathbb{R}$ be a compact set. Then C is closed. Hence, compact sets in $\mathbb{R}$ are closed.

Construction of Cantor set: The Cantor ternary set is created by repeatedly deleting the open middle thirds of a set of line segments. One starts by deleting the open middle third $(1/3, 2/3)$ from the interval $[0, 1]$, leaving two line segments: $C_{1=}[0, 1/3] \cup [2/3, 1]$.Next, the open middle third of each of these remaining segments is deleted, leaving four line segments: $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. Continue in this way always removing the middle third of each segment to get $C_3, C_4, \ldots\ldots\ldots$. Note that $C_1 \supseteq C_2 \supseteq C_3 \supseteq C_4 \ldots\ldots\ldots\ldots$ And for each $k \in \mathbb{N}$. C_k is the union of 2^k closed intervals, each of length 3^{-k}. Let $C = \bigcap_{n=1}^{\infty} C_n$.Then C is the **Cantor set**. The set is constructed inductively as follows: Begin with the unit interval denoted C_0 for notational purpose, and remove the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ called the "Middle third" [15]. Where the nth set formula is given as: $C_n = \frac{C_{n-1}}{3} \cup (\frac{2}{3} + \frac{C_{n-1}}{3})$

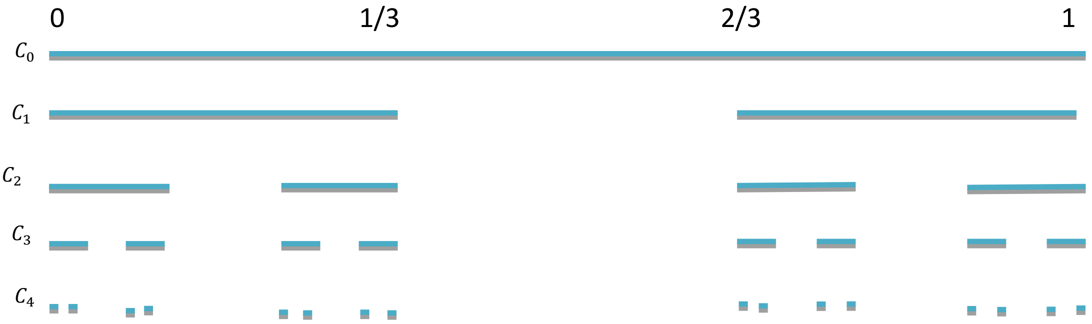

Figure 2.1: The first five sets formed in the construction of the Cantor middle-thirds set.

III. Some Basic Theorems and Properties

In this section we will discuss some basic theorems and properties which is helpful for main results of this paper.

The Proximity Theorem 3.1: Let $s, t \in \Sigma$ and suppose $s_i = t_i$ for $i = 0,1, \ldots n.$ Then $d[s, t] \leq \frac{1}{2^n}$. Conversely, if $d[s, t] < \frac{1}{2^n}$, then $s_i = t_i$ for $i \leq n$.

Proof: If $s_i = t_i,\ for\ i \leq n,$ then $d[s, t] = \sum_{i=0}^{n} \frac{|s_i - s_i|}{2^i} + \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i}$

$$= \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i} \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n}.$$

On the other hand, if $s_j \neq t_j$ for some j$\leq n$, then we must have $d[s, t] \geq \frac{1}{2^j} \geq \frac{1}{2^n}$.

Consequently, if $d[s, t] < \frac{1}{2^n}$, then $s_i = t_i$ for $i \leq n$.

Theorem 3.2: The shift map[1] is continuous.

Proof: Suppose we are given $\in > 0$ and $\boldsymbol{s} = (s_0 s_1 s_2 \ldots\ldots)$. We will show that σ is continuous at $\boldsymbol{s}$. Since $\in > 0$, we may pick n such that $\frac{1}{2^n} < \in$. We then choose $\delta = \frac{1}{2^{n+1}}$. If t is a point in Σ and $d[\boldsymbol{t}, \boldsymbol{s}] < \delta$, then by the Proximity Theorem we must have $s_i = t_i$ for $i = 0,1, \ldots n + 1$. That is, $\boldsymbol{t} = (s_0 \ldots\ldots s_{n+1} t_{n+2} t_{n+3} \ldots\ldots)$. Now $\sigma(\boldsymbol{t}) = s_1 \ldots\ldots s_{n+1} t_{n+2} t_{n+3} \ldots\ldots)$ has entries that agree with those of $\sigma(\boldsymbol{s})$ in the first $n + 1$ spots. Thus, again by the Proximity Theorem, $d[\sigma(\boldsymbol{s}), \sigma(\boldsymbol{t})] \leq \frac{1}{2^n} < \in$. Therefore, σ is continuous at $\boldsymbol{s}$. Since $\boldsymbol{s}$ was arbitrary, Hence the shift map is continuous.

Proposition 3.1: The shift map has the following properties:

(i). The set of periodic points of the shift map is dense in Σ_2.

Proof: Suppose $s = s_0 s_1 s_2 \dots\dots$ in a period point of σ with period k.
Then $\sigma^n(\sigma^k(s)) = \sigma^n(s)$ $[\because\ \sigma^k(s) = s]$.
Since $\sigma^n(s)$ "forgets" the first n digits of s, we see that,

$$\begin{aligned}\sigma^n(\sigma^k(s_0 s_1 s_2 \dots\dots)) &= s_{n+k} s_{n+k+1} s_{n+k+2} \dots\dots \\ &= s_n s_{n+1} s_{n+2} \dots\dots \\ &= \sigma^n(s_0 s_1 s_2 \dots\dots)\end{aligned}$$

and $s_{n+k} = s_n\ \forall\ n$.
This implies that s is a periodic point with period k iff s is a sequence formed by repeating the k digits $s_0 s_1 \dots s_{k-1}$ infinitely often.
To prove that the periodic points of σ are dense in Σ_2, we must show that for all points t in Σ_2 and all $\epsilon > 0$, there is a periodic point of σ contained in $N_\epsilon(t)$.
We need to find a sequence in $N_\epsilon(t)$ that is formed by repeating the limit k digits of the sequence infinitely often. But if $t = t_0 t_1 t_2 t_3 \dots\dots$ and are choose n so that $\frac{1}{2^n} < \epsilon$,then we can let $s = t_0 t_1 \dots t_n t_0 t_1 \dots t_n t_0 t_1 \dots$
As t and s agree on the first $n + 1$ digits, so $d(s,t) \le \frac{1}{2^n} < \epsilon$.
Thus s is in $N_\epsilon(t)$ and s is a periodic point by construction.

(ii). The shift map has 2^n periodic points of period .

Proof: Suppose $s = s_0 s_1 s_2 \dots\dots$ is a periodic point of σ with period n. Then $\sigma^k(\sigma^n(s)) = \sigma^k(s)$.
Now $\sigma^k(\sigma^n(s_0 s_1 s_2 \dots\dots)) = s_{n+k} s_{n+k+1} s_{n+k+2} \dots\dots$

$$\begin{aligned}&= s_k s_{k+1} s_{k+2} \dots\dots \\ &= \sigma^k(s_0 s_1 s_2 \dots\dots)\end{aligned}$$

and $s_{n+k} = s_k\ \forall\ k$.
This implies that s is a periodic point with period n iff s is a sequence formed by repeating the n digits $s_0 s_1 s_2 \dots\dots s_{n-1}$ infinitely often i.e. $s = s_0 s_1 \dots\dots s_{n-1} s_0 s_1 \dots\dots s_{n-1} s_0 \dots\dots$. Now for each s, the repeating sequence is $s_0 s_1 \dots\dots s_{n-1}$. There are 2^n distinct finite sequence of $0's\ 1's$ of length n. Thus there are 2^n periodic points of period n for σ.

Proposition 3.2: Let $s \in \Sigma_2$ and $\epsilon > 0$ then there is $t \in \Sigma_2$ and $n \in \mathbb{N}$ such that $d(s,t) < \epsilon$ and $d(\sigma^n(s),$ *σnt=2* for $n \in \mathbb{N}$ *(n>N)*.

Proof: Let $s = s_0 s_1 s_2 \dots\dots \in \Sigma_2$ and $\epsilon > 0$. We will show that $t \in \Sigma_2$ and $n \in \mathbb{N}$ such that $d(s,t) < \epsilon$ but $d(\sigma^n(s), \sigma^n(t)) = 2$ whenever $n > N$.
Let $t = t_0 t_1 t_2 \dots\dots$ and $n \in \mathbb{N}$ so that $\frac{1}{2^n} < \epsilon$. Consider that the first $N + 1$ digits of t are same as the first $N + 1$ digits of s and other digits of t are all different from the corresponding digits of s.
i.e. $s_i = t_i$ iff $i \le N$. So $d(s,t) \le \frac{1}{2^n} < \epsilon$.
If $n > N$ then $\sigma^n(s)$, $\sigma^n(t)$ differ at each digit. Since $\sigma^n(s) = s_n s_{n+1} s_{n+2} \dots\dots$ and $\sigma^n(t) = t_n t_{n+1} t_{n+2} \dots\dots$
Then $d(\sigma^n(s), \sigma^n(t) = \sum_{i=0}^{\infty} \frac{|s_{n+i} - t_{n+i}|}{2^i} = \sum_{i=0}^{\infty} \frac{1}{2^i} = 2$. $\therefore\ d(\sigma^n(s), \sigma^n(t) = 2$.

IV. Results and Discussions

In this section, we will prove the shift map $\sigma: \sum \to \sum$ is a chaotic dynamical system and the shift map $\sigma : \Sigma \to \Sigma$ is homeomorphism. Then we will prove Cantor's Intersection Theorem and also we will solve a problem about cantor set of Logistic Function.

Theorem 4.1: Shift map $\sigma : \Sigma \to \Sigma$ is chaotic [1].

Proof: The shift map $\sigma : \Sigma \to \Sigma$ is chaotic if and only if, the following conditions are satisfying
(i) Periodic point of σ are dense in Σ.
(ii) σ is topologically transitive on Σ.
(iii) σ is sensitive dependence on initial condition.

(i): Let $x = \dots x_{-1} x_0 x_1 \dots) \in \Sigma$ and an open set U be given. Then for $x \in U$ there exists $\epsilon > 0$ such that $U(x,\epsilon) \subset U$. we choose $x \in \mathbb{N}$ such that $\frac{1}{2^{n-2}} < \epsilon$. We put $y = (x_{-(n-1)} \dots x_{-1} x_0 x_1 \dots x_{n-1}) \in \Sigma$.
Then $y \in P_{2n-1}(\sigma) \subset Per\ (\sigma)$
Then $d(x,y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i} + \sum_{i=1}^{\infty} \frac{|x_{-i} - y_{-i}|}{2^i}$

$$= \sum_{\mathrm{i}=0}^{\mathrm{n}-1} \frac{|x_i - y_i|}{2^{\mathrm{i}}} + \sum_{\mathrm{i}=\mathrm{n}}^{\infty} \frac{|x_i - y_i|}{2^{\mathrm{i}}} + \sum_{\mathrm{i}=1}^{\mathrm{n}-1} \frac{|x_{-i} - y_{-i}|}{2^{\mathrm{i}}} + \sum_{\mathrm{i}=\mathrm{n}}^{\infty} \frac{|x_{-i} - y_{-i}|}{2^{\mathrm{i}}}$$

$$= \sum_{\mathrm{i}=\mathrm{n}}^{\infty} \frac{|x_i - y_i|}{2^{\mathrm{i}}} + \sum_{\mathrm{i}=\mathrm{n}}^{\infty} \frac{|x_{-i} - y_{-i}|}{2^{\mathrm{i}}} \le \sum_{i=n}^{\infty} \frac{1}{2^i} + \sum_{i=n}^{\infty} \frac{1}{2^i}$$

$$= 2\sum_{i=n}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n-2}} < \epsilon$$

Thus, $y \in U(x, \epsilon) \subset U$ Therefore, $y \in Per\,(\sigma) \cap U$

$\Rightarrow Per\,(\sigma) \cap U \neq \emptyset$

$\therefore Per\,(\sigma)$ is dense in Σ.

(ii): Let U_1 and U_2 be given two non-empty open sets in Σ. Then there exists $x\ in\ U_1$ and $\epsilon_1 > 0$ such that $U_1(x, \epsilon_1) \subset U_1$ and there exists y in U_2 and $\epsilon_2 > 0$ such that $U_2(y, \epsilon_2) \subset U_2$. We take $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ then

$$U_1(x, \epsilon) \subset U_1 \text{ and } U_2(y, \epsilon) \subset U_2$$

Let $x = (\ldots x_{-1} x_0 x_1 \ldots) \in \Sigma$ and $y = (\ldots y_{-1} y_0 y_1 \ldots) \in \Sigma$ are given.

For $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{2^{n-1}} < \epsilon$.

Let z be a sequence $(z_i)_{i \in \mathbb{Z}}$ such that,

$$z_i = \begin{cases} x_i \ \ for \ \ -n \le i \le n \\ y_{i+2n+1} \ \ for \ \ -3n-1 \le i \le -n-1 \end{cases}$$

Then $d(z, x) = \sum_{\mathrm{i}=0}^{\infty} \frac{|\mathrm{z_i} - \mathrm{x_i}|}{2^{\mathrm{i}}} + \sum_{i=1}^{\infty} \frac{|\mathrm{z_{-i}} - \mathrm{x_{-i}}|}{2^{\mathrm{i}}} \le \frac{1}{2^{n-1}} < \epsilon$

Let $= \sigma^{2n+1}(z) = (s_i)_{i \in z}$. Then

$$s_i = z_{i-(2n+1)} = \begin{cases} y_{i-(2n+1)+(2n+1)};\ -3n-1 \le i-(2n-1) \le -n-1 \\ y_i \quad ; \quad -n \le i \le n. \end{cases}$$

Thus $d(\sigma^{2n+1}(z), y) = d(s, y) = \sum_{\mathrm{i}=0}^{\infty} \frac{|\mathrm{s_i} - \mathrm{y_i}|}{2^{\mathrm{i}}} + \sum_{\mathrm{i}=1}^{\infty} \frac{|\mathrm{s_{-i}} - \mathrm{y_{-i}}|}{2^{\mathrm{i}}} \le \frac{1}{2^{\mathrm{n}-1}} < \epsilon$

Since $d(z, x) < \epsilon \le \epsilon_1$, we have $, z \in U_1(x, \epsilon_1) \subset U_1$.

Thus, $\sigma^{2n+1}(z) \in \sigma^{2n+1}\big(U_1(x, \epsilon_1)\big) \subset \sigma^{2n+1}(U_1)$.

Since $d(\sigma^{2n+1}(z), y) < \epsilon \le \epsilon_2$ we have $\sigma^{2n+1}(z) \in U_2(y, \epsilon_2) \subset U_2$.

Then $\sigma^{2n+1}(z) \epsilon\ \sigma^{2n+1}\,(U_1) \cap (U_2) \Rightarrow \sigma^{2n+1}(U_1) \cap U_2 \neq \emptyset$.

$$\Rightarrow \sigma^k(U_1) \cap U_2 \neq \emptyset \quad \text{where } k = 2n+1$$

Hence σ is one-sided topologically transitive.

(iii): Let $x = (\ldots x_{-1} x_0 x_1 \ldots) \in \Sigma$ and $\epsilon > 0$ be given. We choose $\mathrm{x} \in \mathbb{N}$ such that $\frac{1}{2^n} < \epsilon$.

We put, $y = (\ldots y_{-1} y_0 y_1 \ldots) \in \Sigma$ such that $y_i = x_i$ for $i \le n$. and $y_{n+1} \neq x_{n+1}$.

Then $d(x, y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i} + \sum_{i=1}^{\infty} \frac{|x_{-i} - y_{-i}|}{2^i}$

$$= \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i} + \sum_{i=1}^{\infty} \frac{|x_{-i} - y_{-i}|}{2^i} + \sum_{i=n+1}^{\infty} \frac{|x_i - y_i|}{2^i}$$

$$= \sum_{i=n+1}^{\infty} \frac{|x_i - y_i|}{2^i} \le \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n} < \epsilon$$

But $d\big(\sigma^{n+1}(x), \sigma^{n+1}(y)\big)$

$$= \sum_{i=0}^{\infty} \frac{|x_{i+n+1} - y_{i+n+1}|}{2^i} + \sum_{i=1}^{\infty} \frac{|x_{-(i+n+1)} - y_{-(i+n+1)}|}{2^i}$$

$$= \sum_{i=0}^{\infty} \frac{|x_{i+n+1} - y_{i+n+1}|}{2^i} \ge \frac{|x_{n+1} - y_{n+1}|}{2^0} = 1$$

Thus σ is sensitive dependence on initial condition.

Hence shift map is chaotic.

Theorem 4.2: The shift map $\sigma : \Sigma \to \Sigma$ is homeomorphism [3].

Proof: It is sufficient to show that (i) σ is one-one (ii) σ is onto (iii) σ is continuous (iv) σ^{-1} is continuous.

(i) σ is one-one: Suppose that $\sigma(x) = \sigma(y)$ where $x = (x_n)_{n \epsilon \mathbb{Z}}$, $y = (y_n)_{n \epsilon \mathbb{Z}} \in \Sigma$. We put $z = \sigma(x) = \sigma(y)$, where $z = (z_n)_{n \epsilon \mathbb{Z}}$; $\mathbb{Z}_{\mathrm{n}} = x_{n-1} = y_{n-1}$. $\forall\ n \epsilon \mathbb{Z}$

Then, $x_n = y_n \quad \forall\ n \epsilon \mathbb{Z}$. Thus $x = y$ and hence σ is one-one.

(ii) σ is onto: Let $y = (y_n)_{n \epsilon \mathbb{Z}} \in \Sigma$, put $x = (x_n)_{n \epsilon \mathbb{Z}}$ where $x_n = y_{n-1}$

Then $\sigma(x) = y$ and hence σ is onto.

(iii) Continuity of σ: Let, $x' = (x'_n)_{n\epsilon\mathbb{Z}} \epsilon \Sigma$

By the definition of σ, we have , $\sigma(x') = y' = (y'_n)_{n\epsilon\mathbb{Z}} \epsilon \Sigma$,where $y'_n = x'_{n-1}$.

Now $d\big(\sigma(x),\ \sigma(x')\big) = d(y, y')$

$$= \sum_{n=0}^{\infty} \frac{|y_n - y'_n|}{2^n} + \sum_{n=1}^{\infty} \frac{|y_{-n} - y'_{-n}|}{2^n}$$

$$= \sum_{n=0}^{\infty} \frac{|x_{n-1} - x'_{n-1}|}{2^n} + \sum_{n=1}^{\infty} \frac{|x_{-n-1} - x'_{-n-1}|}{2^n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{|x_{n-1} - x'_{n-1}|}{2^{n-1}} + 2\sum_{n=1}^{\infty} \frac{|x_{-n-1} - x'_{-n-1}|}{2^{n+1}}$$

$$= \frac{1}{2}\frac{|x_{-1} - x'_{-1}|}{2^{-1}} + \frac{1}{2}\sum_{m=0}^{\infty} \frac{|x_m - x'_m|}{2^m} + 2\sum_{l=2}^{\infty} \frac{|x_{-l} - x'_{-l}|}{2^l}$$

$$= \frac{1}{2}\frac{|x_{-1} - x'_{-1}|}{2^{-1}} + 2\sum_{l=2}^{\infty} \frac{|x_{-l} - x'_{-l}|}{2^l} + \frac{1}{2}\sum_{m=0}^{\infty} \frac{|x_m - x'_m|}{2^m}$$

$$= 2\sum_{l=1}^{\infty} \frac{|x_{-l} - x'_{-l}|}{2^l} + \frac{1}{2}\sum_{m=0}^{\infty} \frac{|x_m - x'_m|}{2^m}$$

$$\leq 2\sum_{m=0}^{\infty} \frac{|x_m - x'_m|}{2^m} + 2\sum_{l=1}^{\infty} \frac{|x_{-l} - x'_{-l}|}{2^l}$$

$$= 2d(x, x')$$

Now, for any $\epsilon > 0$ given, put $\delta = \frac{\epsilon}{2}$.

If $d(x, x') < \delta$ then $d\big(\sigma(x), \sigma(x')\big) < 2\delta = 2.\frac{\epsilon}{2} = \epsilon$.

Hence σ is continuous function.

(iv) Continuity of σ^{-1}: Let, $x' = (x'_n)_{n\epsilon\mathbb{Z}} \epsilon \Sigma$ we put $\sigma^{-1}(x) = y$ then $\sigma(y) = x$. Thus by definition of σ, we have $y_n = x_{n+1}$.

Again, put $\sigma^{-1}(x') = y' \Rightarrow \sigma(y') = x'$ and by the definition of σ, we have, $y'_n = x'_{n+1}$.

Now $d\big(\sigma^{-1}(x), \sigma^{-1}(x')\big) = d(y, y')$

$$= \sum_{n=0}^{\infty} \frac{|y_n - y'_n|}{2^n} + \sum_{n=1}^{\infty} \frac{|y_{-n} - y'_{-n}|}{2^n}$$

$$= \sum_{n=0}^{\infty} \frac{|x_{n+1} - x'_{n+1}|}{2^n} + \sum_{n=1}^{\infty} \frac{|x_{-n+1} - x'_{-n+1}|}{2^n}$$

$$= 2\sum_{n=0}^{\infty} \frac{|x_{n+1} - x'_{n+1}|}{2^{n+1}} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{|x_{-n+1} - x'_{-n+1}|}{2^{n-1}}$$

$$= 2\sum_{m=1}^{\infty} \frac{|x_m - x'_m|}{2^m} + \frac{1}{2}\sum_{l=0}^{\infty} \frac{|x_{-l} - x'_{-l}|}{2^l}$$

$$= 2\sum_{m=1}^{\infty} \frac{|x_m - x'_m|}{2^m} + \frac{|x_0 - x'_0|}{2} + \frac{1}{2}\sum_{l=1}^{\infty} \frac{|x_{-l} - x'_{-l}|}{2^l}$$

$$\leq \frac{2|x_0 - x'_0|}{2} + 2\sum_{m=1}^{\infty} \frac{|x_m - x'_m|}{2^m} + \frac{1}{2}\sum_{l=1}^{\infty} \frac{|x_{-l} - x'_{-l}|}{2^l}$$

$$= 2\sum_{m=0}^{\infty} \frac{|x_m - x'_m|}{2^m} + \frac{1}{2}\sum_{l=1}^{\infty} \frac{|x_{-l} - x'_{-l}|}{2^l}$$

$$\leq 2\sum_{m=0}^{\infty} \frac{|x_m - x'_m|}{2^m} + 2\sum_{l=1}^{\infty} \frac{|x_{-l} - x'_{-l}|}{2^l}$$

$$= 2d(x, x')$$

Now for any $\epsilon > 0$ given, we put $\delta = \frac{\epsilon}{2}$.

If $d(x,x') < \delta$ then $d(\sigma^{-1}(x), \sigma^{-1}(x')) < 2\delta = 2.\frac{\epsilon}{2} = \epsilon$.
Therefore σ^{-1} is continuous function.
Hence, we conclude that the shift map $\sigma : \Sigma \to \Sigma$ is homeomorphism.

Theorem 4.5: (Cantor's Intersection Theorem) Let (X,d) be a complete metric space and let (F_n) be a decreasing sequence of nonempty closed subsets of X such that $d(F_n) \to 0\ as\ n \to \infty$ (or diam $(F_n) \to$ *0* as *n→∞.* Then *F=n=1∞Fn* contains exactly one point.

Proof: Since $F_n \neq \emptyset$ for each $n \in \mathbb{N}$, we can choose a sequence of points (x_n) such that $x_n \in F_n$, for $n = 1,2,3, \dots \dots$ we shall show (x_n) is a Cauchy-sequence in X. Now (F_n) is a decreasing sequence that is $F_{n+1} \subset F_n$ for all n, therefore $x_n, x_{n+1} \dots \dots$ all lie in F_n. Moreover $d(F_n) \to 0$ as $n \to \infty$. Therefore given $\in > 0$ there exist a positive integer n_0 such that $d(F_n) < \in\ \forall\ n \geq n_0$. $x_{n_0+1}, x_{n_0}, x_{n_0+2}, \dots \dots$ all lie in F_{n_0}. Thus for positive integer $m,n \geq n_0$, we have $d(x_n, x_m) \leq d(F_{n_0}) < \in$. Therefore, (x_n) is a Cauchy sequence in X. Since (X,d) is complete there exist a point $x \in X$ such that $\lim_{n\to\infty} x_n = x$. If possible $x \notin \bigcap_{n=1}^{\infty} F_n$. Then there exist a positive integer n, such that $x \in F_m$. Since F_m is closed and $\in F_m$, then $d(x_n, F_m) > 0$. Let $d(x_n, F_m) = r > 0$ then $d(x,y) \geq r\ \forall\ y \in F_m$. Thus the open spheres $s\left(x, \frac{r}{2}\right)$ and F_m are closely disjoint and therefore $n > m\ \Rightarrow F_n \subset F_m$ and this implies $x_n \in F_m (\because\ x_n \in F_n) \Rightarrow x_n \notin s\left(x, \frac{r}{2}\right)$. This is possible, since (x_n) converges to x. Hence $x \in \bigcap_{n=1}^{\infty} F_n$.
Now to show that $x \in \bigcap_{n=1}^{\infty} F_n$ is unique. If possible, Let y be another point in $\bigcap_{n=1}^{\infty} F_n$. Then $y \in F_n$, for every $n. \Rightarrow d(x,y) \leq d(F_n)$ for every n (by definition of the diameter). But, since it is given that $d(F_n) \to 0$ as $n \to \infty$. Therefore on taking limit, as $n \to \infty$ $d(x,y) \leq 0$. But $d(x,y) \geq 0$ is always true. Hence $d(x,y) = 0$ and so $x = y$. Thus $\bigcap_{n=1}^{\infty} F_n$ consists of exactly one point.

Problem 4.1: If $a > (2 + \sqrt{5}\,)$, then the set $C = \bigcap_{n=1}^{\infty} C_n$ is a cantor set.

Proof: Here $h(x) = ax(1-x)$, where $a > (2 + \sqrt{5})$. We consider the following Lemma :
Let $a > (2 + \sqrt{5})$ and $h(x) = ax(1-x)$. There is $\lambda > 1$ such that $|h'(x)| > \lambda$ whenever x is in C_1. Further the length of each interval in C_n is less than $\left(\frac{1}{\lambda}\right)^n$. since 0 is a fixed point of h , it is clearly in C. Then C is not empty.
To prove that C is a cantor set we need to show that.

i) C is closed and bounded.
ii) C contains no intervals and
iii) Every point in C is an accumulation point of C .

i)Since C is the intersection of closed sets then it is closed. As C is contained in $[0,1]$. It is also bounded.

ii) If C contains the open interval (x,y) with length $|x-y|$, then for each n , (x,y) must be contained in one of the intervals of C_n. However lemma (i), implies that there is $\lambda > 1$ such that the length of an interval in C_n is less than $\left(\frac{1}{\lambda}\right)^n$, since we can find n_0 such that $|x-y| > \left(\frac{1}{\lambda}\right)^{n_0}$ the interval (x,y) cannot possibly fit into an interval in C_{n_0}. Hence C contains no open intervals.

iii) Suppose x is a point in C and let $N_\delta(x) = (x-\delta, x+\delta)$ be a nbd of x. We must show that $N_\delta(x)$ contains a point in C other than x. If a is an end point of one of the intervals in C_n , then a in C since $h^{n+1}(a) = 0$. Now for each n , x must be contained in one of the intervals of C_n. We let λ be an in lemma (i) and choose n large through so that $\left(\frac{1}{\lambda}\right)^n < \delta$. Then the entire interval of C_n must be in $N_\delta(x)$ since the length of each interval in C_n is less than$\left(\frac{1}{\lambda}\right)^n$. Since both of the end points of the interval one in $N_\delta(x)$ and at least one of them is not x. Then every point in C is an limit point of C .

V. Conclusion

In this paper, we have introduced the concept of shift map and cantor set. Then we have discussed about cantor set and illustrated the construction of this set. After that we have proved some properties of shift map. Also we have proved shift map is chaotic, homeomorphism and also proved cantor's Intersection theorem. Finally, we have solved a problem about cantor set of Logistic Function.

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