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Some Facts about the Beauty of Ramanujan Number 1729

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Abstract: *In this article, some facts about the beauty of Ramanujan number 1729 are mentioned. Ramanujan number 1729 is the smallest positive integer which can be expressed as the sum of cubes of two numbers in two different ways. The Diophantine equation $N = X^3+Y^3= W^3+ Z^3$ for 1729 is solved and the relations among the terms of this equation are derived. Different types of representations of 1729 and magic squares containing 1729 are given in this article. 1729 can be expressed in terms of Fibonacci numbers, but it is not a Fibonacci number. 1729 is a Harshad number.*

Key words: *Ramanujan number, Diophantine equation, Fibonacci number, Harshad number, Magic square.*

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I. Introduction

Srinivasa Ramanujan (22 December 1887 - 26 April 1920) was an Indian mathematician who lived during the British rule in India. The number 1729 is the number of a taxi which the British mathematician Prof.G.H. Hardy had hired when he came to see ailing Ramanujan in hospital. When Prof. Hardy remarked to Ramanujan that the taxi had a dull number 1729, Ramanujan immediately responded that it is a very interesting smallest number which can be expressed as the sum of cubes of two numbers in two different ways. Since then 1729 is called Ramanujan number. The number 1729 is also known as Hardy – Ramanujan number.

$$1729 = 12^3 + 1^3 = 10^3 + 9^3 \tag{1}$$

$$12^3 + 1^3 = 1728 + 1 = 1729$$

$$10^3 + 9^3 = 1000 + 729 = 1729$$

Ramanujan knew the following formula for the sum of two cubes expressed in two different ways giving 1729.

$$(x^2 + 9xy - y^2)^3 + (12x^2 - 4xy + 2y^2)^3 = (9x^2 - 7xy - y^2)^3 + (10x^2 + 2y^2)^3$$

$$for\ x = 1\ and\ y = 1 \tag{2}$$

For $x = 1$ and $y = 1$, LHS of (2) = $9^3 + 10^3 = 1729$ and RHS of (2) = $1^3 + 12^3 = 1729$

On page 82 of Ramanujan's Note Book, the following relation was mentioned regarding 1729.

$$\alpha_n^3 + \beta_n^3 = \gamma_n^3 + (-1)^n \tag{3}$$

Where, $\alpha_0 = 9$, $\beta_0 = -12$ & $\gamma_0 = -10$. Putting $n = 0$ in (3), we obtain 1729.

$$\alpha_0^3 + \beta_0^3 = \gamma_0^3 - 1 \text{ or } 9^3 - 12^3 = -10^3 - 1 \text{ or } 10^3 + 9^3 = 12^3 + 1^3 = 1729$$

II. Solution of Diophantine Equation for Ramanujan Number 1729

Since Ramanjan number 1729 is equal to the sum of the cubes of two numbers expressed in two different ways, it can be written as

$$N = X^3 + Y^3 = W^3 + Z^3 \tag{4}$$

Where, $N = 1729$.

The above equation (4) is known as Diophantine equation. Algebraic equations requiring solutions in integers are called Diophantine equations. They are named after the mathematician of Alexandria, Diophantus.

$$\text{Let, } N = pqr = \alpha^3 + \beta^3 \tag{5}$$

$$\Rightarrow pqr = (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta) \tag{6}$$

Comparing both sides of (6), we can write

$$q = (\alpha + \beta) \tag{7}$$

$$pr = (\alpha^2 + \beta^2 - \alpha\beta) \tag{8}$$

Using the value of β from (7) in (8), we have

$$pr = \alpha^2 + (q - \alpha)^2 - \alpha(q - \alpha)$$

$$\Rightarrow 3\alpha^2 - 3q\alpha + (q^2 - pr) = 0 \tag{9}$$

The above equation (9) is a quadratic equation in α and its solutions are given by

$$\alpha = \frac{3q \pm \sqrt{9q^2 - 12(q^2 - pr)}}{6} = \frac{3q \pm \sqrt{12pr - 3q^2}}{6} \tag{10}$$

Using the value of α from (7) in (8), we get

$$pr = (q-\beta)^2 + \beta^2 - (q-\beta)\beta$$

$$\Rightarrow 3\beta^2 - 3q\beta + (q^2 - pr) = 0 \qquad (11)$$

The above equation (11) is a quadratic equation in β and its solutions are given by

$$\beta = \frac{3q \pm \sqrt{9q^2 - 12(q^2 - pr)}}{6} = \frac{3q \pm \sqrt{12pr - 3q^2}}{6} \qquad (12)$$

Since $N = 1729 = 7 \times 13 \times 19 = pqr$ (using 5), we have $p = 7, q = 13$ & $r = 19$. Substituting these values in (10), we get

$$\alpha = \frac{3\times 13 \pm \sqrt{12\times 7\times 19 - 3\times (13)^2}}{6} = \frac{39 \pm 33}{6} = 12\ \&1 \qquad (13)$$

These are values of X & Y in (4). That is, $X = 12$ and $Y = 1$.
As $N = 1729 = 7 \times 19 \times 13 = pqr$, we get $p = 7, q = 19$ & $r = 13$. Substituting these values in (12), we get

$$\beta = \frac{3\times 19 \pm \sqrt{12\times 7\times 13 - 3\times (19)^2}}{6} = \frac{57 \pm 3}{6} = 10\ \&\ 9 \qquad (14)$$

These are values of W& Z in (4). That is, $W = 10$ and $Z = 9$. Thus, (13) and (14) are the solutions of Diophantine equation for Ramanujan number 1729.

III. Relations Among the Terms of Diophantine Equation for 1729

The Diophantine equation is

$$N = X^3 + Y^3 = W^3 + Z^3 \qquad (15)$$

Comparing (15) with (1), we get $X = 12, Y = 1, W = 10$ & $Z = 9$ for the Ramanujan number $N = 1729$..
The Eqn.(15) can be written as

$$Z^3 - Y^3 = X^3 - W^3 \qquad (16)$$

Consider the identity

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2) \qquad (17)$$

Using (17) in (16), we have

$$(Z-Y)(Z^2 + ZY + Y^2) = (X-W)(X^2 + XW + W^2) \qquad (18)$$

The above equation is satisfied if

$$(Z-Y) = V(X-W) \qquad (19)$$

and $$(Z^2 + ZY + Y^2) = \frac{(X^2 + XW + W^2)}{V} \qquad (20)$$

The value of V for the number 1729 can be calculated by substituting the values of X, Y, Z & W in (19).

$$V = \frac{Z-Y}{X-W} = \frac{9-1}{12-10} = 4$$

Substituting $V = 4$ in (19) and (20), we obtain

$$(Z-Y) = 4(X-W) \qquad (21)$$

$$4(Z^2 + ZY + Y^2) = (X^2 + XW + W^2) \qquad (22)$$

Modifying LHS, the above Eqn.(22) can be written as

$$3(Z+Y)^2 + (Z-Y)^2 = (X^2 + XW + W^2) \qquad (23)$$

$$\text{Let,} \quad P = Z + Y \qquad (24)$$

Substituting (21) & (24) in LHS of (23), we get

$$3P^2 + 16(X^2 + W^2 - 2XW) = (X^2 + XW + W^2)$$

$$\Rightarrow 3(P^2 + 5X^2 + 5W^2 - 11XW) = 0$$

$$\Rightarrow (P^2 + 5X^2 + 5W^2 - 11XW) = 0 \qquad (25)$$

Adding (21) and (24), we have

$$(Z-Y) + (Z+Y) = 4(X-W) + P$$

$$\Rightarrow Z = \frac{P + 4(X-W)}{2} \qquad (26)$$

Subtracting (21) from (24), we get

$$\Rightarrow (Z+Y) - (Z-Y) = P - 4(X-W)$$

$$\Rightarrow Y = \frac{P - 4(X-W)}{2} \qquad (27)$$

$Z = 9$ & $Y = 1$ for Ramanujan number 1729. Using these values in (24), we obtain

$$P = 9 + 1 = 10$$

The Eqn. (25) is a quadratic equation in the variables P, X & W with solutions $P = 10, X = 12$ & $W = 10$. The Eqns. (24), (25), (26) & (27) are the relations among the terms X, Y, W & Z of Diophantine equation (15) for Ramanujan number 1729.

IV. Ramanujan -Type Numbers

The positive integers (except 1729) which can be expressed as the sum of the cubes of two numbers in two different ways are called Ramanujan-type numbers. The following are few examples.

$$\begin{aligned}
4104 &= 16^3 + 2^3 = 15^3 + 9^3 \\
&= 4096 + 8 = 3375 + 729 \\
13832 &= 24^3 + 2^3 = 20^3 + 18^3 \\
&= 13824 + 8 = 8000 + 5832 \\
20683 &= 27^3 + 10^3 = 24^3 + 19^3 \\
&= 19683 + 1000 = 13824 + 6859 \\
32832 &= 32^3 + 4^3 = 30^3 + 18^3 \\
&= 32768 + 64 = 27000 + 5832 \\
4673088 &= 164^3 + 64^3 = 167^3 + 25^3 \\
&= 4410944 + 262144 = 4657463 + 15625 \\
9261729 &= 210^3 + 9^3 = 172^3 + 161^3 \\
&= 9261000 + 729 = 5088448 + 4173281
\end{aligned}$$

V. Integers in Terms of the Digits of 1729

Any integer can be expressed in terms of the four digits 1, 7, 2 and 9 of 1729. The following are few examples.

0 = 1(7 + 2- 9)
1 = 1+ 7 +2 – 9
$2 = 1 + \{(7+2) \div 9\}$
3 = - 1 - 7 +2 +9
4 = -1×7 +2 + 9
5 = 1 – 7 +2 + 9
6 = 17 – 2 – 9
7 = (1 + 7)2 – 9
$8 = 1^7 \times 2^{\sqrt{9}}$
$9 = 1^7 + 2^{\sqrt{9}}$
10 = 17 + 2 – 9

$20 = 1 \times 7 \times 2 + (\sqrt{9})!$
$30 = 1^7 + 29$
$32 = 1 + 7 + [-2 + (\sqrt{9})!]!$
$40 = 1 \times 7^2 - 9$
$42 = -1 + 7^2 - (\sqrt{9})!$
$50 = 1 + 7(-2 + 9)$
$60 = (1 + 7 + 2) \times (\sqrt{9})!$
$70 = 1 + 72 - \sqrt{9}$
$80 = -1 + 72 + 9$
$90 = (1 + 7 + 2) \times 9$
$121 = 1 + (7 \times 2 - 9)!$

The negative integer can be expressed in terms of digits 1, 7, 2 & 9 by putting minus sign (-) before the representation for the corresponding positive integer.

VI. Representations of 1729

(a) 1729 is the smallest number which can be represented in quadratic form $(a^2 + ab + b^2)$in four different ways with a and b as positive integers. The integer pairs (a, b) are (25, 23), (32,15), (37,8) and (40,3).

$$\begin{aligned}
1729 &= 25^2 + 25 \times 23 + 23^2 = 625 + 575 + 529 \\
&= 32^2 + 32 \times 15 + 15^2 = 1024 + 480 + 225 \\
&= 37^2 + 37 \times 8 + 8^2 = 1369 + 296 + 64 \\
&= 40^2 + 40 \times 3 + 3^2 = 1600 + 120 + 9
\end{aligned}$$

(b) The number 1729 can be expressed in the form $(a^2 - b^2)$. The following are few examples.

$$\begin{aligned}
1\,729 &= 55^2 - 36^2 = 3025 - 1296 \\
&= 73^2 - 60^2 = 5329 - 3600 \\
&= 127^2 - 120^2 = 16129 - 14400 \\
&= 865^2 - 864^2 = 748225 - 746496
\end{aligned}$$

(c) 1729 can be represented in terms of each digit from 1 to 9 as given below.

$$\begin{aligned}
1729 &= (11 + 1)^{1+1+1} + 1 \\
&= (2/2 + 2) \times (22 + 2)^2 + 2/2 \\
&= (3 \times 3 + 3)^3 + 3/3 \\
&= 4 \times (4 \times 44 + 4^4) + 4/4 \\
&= 55 \times (5 \times 5 - 5) + (5^5 - 5)/5 + 5 \\
&= 6 \times 6 \times (6 \times 6 + 6 + 6) + 6/6 \\
&= 7 \times 7 \times (7 \times 7 - 7 - 7) + 7 + 7 \\
&= 8 \times \{8 \times (8 + 8) + 88\} + 8/8 \\
&= 9 \times 9 \times 9 + 999 + 9/9
\end{aligned}$$

(d) 1729 can be expressed in terms of the numbers 1 to 10 either in increasing or decreasing order.

$$\begin{aligned}
1729 &= 1 + 2^3 + [-4 + 56 + \{(7 + 8)/\sqrt{9}\}!] \times 10 \\
&= 10 \times (98 + 7 + 65 + \sqrt{4}) + 3^2 \times 1
\end{aligned}$$

(e) 1729 can be written in terms of the digits 1 to 5, 1 to 6, 1 to 7, 1 to 8 and 1 to 9 either in increasing or decreasing order.

$$\begin{aligned} 1729 &= 12^3 - 4 + 5 && = 54 \times 32 + 1 \\ &= 12^3 + (-4+5)^6 && = 6 \times (5+4) \times 32 + 1 \\ &= 123 \times (4 \times 5 - 6) + 7 && = (7-6) \times (54 \times 32 + 1) \\ &= -1 + (2 + 34 + 5) \times 6 \times 7 + 8 && = 8 - 7 + 6 \times (5+4) \times 32 \times 1 \\ &= 12 - 3 + 4^5 - 6 + 78 \times 9 && = (98 - 7) \times (6 \times 5 - 4 \times 3 + 2 - 1) \end{aligned}$$

(f) 1729 can be written with any of the digits 4, 5, 6, 7, 8 and 9 at the beginning and 0 at the end.

$$\begin{aligned} 1729 &= (4 \times 3)^{2+1} + 0! \\ &= 54 \times 32 + 1 \times 0! \\ &= 6! - 5 - 4 - 3! + 2^{10} \\ &= (7+6) \times (-5 - 4! \times 3 + 210) \\ &= 8 - 7 + (6+5) \times 4^3 + 2^{10} \\ &= 9 + 8 \times (7 \times 6 \times 5 - 4 - 3 + 2 + 10) \end{aligned}$$

(g) 1729 can be represented in terms of digits 1 to 9 or 1 to 6 as power series.

$$\begin{aligned} 1729 &= 1^5 + 2^8 + 3^9 + 4^4 + 5^1 - 6^7 + 7^2 + 8^6 - 9^3 \\ &= 1^3 + 2^6 + 3^2 + 4^5 + 5^4 + 6^1 \end{aligned}$$

(h) 1729 can be written as a sum with same power 2 or 3.

$$\begin{aligned} 1729 &= 6^2 + 18^2 + 37^2 && 1729 = 1^3 + 12^3 \\ &= 8^2 + 12^2 + 39^2 && \quad = 9^3 + 10^3 \\ &= 8^2 + 24^2 + 33^2 && \quad = 1^3 + 6^3 + 8^3 + 10^3 \\ &= 10^2 + 27^2 + 30^2 && \quad = 1^3 + 3^3 + 4^3 + 5^3 + 8^3 + 10^3 \\ &= 12^2 + 17^2 + 36^2 \\ &= 18^2 + 26^2 + 27^2 \end{aligned}$$

(i) 1729 can be represented as product decomposition with power 3.

$$\begin{aligned} 1729 &= (6^3 - 5^3) \times (3^3 - 2^3) \\ &= (4^3 + 3^3) \times (3^3 - 2^3) \end{aligned}$$

(j) 1729 can be written in terms of the digits of date of birth of Ramanujan 22-12-1887.

$$1729 = (-2 + 21) \times (-2 - 1 + 8 + 8) \times 7$$

(k) 1729 can be expressed in terms of the digits of date of death of Ramanujan 26-04-1920.

$$1729 = \{-(2 + 60) \times 4 + 1\} \times (-9 + 2 + 0)$$

VII. Magic Squares Containing 1729

As a young student, Srinivasa Ramanujan was interested in magic squares. In a magic square, each row sum = each column sum = each diagonal sum. The 4 × 4 Magic Square-1 contains Ramanujan's date of birth 22-12-1887 in the first row and Ramanujan number 1729 in the second row. The sum of numbers in each row or column or along the diagonal in this magic square is 139, which is a prime number.

In 4 × 4 Magic Square-2, Ramanujan's date of birth 22-12-1887 and date of death 26-04-1920 are included in the first row. The number 1729 is present in the second row. In this magic square, each row sum = each column sum = each diagonal sum = 8623, which is a prime number.

22	12	18	87
88	17	29	5
20	14	79	26
9	96	13	21

Magic square -1

2212	1887	2604	1920
1914	2610	1729	2370
1918	2395	1889	2421
2579	1731	2401	1912

Magic Square - 2

VIII. Few Other Facts on 1729

- There are 8 divisors of 1729. The divisors are 1,7,13,19,91,133,247 and1729.
- 1729 is not a perfect number. Perfect number is a positive integer that is equal to the sum of its proper divisors, excluding the number itself. Since the sum of the proper divisors of 1729 is 511 (=1+7+13 +19 + 91+133 +247), it is not a perfect number.
- 1729 is not a Fibonacci number. The sequence of numbers introduced by famous Italian mathematician Leonardo Pisano (1170 -1250) is known as Fibonacci sequence. Fibonacci was his nick name. The Fibonacci sequence is given by

1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,987,1597,2584,..........

Each term in the Fibonacci sequence after the first two is the sum of the immediately preceding two terms. The 17^{th} and 18^{th} Fibonacci numbers are 1597 and 2584 respectively. So, 1729 is not a Fibonacci number. The n^{th} term in Fibonacci sequence, denoted by F_n, is called n^{th} Fibonacci number. 1729 is equal to the sum of 5 Fibonacci numbers as given below.

$$1729 = F_2 + F_6 + F_9 + F_{11} + F_{17} = 1 + 8 + 34 + 89 + 1597$$

1729 can be also expressed in terms of Fibonacci numbers as mentioned below.

$$1729 = 1 + (F_7 - F_2)^{\sqrt{9}} = 1 + (13 - 1)^3 = 1 + 12^3$$

- 1729 is a Harshad number. Harshad numbers were defined by D.R. Kaprekar(1905 – 1986). 'Harshad' means joy giving. A Harshad number is an integer that is divisible by the sum of its digits. 1729 is divisible by 1+ 7 +2 + 9 = 19 $(1729 \div 19 = 91)$. So, the Ramanujan number 1729 is a Harshad number. Few other examples of Harshad numbers are 21,24, 153,378, 2620, 2924, 6804, etc.
- 1729 is not a prime number.
- The prime numbers obtained with the digits of 1729 are
  17,29,71,127,179,197,271,719,971,2917,7219 and 9721
- 1729 is one of the four positive integers (others being 81, 1458 and the trivial case 1) which, when its digits are added together, produces a sum which, when multiplied by its reversal, gives the original number.

  1+7+2+9 =19; 19 × 91 =1729
- In octal number system, 1729 is written as $(3301)_8$. Converting $(3301)_8$ into decimal number system, we get 1729 as given below.

$$(3301)_8 = 8^3 \times 3 + 8^2 \times 3 + 8^1 \times 0 + 8^0 \times 1$$
$$= 1536 + 192 + 0 + 1 = (1729)_{10}$$

- In hexadecimal number system, 1729 is denoted as $(6C1)_{16}$. In hexadecimal number system C represents 12. The decimal equivalent of $(6C1)_{16}$ is 1729 as shown below.

$$(6C1)_{16} = 16^2 \times 6 + 16^1 \times 12 + 16^0 \times 1$$
$$= 1536 + 192 + 1 = (1729)_{10}$$

- 1729 is a Carmichael number. The Carmichael number is of the form

$$(6n + 1)(12n + 1)(18n + 1); n \geq 1$$

For $n = 1$, the Carmichael number is 7 × 13 × 19 = 1729.

IX. Conclusion

The Diophantine equation for Ramanujan number 1729 was solved and the relations among the terms of this equation were derived. Different representations of 1729 and magic squares containing this number were given. The relations of 1729 with other number systems were mentioned.

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