A Cooperative Game Theory Application and Economical Behavior in an Aspect of Bangladesh with a Real- Life Problem.

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Abstract: Cooperative game theory made a great impression on several aspects of life sciences and the impression of scientific theory had started fifty years ago. This research paper illustrates a top-level view of the implementation of skills gained by students in Mathematics for Business and Economics to unravel problems of the game theory. Economically speaking, people have choices, demands, wishes, and desires. While it's expected that individuals will act in a very rational way to comply with social norms, the method is complicated thanks to external factors like prices, production, gains, and expenses. The act of constructing a rational decision regarding choices involves what's said as a social exchange economy.
Key Word: Game Theory, Cooperative Game, Strategy, Player, Zero Sum Game, Penalty, Equilibrium, Payoff Matrix, Probability.

I. Introduction

Game theory is an autonomous discipline that is used in applied mathematics, social sciences, most remarkably in mathematics, economics, as well as in computer science, biology, engineering, international relations, philosophy, and political science. It was displayed in economics and mathematics to measure economic behaviors including behaviors of firms, markets, and consumers. The cooperative game theory can be contemplated as a modeling procedure that is used to analyze and explain the actions of all players joined in competitive situations and to compare and determine the relative optimality of distinct strategies. The first inspect of games in terms of economics was by Cournot in 1838 on pricing and production but John von Neumann is considered as the founder of the modem game theory. In cooperative game theory, it can be said that the set of bounded computational capacity of equilibrium payoffs carries only one valuation, that the valuation of the game with penalty approaches the valuation of the one-shot game as the penalty goes to zero. The goal is to anticipate moves to ready, which is able to cause ultimate victory.

II. Mathematical Formulation

Let us consider two players' X and Y (two-person zero-sum game). Each chooses one of the possible strategies $X_i(i = 1, m)$ and $Y_j(j = 1, n)$.

Let $\Psi_1(X_i; Y_j)(i = 1, m; j = 1, n)$ when Player X wins; and $\Psi_2(X_i; Y_j)(i = 1, m; j = 1, n)$ when Player Y wins.

Since we have a zero-sum game, $\Psi_1(X_i; Y_j) + \Psi_2(X_i; Y_j) = 0$.

We have $\Psi_1(X_i; Y_j) = \Psi(X_i; Y_j)$ and $\Psi_2(X_i; Y_j) = -\Psi(X_i; Y_j)$

Thus, the goal of Player X is to maximize the value $\Psi(X_i; Y_j)$ and Player Y minimize it. Suppose that $\Psi(X_i; Y_j) = a_{ij}$, we have the matrix X:

$$X = \begin{pmatrix} a_{11} & a_{12} & \ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots\ldots & a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & \\ a_{m1} & a_{m2} & \ldots\ldots\ldots & a_{mn} \end{pmatrix} \ldots\ldots\ldots\ldots\ldots (1)$$

where rows correspond to the strategies X_i and columns to the strategies Y_j. The matrix X is called the payoff matrix. An element of this matrix a_{ij} is when Player X wins if he/she had chosen X_i strategy, and Player Y had chosen the Y_j strategy. Player X has to choose the strategy to maximize his/her minimum payoffs, such as:

$$A = \begin{smallmatrix} max & min \\ i & j \end{smallmatrix} a_{ij} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The amount of the loss is called the upper value of the game:

$$B = \begin{smallmatrix} min & max \\ j & i \end{smallmatrix} a_{ij} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

If $\begin{smallmatrix} max \\ i \end{smallmatrix} \begin{smallmatrix} min \\ j \end{smallmatrix} a_{ij} = \begin{smallmatrix} min \\ j \end{smallmatrix} \begin{smallmatrix} max \\ i \end{smallmatrix} a_{ij} = v$, the game is called well-defined. But if $A \neq B, A \leq v \leq B$, its maximin – minimax strategies are not optimal. This means that each party can improve its results by choosing a different approach. The optimal solution for such a game is mixed strategies, which are the specific combinations of the original "pure" strategies.

For Player X vector $= (p_1, p_2, p_3, \dots \dots, p_m)$, where $\sum_{i=1}^{m} p_i = 1$

For Player Y vector $= (q_1, q_2, q_3, \dots \dots, q_n)$, where $\sum_{j=1}^{n} q_j = 1$

$$p_i \geq 0 (i = 1, m); q_j \geq 0 \ (j = 1, n)$$

It appears that when the mixed strategies are used, for every finite game one can find a couple of stable optimal strategies. The main theorem of game theory determines the existence of a solution. The main theorem of game theory says that every finite game has at least one solution which is possible in mixed strategies.

Suppose we have a finite matrix game with the payoff matrix (1). According to the theorem, optimal mixed strategies for players are determined by vectors

$P^* = (p_1{}^*, p_2{}^*, p_3{}^*, \dots \dots, p_m{}^*)$ and $Q^* = (q_1{}^*, q_2{}^*, q_3{}^*, \dots \dots, q_m{}^*)$, which give the possibility to receive the win: $A \leq v \leq B$.

Using the optimal mixed strategy should provide Player X with a certain payoff that is not less than the value of the game. Mathematically, this condition is written as:

$$\sum_{i=1}^{m} a_{ij} p_i{}^* \geq v(j = 1, n) \ \dots \dots \dots \dots \dots (4)$$

On the other hand, the use of an optimal mixed strategy for Player Y should provide, for any strategy that Player X chooses, the loss that is not exceeding the value of the game, that is:

$$\sum_{i=1}^{m} a_{ij} q_j{}^* \leq v(j = 1, n) \ \dots \dots \dots \dots \dots \dots (5)$$

These relationships are used to find the solution of the game. We need to find the mixed strategies and the value of the game. Denote the desired probabilities value for the Player X in the "pure" strategies with $P^* = (p_1{}^*, p_2{}^*)$ and for player Y with $Q^* = (q_1{}^*, q_2{}^*)$.

According to the main game theory theorem, if the player sticks to its optimal strategy, the payoff will be equal to the value of the game.

So if Player X sticks to his/her optimal strategy $P^* = (p_1{}^*, p_2{}^*)$, then:

$$\begin{cases} a_{11}p_1{}^* + a_{21}p_2{}^* = v \\ a_{12}p_1{}^* + a_{22}p_2{}^* = v \end{cases} \ \dots \dots \dots \dots \dots \dots (6)$$

Since $p_1{}^* + p_2{}^* = 1$, then $p_2{}^* = 1 - p_1{}^*$. Substituting this expression into the system (6), we obtain:

$$\begin{cases} a_{11}p_1{}^* + a_{21}(1 - p_1{}^*) = v \\ a_{12}p_1{}^* + a_{22}(1 - p_1{}^*) = v \end{cases}$$

Or, $a_{11}p_1{}^* + a_{21}(1 - p_1{}^*) = a_{12}p_1{}^* + a_{22}(1 - p_1{}^*)$

Solving this equation, we have,

$$p_1{}^* = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \ \dots \dots \dots \dots \dots (7)$$

And,

$$p_2{}^* = 1 - p_1{}^* = 1 - \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} \ \dots \dots \dots \dots \dots (8)$$

Player Y sticks to his/her optimal strategy $Q^* = (q_1{}^*, q_2{}^*)$ then:

$$\begin{cases} a_{11}q_1{}^* + a_{12}q_2{}^* = v \\ a_{21}q_1{}^* + a_{22}q_2{}^* = v \end{cases} \ \dots \dots \dots \dots \dots . (9)$$

Since $q_1{}^* + q_2{}^* = 1$, then $q_2{}^* = 1 - q_1{}^*$. Substituting this expression into the system (9), we obtain:

$$\begin{cases} a_{11}q_1{}^* + a_{12}(1 - q_1{}^*) = v \\ a_{21}q_1{}^* + a_{22}(1 - q_1{}^*) = v \end{cases}$$

Or, $a_{11}q_1{}^* + a_{12}(1 - q_1{}^*) = a_{21}q_1{}^* + a_{22}(1 - q_1{}^*)$

Solving this equation, we have,

$$q_1{}^* = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} \ \dots \dots \dots \dots \dots (10)$$

And,

$$q_2{}^* = 1 - q_1{}^* = 1 - \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} = \frac{a_{11} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \ \dots \dots \dots (11)$$

The value of the game $\boldsymbol{v}$ could be found by put the values $p_1{}^*, p_2{}^* \ or \ (q_1{}^*, q_2{}^*)$ to any of the equations (6) or (9):

$$v = \frac{a_{22}a_{11} - a_{12}a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \ \dots \dots \dots \dots \dots (12)$$

III. Application of the Model

The application of the above model to the real situation we will derive pure and mixed strategies Nash Equilibrium with the corresponding geometric representation, and finally, the recommendations according to the optimal strategy will be made. To find the optimal strategy for the company, the first step would be to build the payoff matrix (1). In this part of the paper we consider the zero-sum game – a game in which one player's gains result only from another player's equivalent losses. We will consider this model from the point of view of the company AKS. Let us assume that some Bangladeshi steel company invests in two new projects and the company is organizing a tender to find the contractors for the engineering and construction. The first project (X_1) is the engineering and erection of a new rolling mill in the central region of Bangladesh. The second project (X_2) is an upgrade and adjustment of the existing rolling mill in the eastern region of Bangladesh.

The first project is more expensive and, of course, is more valuable to the companies participating in the tender. Three engineering companies which are represented in this segment and interested in the Bangladeshi market are participating in the tender. Let us assume that these companies are: AKS, RSRM and GPH Ispat. RSRM is the clear favorite in getting a larger project in the Bangladeshi market as its market share is much larger than that of AKS or GPH Ispat. In the proposed model we consider two possible scenarios: when RSRM and AKS enter into a strategic interaction and are competing for the projects – scenario (Y_1); and the scenario (Y_2), when now GPH Ispat and AKS are competing for the projects. In the payoff matrix (1), X_1 and X_2 are projects 1 and 2 (Table 1). The expected return which the companies could get from project X_1 is 1,00,000 BDT, the expected return for the second project X_2 is approximately 70,000 BDT. Strategy Y_1 is a situation in which AKS and RSRM are vying on the market, while strategy Y_1 is a situation in which AKS and GPH Ispat are competing for the projects. First, let us consider situation Y_1. In situation X_1Y_1, when there are RSRM and AKS on the market, the probability of obtaining an order for constructing a plant for RSRM is much higher than that of AKS. This is due to the fact that RSRM is better known and it has extensive experience in the construction of plants from scratch. The probability of receiving an order for RSRM will be 80% and 20% for AKS. For the X_2Y_1 situation, AKS has a slightly higher chance of getting this project because AKS has significant experience in reconstructing the factories and setting up equipment and has firmly established itself in this area. In this case the probability of receiving an order for RSRM will be 55% and for AKS 45%. This means that in the cell X_1Y_1 in the payoff matrix (1) we write: $1{,}00{,}000\ BDT \times (0.2) = 20{,}000\ BDT$. And in the cell X_2Y_1: $70{,}000\ BDT \times (0.45) = 31{,}500\ BDT$. The values in cells are the expected payoffs for AKS and the expected losses for RSRM at the same time.

Now let us consider situation Y_2. In this situation AKS and GPH Ispat are vying in the market. These two firms are relatively new to the Bangladeshi market, and therefore the probability of receiving an order for constructing a new plant for both firms will be the same, at 50%. However, in the situation X_2Y_2, GPH Ispat has a slight advantage as it has more experience in adapting in the Eastern region of Bangladesh, so the probability of receiving an order for AKS is 40% and for GPH Ispat 60%.

Table-1: The playoff matrix of the game

i	X_i	Y_j	
		$Y_1\left(\frac{\text{AKS}}{\text{RSRM}}\right)$	$Y_2\left(\frac{\text{AKS}}{\text{GPH Ispat}}\right)$
1	$X_1(100000\ BDT)$	0.2658	0.0748
2	$X_2(70000\ BDT)$	0.4986	0.0636

This means that in the cell X_1Y_2 in the payoff matrix (1) we write:

$$100\,000\ BDT \times (0.5) = 50\,000\ BDT$$

And in the cell X_2Y_2:

$$70\,000\ BDT \times (0.4) = 28\,000\ BDT.$$

Thus finally in Table 1 we obtain the following payoff matrix (1).
Make sure that the game does not have a saddle point taking into account $(2) - (3)$:

$$max\{min(20000; 50000);\ min(31500; 28000)\} = max\{20000; 28000\} = 28000 = A,$$
$$min\{max(20000; 31500);\ max(50000; 28000)\} = min\{31500; 50000\} = 315000 = B.$$

If $A \neq B$, this means that the game does not have a saddle point. Taking into account $(7) - (11)$ we obtain:

$$p_1{}^* = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} = \frac{28000 - 31500}{20000 + 28000 - 50000 - 31500} = 0.104$$

And,

$$p_2{}^* = 1 - 0.104 = 0.896;$$

$$q_1{}^* = \frac{a_{22-}a_{12}}{a_{11} + a_{22-} a_{12} - a_{21}} = \frac{28000 - 50000}{20000 + 28000 - 50000 - 31500} = 0.65$$

And,

$$q_2{}^* = 1 - 0.65 = 0.35;$$

The value of the game is from (12) is:

$$v = \frac{a_{22}a_{11} - a_{12}a_{21}}{a_{11} + a_{22-} a_{12} - a_{21}} = \frac{28000 \times 20000 - 50000 \times 31500}{20000 + 28000 - 50000 - 31500} = 30298.5$$

IV. Discussion

The solution of the game means that AKS should mix its pure strategies, which consist of choosing the X_1 project jointly with RSRM with the probability 0.104 and the X_2 project jointly with RSRM with the probability 0.896. Project X_1 together with GPH Ispat, AKS should choose with probability 0.65 and project X_2 with GPH Ispat, AKS chooses with the probability 0.35. Under these conditions, the expected gain will be equal to the value of the game which is 30,298.5 BDT.

This means that if AKS has to choose which strategy to follow, it would give preference to the strategies with high probabilities. That is, X_2 together with RSRM, and X_1 together with GPH Ispat. These results are summarized in Table 2. Note on the *x*-axis the segment with the length equal to one (Figure).

The left part of the segment corresponds to strategy X_1 and the right end corresponds to strategy X_2. All the intermediate points of the segment correspond to the mixed strategies of AKS. The probability $p_1 = 0.104$ of strategy X_1 will be equal to the distance from point *P* to the right end of the segment, and the probability $p_2 = 0.896$ of X_2 strategy is the distance to the left end of the segment. Let us draw two perpendiculars to the *x*-axis through the points X_1 and X_2 axis I and axis II. The first will correspond to the gains that a firm could get choosing strategy X_1 and the second to strategy X_2.

Table-2: Mixed strategy distribution for AKS

i	Competing firm	Projects	
		X_1	X_2
1	RSRM	$p_1{}^* = 0.104$	$p_2{}^* = 0.896$
2	GPH Ispat	$q_1{}^* = 0.650$	$q_2{}^* = 0.350$

One can also give a clear geometric interpretation for the solution of the 2×2 game. Following the instructions above one can get Figure:
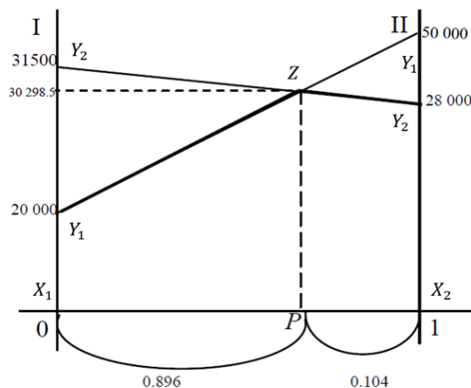
Let us draw the lines which correspond to strategy Y_1. The length of the segment X_1Y_1 equals $a_{11} = 20,000$, while the length of the segment X_2Y_1 equals $a_{12} = 50,000$. Similarly, draw the lines which correspond to strategy Y_2.

We need to find the optimal strategy *P**, for which the minimum payoff of AKS will be maximized. To do this, let us draw a bold line (Figure) which corresponds to the lower bound of winning if strategies Y_2 and Y_1 are selected, i.e. the kinked line Y_1ZY_2. This line includes all the minimum gains of AKS at any of its mixed

strategies. It is obvious that the best possible minimum value in this example is at point Z, and generally corresponds to the point where the curve that indicates the minimum payoff of AKS gets its maximum value. The ordinate of this point is the value of the game and equals $v = 30,298.5$. The distance from the left end of the segment p_2 and the distance to the right end of thesegment p_1 respectively, are equal to the probabilities of strategies X_2 and X_1. As a result, it should be said that after considering the theoretic entry situation we came to the conclusion that for AKS the optimal solution would be to mix its pure strategies, i.e. while competing with RSRM, the choose X_1 project with the probability 0.104 and the X_2 project with theprobability 0.896. Project X_1 together with GPH Ispat, AKS should choose with probability 0.65 and project X_2 with GPH Ispat, AKS chooses with the probability 0.35. Under these conditions, the expected gain will be equal to the value of the game which is 30,298.50 BDT. This means that if AKS has to choose which strategy to follow, it will give preference to the strategies with high probabilities that is X_2 together with RSRM, and X_1 together with GPH Ispat.

V. Conclusion

This paper gives a mild and gentle intro to cooperative game theory in the economy. Cooperative game theory could be a structural, instead of procedural theory. It put up unsatisfying for business strategy whereas the business strategy is usually distressed with what a firm does. Elementary terms and components of game theory and also the most significant solution concepts are induced with some sample implementations. One can see that game-theoretic models are often accustomed to the analysis of many interesting economical phenomena, including legislative bargaining in the economy, conclusions to go to analysis about international business, and formation of coalition governments in comparative economy.

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