IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 16, Issue 3 Ser. II (May – June 2020), PP 19-24
www.iosrjournals.org

On $\mathcal{F}_{[\lambda,\mu]}$ −Regular Four-Dimensional Matrices for $[\lambda, \mu]$-Almost Convergence of Double Sequences

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Abstract:
Background: *A double sequence $A = (a_{ijkl})$ is said to belong to the class (X, Y), where X and Y are two sequence spaces, if any sequence $x = \{x_{mn}\}$ in X is transformed to a sequence $y = \{y_{mn}\}$ in Y by the matrix transformation $y_{mn} = \sum_{k=0}^{\infty}\sum_{j=o}^{\infty} a_{mnjk}\ x_{jk}$ such that the sequence $\{y_{mn}\}$exists and converges in the Pringsheim sense.A t sequence $x = \{x_{mn}\}$ of real is said to be $[\lambda,\mu]$-almost convergent (briefly, $\mathcal{F}_{[\lambda,\mu]} - convergent$) to some number l if $x \in \mathcal{F}_{[\lambda,\mu]}$, where*

$$\mathcal{F}_{[\lambda,\mu]} = \{\ x = \{x_{mn}\}: p - lim_{ij\to\infty}\, \Omega_{i,j,s,t}\,(x) = L\ exists, uniformly\ in\ s,t; L = \mathcal{F}_{[\lambda,\mu]} - lim\ x\},$$
$$and, \Omega_{i,j,s,t}(x) = \frac{1}{\lambda_i\mu_j}\sum_{m\in J_i}\sum_{n\in I_j} x_{m+s,n+t}.$$

Materials and methods*: For double sequences the Cauchy's criterion of convergence has been modified by Pringsheim. Similarly, the necessary and sufficient conditions for the regularity of an infinite four dimensional matrix is a given by Robison. These concepts has been utilized to generalize the concept of $[\lambda,\mu]$-almost convergence double sequencesthrough de la Vallèe-Poussin mean and characterized some four-dimensional infinites matrices. We collect the relevant publications in this field and apply the same technique as applied in these papers to generalize the known results.*
Results*: In this paper we characterizeinfinite four-dimensional matrices which transform the sequence belonging to the space of bounded double sequence into the space of generalized almost convergence double sequence (i.e. $A = (a_{pqjk}) \in (\mathsf{C}_{bp}, \mathcal{F}_{[\lambda,\mu]})$). We introduced the concept of $[\lambda,\mu]$-almost Cauchy double sequences. It has also been proved that the space generalized almost convergence double sequence is regular (i.e. $\mathcal{F}_{[\lambda,\mu]} - regular$).*
Conclusion*: The condition $sup_{m,n,s,t,j,k}\left|\frac{1}{\lambda_m\mu_n}\sum_{p\in J_m}\sum_{q\in I_n} a_{p+s,q+t,j,k}\right| < \infty$ has been found to be necessary and sufficient for a four-dimensional matrix$A = (a_{pqjk}) \in (\mathsf{C}_{bp}, \mathcal{F}_{[\lambda,\mu]})$ to be$[\lambda,\mu]$-almost convergent. Again the necessary and sufficient conditions have been established for amatrix $A = (a_{pqjk})$ to be$\mathcal{F}_{[\lambda,\mu]} - regular$.*
Keywords: *$[\lambda,\mu]$- almost convergence, $\mathcal{F}_{[\lambda,\mu]} - regular$ matrix, $[\lambda,\mu]$-Cauchy double sequences, $[\lambda,\mu]$-almost coercive matrix.*

Date of Submission: 06-05-2020 Date of Acceptance: 19-05-2020

I. Introduction

The definition of almost convergence of the sequences of real numbers $x = \{x_n\}$ was given by Lorentz (1948)[1]as follows:
A sequence $x = \{x_n\}$ is said to be almost convergent to l if for every$\varepsilon > 0$, there exists N ∈ ℕ, such that $\left|\frac{1}{n}\sum_{i=0}^{n-1} x_{n+i} - l\right| < \varepsilon$ for all i > N. We write$f - limit x = l$.
Moricz and Rhoades (1988)[2] extended the concept of almost convergenceof a sequence $x = \{x_n\}$ to double sequences of real numbers $x = \{x_{mn}\}$. The sequence $x = \{x_{mn}\}$ almost converges to$l$, if for all$\varepsilon > 0$, there exists$N \in \mathbb{N}$, such that

$$\left|\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1} x_{m+i,n+j} - l\right| < \varepsilon, \text{ for all p, q > N and for all (m, n)} \in \mathbb{N}\times\mathbb{N}. \quad (1.1)$$

Moricz and Rhoades also characterized some matrix classes involving this concept.

As in the case of single sequences, every almost convergent double sequence is bounded. But a convergent double sequence need not be bounded. Thus, a convergent double sequence need not be almost convergent. However, every bounded convergent double sequence is almost convergent.

The idea of almost convergence is narrowly connected with the Banach limits; that is, a sequence $x_n \in \ell_\infty$ is almost convergent to l if all of its Banach limits are equal. As an application of almost convergence,

Mohiuddine (2011)[3] obtained some approximation theorems for sequence of positive linear operator through this notion.
Let $\{A = a_{pqmn}\,,\ p, q = 0, 1, 2, \ldots\}$ be a doubly infinite matrix of real numbers for all m, n = 0,1,2, ⋯. The sums

$$y_{pq} = (Ax)_{pq} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{pqmn}\ x_{mn} \qquad (1.2)$$

called the A-mean of the sequence $x = \{x_{jk}\}$, yield a method of summability. More exactly, we say that a sequence $x = \{x_{jk}\}$ is A-summable to the limit l if the A-mean exists for all j, k = 0,1,2, … in the sense of Pringsheim

$$\lim_{m,n\to\infty} \sum_{j=0}^{m}\sum_{k=0}^{n} a_{pqjk}\ x_{jk} = y_{pq}\ and\ \lim_{pq\to\infty} y_{pq} = l. \qquad (1.3)$$

We say that a matrix A is bounded regular if every bounded and convergent sequence $x = \{x_{jk}\}$ is A-summable to the same limit and the A-means are bounded.(Başarir M. 1995)[4]
Let $\lambda = (\lambda_m : m = 0,1,2, \ldots)\, and\, \mu = (\mu_n : n = 0,1,2, \ldots)$ be two nondecreasing sequences of positive real numbers with each tending to ∞ such that $\lambda_{m+1} \le \lambda_m + 1, \lambda_1 = 0, \mu_{n+1} \le \mu_n + 1, \mu_1 = 0$ and

$$\mathfrak{I}_{mn}(x) = \frac{1}{\lambda_m \mu_n} \sum_{j\in J_m}\sum_{k\in I_n} x_{jk} \qquad (1.4)$$

is called the $double\, de\, la\, Vallée - Poussin\, mean$, where $J_m = [m - \lambda_m + 1, m]$ and $I_n = [n - \mu_n + 1,\ n]$. We denote the set of all $\lambda\, and\, \mu$ type sequence by using the symbol $[\lambda, \mu]$.
Quite recently, Mohiuddine and Alotaibi (2014)[5] presented a generalization of the notion of almost convergent double sequences with the help of de la Vallèe-Poussin mean and called it $[\lambda, \mu]$-almost convergent.They obtained some useful results using this concept.

II. Material and Methods

We recall some concepts and results on the almost convergent double sequences through the notion of de la Vallèe-Poussin mean and infinite four-dimensional matrices. These results will be used in this paper.
Theorem 2.1 (Robison, 1999)[6]: Necessary and sufficient conditions for the matrix $A = (a_{pqmn})$ to be regular are:

i. $\lim_{p,q\to\infty} a_{p,q,m,n} = 0$, for each m and n
ii. $\lim_{p,q\to\infty} \sum_{m=1}^{p}\sum_{n=1}^{q} a_{pqmn} = 1$
iii. $\lim_{p,q\to\infty} \sum_{m=1}^{p} |a_{pqmn}| = 0$, for each n,
iv. $\lim_{p,q\to\infty} \sum_{n=1}^{q} |a_{pqmn}| = 0$, for each m,
v. $\sum_{m=1}^{p}\sum_{n=1}^{q} |a_{pqmn}| \le D < \infty$, where, D is some constant.

Definition 2.2 (Mohiuddine and Alotaibi, 2014)[5]: A double sequence $x = \{x_{mn}\}$ of real is said to be $[\lambda, \mu]$-almost convergent (briefly, $\mathcal{F}_{[\lambda,\mu]} - convergent$) to some number l if $x \epsilon \mathcal{F}_{[\lambda,\mu]}$, where
$\mathcal{F}_{[\lambda,\mu]} = \{\, x = \{x_{mn}\} : p - \lim_{ij\to\infty} \Omega_{i,j,s,t}(x) = L\ exists, uniformly\ in\ s, t; L = \mathcal{F}_{[\lambda,\mu]} - \lim x\}$,
Where

$$\Omega_{i,j,s,t}(x) = \frac{1}{\lambda_i \mu_j} \sum_{m\in J_i}\sum_{n\in I_j} x_{m+s,n+t}.$$

Denote by $\mathcal{F}_{[\lambda,\mu]}$, the space of all $[\lambda, \mu]$-almost convergent sequence $\{x_{mn}\}$. Note that $\mathcal{C}_{BP} \subset \mathcal{F}_{[\lambda,\mu]} \subset \ell_\infty$.
Definition 2.3 (Mohiuddine and Alotaibi, 2014)[5]: A four-dimensional matrix $A = (a_{pqnm})$ is said to be $[\lambda, \mu]$-almost regular if $Ax \in \mathcal{F}_{[\lambda,\mu]}$ for all $x = \{x_{mn}\} \in \mathcal{C}_{BP}$, where $\mathcal{C}_{BP}$ denotes the set of all bounded convergent double sequences in the Pringsheim sense, with $\mathcal{F}_{[\lambda,\mu]} - limit Ax = \lim x$, and one denotes this by $A \in (\mathcal{C}_{BP}, \mathcal{F}_{[\lambda,\mu]})reg.$
Definition 2.4 (Mohiuddine and Alotaibi, 2014)[7]: A matrix $A = (a_{pqmn})$ is said to be of class $(\mathcal{F}_{[\lambda,\mu]}, \mathcal{F}_{[\lambda,\mu]})$ if it maps every $\mathcal{F}_{[\lambda,\mu]}$-convergent double sequence into $\mathcal{F}_{[\lambda,\mu]}$-convergent double sequence; that is, $Ax \in \mathcal{F}_{[\lambda,\mu]}$ for all $x = \{x_{mn}\} \in \mathcal{F}_{[\lambda,\mu]}$. In addition, if $\mathcal{F}_{[\lambda,\mu]} - limit Ax = \mathcal{F}_{[\lambda,\mu]} - limit\ x$, then A is $\mathcal{F}_{[\lambda,\mu]}$ −regular and, in symbol, one will write $A \in (\mathcal{F}_{[\lambda,\mu]}, \mathcal{F}_{[\lambda,\mu]})reg.$
Definition 2.5 (Ĉunjalo, 2008)[8]: The sequence $x = \{x_{mn}\}$ is almost Cauchy, if for all $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$\left| \frac{1}{p_1 q_1} \sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1} x_{m_1+i,n_1+j} - \frac{1}{p_2 q_2} \sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1} x_{m_2+i,n_2+j} \right| < \varepsilon$$

for all $p_1, p_2, q_1, q_2 > k\ and\ for\ all\ (m_1, n_1), (m_2, n_2) \in \mathbb{N} \times \mathbb{N}$
It is known that a double sequence $x = \{x_{mn}\}$ of real number is Cauchy sequence if and only if it convergent
The equivalent of almost convergence is almost Cauchy condition.

Definition 2.6(Mohiuddine and Alotaibi, 2014)[5]: A matrix $A = (a_{pqjk})$ is said to be $[\lambda, \mu]$-almost coercive if it maps every C_{BP}-convergent double sequence $x = \{x_{jk}\}$ into $\mathcal{F}_{[\lambda,\mu]}$-convergent double sequence, that is, $Ax \in \mathcal{F}_{[\lambda,\mu]}$ for all $x = \{x_{jk}\} \in C_{bp}$. We denote this by $A = (a_{pqjk}) \in (C_{bp}, \mathcal{F}_{[\lambda,\mu]})$.
Lemma 2.1 (Ĉunjalo, 2008)[8]: The sequence $x = \{x_{mn}\}$ is almost convergent if and if it is almost Cauchy.
Theorem 2.2(Mohiuddine and Alotaibi, 2014)[7]: A matrix $A = (a_{pqmn}) \in (\mathcal{F}_{[\lambda,\mu]}, \mathcal{F}_{[\lambda,\mu]})$ if and only if

i. $\| A \| = sup_{pq} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |a_{pqmn}| < \infty$
ii. $a = (\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{pqmn})_{p,q=1}^{\infty} \in \mathcal{F}_{[\lambda,\mu]}$,
iii. $A(S-1) \in (\mathcal{L}_\infty, \mathcal{F}_{[\lambda,\mu]})$, where S is the shift operator.

III. Results

We establish the following results:
Theorem3.1: A four-dimensional matrix $A = (a_{pqjk}) \in (C_{BP}, \mathcal{F}_{[\lambda,\mu]})$ is $[\lambda, \mu]$-almost coercive if and only if

$$sup_{m,n,s,t,j,k} \left| \frac{1}{\lambda_m \mu_n} \sum_{p \in J_m} \sum_{q \in I_n} a_{p+s,q+t} x_{jk} \right| < \infty$$

Proof:*Sufficiency.*
Suppose that $A = (a_{pqjk}) \in (C_{bp}, \mathcal{F}_{[\lambda,\mu]})$. Then there exist $x = \{x_{jk}\} \in C_{bp}$, such that $Ax \in \mathcal{F}_{[\lambda,\mu]}$. Here $A_{pq} \in \mathcal{F}_{[\lambda,\mu]}$ for each $p, q \in \mathbb{N}$. Therefore,

$$\|Ax\|_\infty = sup_{m,n,s,t,j,k} \left| \frac{1}{\lambda_m \mu_n} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p \in J_m} \sum_{q \in I_n} a_{p+s,q+t,j,k} x_{jk} \right| < \infty$$

$$\leq sup_{m,n,s,t,j,k} \left| \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left(\frac{1}{\lambda_m \mu_n} \sum_{p \in J_m} \sum_{q \in I_n} a_{p+s,q+t,j,k} \right) x_{jk} \right| < \infty$$

$$\leq sup_{m,n,s,t,j,k} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left| \frac{1}{\lambda_m \mu_n} \sum_{p \in J_m} \sum_{q \in I_n} a_{p+s,q+t,j,k} \right| |x_{jk}| < \infty$$

Thus, the above condition is sufficient.
Necessity:
Suppose that the condition hold, for all $x = \{x_{jk}\} \in C_{bp}$. Then we have

$$\left| \frac{1}{\lambda_m \mu_n} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p \in J_m} \sum_{q \in I_n} a_{p+s,q+t,j,k} x_{jk} \right| \leq \left| \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left(\frac{1}{\lambda_m \mu_n} \sum_{p \in J_m} \sum_{q \in I_n} a_{p+s,q+t,j,k} \right) x_{jk} \right|$$

$$\leq \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left| \frac{1}{\lambda_m \mu_n} \sum_{p \in J_m} \sum_{q \in I_n} a_{p+s,q+t,j,k} \right| |x_{jk}|$$

We obtain, after taking supremum over m,n,s,t,,j,k. that

$$sup_{m,n,s,t,j,k} \left| \frac{1}{\lambda_m \mu_n} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p \in J_m} \sum_{q \in I_n} a_{p+s,q+t,j,k} x_{jk} \right| \leq sup_{m,n,s,t,j,k} \left| \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left(\frac{1}{\lambda_m \mu_n} \sum_{p \in J_m} \sum_{q \in I_n} a_{p+s,q+t,j,k}\, x_{jk} \right) \right|$$

$$\leq sup_{m,n,s,t,j,k} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left| \frac{1}{\lambda_m \mu_n} \sum_{p \in J_m} \sum_{q \in I_n} a_{p+s,q+t,j,k} \right| |x_{jk}|$$

$$\leq sup_{m,n,s,t,j,k} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left| \frac{1}{\lambda_m \mu_n} \sum_{p \in J_m} \sum_{q \in I_n} a_{p+s,q+t,j,k} \right| M < \infty$$

Then it is derived from the last inequality that $Ax \in \mathcal{F}_{[\lambda,\mu]}$. This completes the proof ■
Definition 3.1: A double sequence $x = \{x_{jk}\} \in \mathcal{F}_{[\lambda,\mu]}$ is said to be $[\lambda, \mu]$-almost Cauchy, if for all $\varepsilon > 0$, there exist $h \in \mathbb{N}$ such that:

$$\left|\Omega_{m_1 n_1 st}(x) - \Omega_{m_2 n_2 st}(x)\right| < \varepsilon, \forall\, m_1, m_2, n_1, n_2 > h$$

Where,
$\mathcal{F}_{[\lambda,\mu]} = \{ x = \{x_{mn}\}: p - \lim_{ij \to \infty} \Omega_{i,j,s,t}(x) = L \; exists, uniformly \; in \; s,t; L = \mathcal{F}_{[\lambda,\mu]} - \lim x\}$,

$$\Omega_{i,j,s,t}(x) = \frac{1}{\lambda_i\mu_j}\sum_{m\in J_i}\sum_{n\in I_j} x_{m+s,n+t}$$

Theorem 3.2: A double sequence $x = \{x_{jk}\} \in \mathcal{F}_{[\lambda,\mu]}$ is $[\lambda,\mu]$-almost convergent, if and only if it is $[\lambda,\mu]$-almost Cauchy.

Proof:

Suppose a sequence $x = \{x_{jk}\} \in \mathcal{F}_{[\lambda,\mu]}$ is $[\lambda,\mu]$-almost convergent. Then,$\forall\, \varepsilon > 0, \exists\, h \in \mathbb{N}$, such that:

$$\left|\frac{1}{\lambda_m\mu_n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} x_{j+s,k+t} - \ell\right| < \varepsilon/2$$

For all $m,n > h\ and\ \forall\,(j,k) \in \mathbb{N}\times\mathbb{N}$

Therefore,

$$\left|\frac{1}{\lambda_{m_1}\mu_{n_1}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1} x_{j_1+s,k_1+t} - \frac{1}{\lambda_{m_2}\mu_{n_2}}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} x_{j_2+s,k_2+t}\right|$$
$$\leq \left|\frac{1}{\lambda_{m_1}\mu_{n_1}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1} x_{j_1+s,k_1+t} - l\right| + \left|\frac{1}{\lambda_{m_2}\mu_{n_2}}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} x_{j_2+s,k_2+t} - l\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

For all $m_1, m_2, n_1, n_2 > h\ \ and\ \forall (j_1,k_1),(j_2,k_2) \in \mathbb{N}\times\mathbb{N}$.

Hence, the sequence $x = \{x_{jk}\}$ is $[\lambda,\mu]$-almost Cauchy.

Conversely, suppose that the sequence $x = \{x_{jk}\} \in \mathcal{F}_{[\lambda,\mu]}$ is $[\lambda,\mu]$-almost Cauchy. Then $\forall\, \varepsilon > 0, \exists\, h \in \mathbb{N}$, $\ni$

$$\left|\frac{1}{\lambda_{m_1}\mu_{n_1}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1} x_{j_1+s,k_1+t} - \frac{1}{\lambda_{m_2}\mu_{n_2}}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} x_{j_2+s,k_2+t}\right| < \frac{\varepsilon}{2}$$

For all $m_1, m_2, n_1, n_2 > h\ \ and\ \forall (j_1,k_1),(j_2,k_2) \in \mathbb{N}\times\mathbb{N}$.

Taking $j_1 = j_2 = j_0\ and\ k_1 = k_2 = k_0$ in relation (1), we obtain that

$\left(\frac{1}{\lambda_m\mu_n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} x_{j_0+s,k_0+t}\right)_{m,n=1}^{\infty}$ is a Cauchy sequence and, therefore convergent.

Let $\lim_{m,n\to\infty}\frac{1}{\lambda_m\mu_n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} x_{j_0+s,k_0+t} = \ell$. Then, $\forall\, \varepsilon > 0, \exists\, h_1 \in \mathbb{N}$, $\ni$

$$\left|\frac{1}{\lambda_m\mu_n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} x_{j_0+s,k_0+t} - \ell\right| < \frac{\varepsilon}{2}, \qquad \forall\, m,n > h_1$$

It follows that:

$$\left|\frac{1}{\lambda_m\mu_n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} x_{j+s,k+t} - \ell\right| \leq \left|\frac{1}{\lambda_m\mu_n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} x_{j+s,k+t} - \frac{1}{\lambda_m\mu_n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} x_{j_0+s,k_0+t}\right| + \left|\frac{1}{\lambda_m\mu_n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} x_{j_0+s,k_0+t} - \ell\right|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

For $\forall\, m,n > \max(h,h_1)\ \ and\ \forall\,(j,k) \in \mathbb{N}\times\mathbb{N}$. It follows that, the sequence $x = \{x_{jk}\}[\lambda,\mu]$-almost converges to ℓ and hence $[\lambda,\mu]$-almost convergent.

This completes the proof of the Theorem.

Theorem 3.3: A matrix $A = (a_{pqjk})$ is $\mathcal{F}_{[\lambda,\mu]} - regular$ if and only if

1. $\|A\|_\infty = sup_{m,n,s,t,j,k}\left|\frac{1}{\lambda_m\mu_n}\sum_{p\in J_m}\sum_{q\in I_n} a_{p+s,q+t,j,k}\right| < \infty$
2. $\lim_{m,n\to\infty}\alpha(m,n,j,k,s,t) = 0$
3. $\lim_{m,n\to\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha(m,n,j,k,s,t) = 1$
4. $\lim_{m,n\to\infty}\sum_{j=0}^{\infty}|\alpha(m,n,j,k,s,t)| = 0, (k \in \mathbb{N}$
5. $\lim_{m,n\to\infty}\sum_{k=0}^{\infty}|\alpha(m,n,j,k,s,t)| = 0, (j \in \mathbb{N}$
6. $\lim_{m,n\to\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\alpha(m,n,j,k,s,t)|$ exists

Where the limits are uniform in s,t and $\alpha(m,n,j,k,s,t) = \frac{1}{\lambda_m\mu_n}\sum_{p\in J_m}\sum_{q\in I_n} a_{p+s,q+t,j,k}$

Proof:

Sufficiency:

Suppose that the conditions (1-6) hold. Define a sequence $x = \{x_{j,k}\} \in C_{bp}$ with $p - \lim_{j,k} x_{jk} = \ell$ (say). Then, by the definition of p-limit, for any given $\varepsilon > 0$, there exist a $\mathbb{N} > 0$, such that $|x_{jk}| < |\ell| + \varepsilon$ whenever $j,k > N$.

Now, we can write

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha(m,n,j,k,s,t)x_{jk}$$

$$=\sum_{j=0}^{N}\sum_{k=0}^{N}\alpha(m,n,j,k,s,t)x_{jk}+\sum_{j=N}^{\infty}\sum_{k=0}^{N-1}\alpha(m,n,j,k,s,t)x_{jk}$$
$$+\sum_{j=0}^{N-1}\sum_{k=N}^{\infty}\alpha(m,n,j,k,s,t)x_{jk}+\sum_{j=N+1}^{\infty}\sum_{k=N+1}^{\infty}\alpha(m,n,j,k,s,t)x_{jk}$$

Hence,

$$\left|\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha(m,n,j,k,s,t)x_{jk}\right|$$

$$\leq\|x\|\sum_{j=0}^{N}\sum_{k=0}^{N}|\alpha(m,n,j,k,s,t)|$$
$$+\|x\|\sum_{j=N}^{\infty}\sum_{k=0}^{N-1}|\alpha(m,n,j,k,s,t)|+\|x\|\sum_{j=0}^{N-1}\sum_{k=N}^{\infty}|\alpha(m,n,j,k,s,t)|+(|\ell|$$
$$+\varepsilon)\left|\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha(m,n,j,k,s,t)\right|$$

Therefore, by letting $m,n\to\infty$ and considering the conditions (1-6), we have

$$\left|\lim_{m,n\to\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha(m,n,j,k,s,t)x_{jk}\right|\leq|\ell|+\varepsilon$$

i.e., $\left|\mathcal{F}_{[\lambda,\mu]}-\lim Ax\right|\leq|\ell|+\varepsilon$. Since ε is an arbitrary, this implies the $\mathcal{F}_{[\lambda,\mu]}-regularity\ of\ A=(a_{pqjk})$.

Necessity:

Suppose that A is$\mathcal{F}_{[\lambda,\mu]}-regular$. Then, by the definition, the A-transform of x exist and $Ax\in\mathcal{F}_{[\lambda,\mu]}$ for each $x\in C_{BP}$. Therefore, Ax is also bounded. So, there exists a positive number M such that

$$sup_{m,n,s,t,j,k}\left|\frac{1}{\lambda_m\mu_n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p\in J_m}\sum_{q\in I_n}a_{p+s,q+t,j,k}x_{jk}\right|<M<\infty$$

for each, $x\in C_{BP}$. Now let us choose a sequence $y=\{y_{jk}\}$ with

$$y_{jk}=\begin{cases}sgn\ a_{pqjk}, & 0\leq j\leq r, 0\leq k\leq r\\ 0, & otherwise\end{cases}\ (p,q=1,2,3\ldots)$$

Then, the necessity of condition (1) follows by considering the sequence $y=\{y_{jk}\}$.

For the necessity of (6), define a sequence $u=\{u_{jk}\}$ by $y=\{y_{jk}\}$, with $\alpha(m,n,j,k,s,t)$ in place ofa_{pqjk}. Then, $P-limAu$, implies (6).

Let us define the sequence e^{il} as follows

$$e_{jk}^{il}=\begin{cases}1, & if\ (j,k)=(i,l)\\ 0, & otherwise;\end{cases}$$

and denote the point wise sum by $s^l=\sum_i e^{il}$ and$r^i=\sum_l e^{il}$ $(i\in\mathbb{N})$. Then, the necessity of condition (2) follows from $\mathcal{F}_{[\lambda,\mu]}-\lim Ae^{il}$.

Also, $\mathcal{F}_{[\lambda,\mu]}-\lim Ar^j=\lim_{m,n\to\infty}\sum_j|\alpha(m,n,j,k,s,t)|=0$, $(k\in\mathbb{N}$ and,

$$\mathcal{F}_{[\lambda,\mu]}-\lim As^k=\lim_{m,n\to\infty}\sum_j|\alpha(m,n,j,k,s,t)|=0,\qquad (j\in\mathbb{N}$$

To verify the conditions (4) and (5), we need to prove that these limits are uniform in s,t. So, let us suppose that (5) does not hold, i.e., for any $j_0\in\mathbb{N}$,

$$\lim_{m,n}sup_{s,t}\sum_k|\alpha(m,n,j_0,k,s,t,|\neq 0$$

Then, there exists an $\varepsilon>0$ and index sequences $(m_i),(n_i)$ such that

$$sup_{s,t}\sum_k|\alpha(m,n,j_0,k,s,t)|\geq\varepsilon\qquad (i\in\mathbb{N})$$

Since,

$$\sum_k |\alpha(m,n,j_0,k,s,t)| \le sup_{p,q} \sum_{j,k} |a_{pqjk}| < \infty$$

And (2) holds, we may find an index sequence (k_i) such that

$$\sum_{k=1}^{k_i} |\alpha(m_i,n_i,j_0,k,s_i,t_i)| \le \frac{\varepsilon}{8}, \qquad (i \in \mathbb{N})$$

And

$$\sum_{k=k_{i+1}+1}^{\infty_i} |\alpha(m_i,n_i,j_0,k,s_i,t_i)| \le \frac{3\varepsilon}{4}, \qquad (i \in \mathbb{N})$$

Now, define a sequence $x = \{x_{jk}\}$ by

$$x_{jk} = \begin{cases} (-1)^i \alpha(m_i,n_i,j_0,k,s_i,t_i), & if\ k_i + 1 \le k \le k_{i+1} (i \in \mathbb{N}); j = j_0 \\ 0, & if\ j \ne j_0 \end{cases}$$

Then, clearly $x \in C_{BP}$ with $\|x\|_\infty \le 1$. But, for even i, we have

$$\frac{1}{\lambda_{m_i}\mu_{n_i}} \sum_{m=s_i}^{s_i+m_i-1} \sum_{n=t_i}^{t_i+n_i-1} (Ax)_{pq} = \sum_k \alpha(m_i,n_i,j_0,k,s_i,t_i) x_{jk}$$

$$\ge \sum_{k=k_{i+1}+1}^{k_i+1} \alpha(m_i,n_i,j_0,k,s_i,t_i) x_{jk} - \sum_{k=1}^{k_i} |\alpha(m_i,n_i,j_0,k,s_i,t_i)|$$

$$- \sum_{k=k_{i+1}+1}^{\infty} |\alpha(m_i,n_i,j_0,k,s_i,t_i)| \ge \sum_{k=k_i+1}^{k_{i+1}} |\alpha(m_i,n_i,j_0,k,s_i,t_i)| - \frac{\varepsilon}{8} - \frac{\varepsilon}{8} \ge \frac{3\varepsilon}{4} - \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

Analogously, for odd i, one can show that

$$\frac{1}{\lambda_{m_i}\mu_{n_i}} \sum_{m=s_i}^{s_i+m_i-1} \sum_{n=t_i}^{t_i+n_i-1} (Ax)_{pq} \le -\frac{\varepsilon}{2}$$

Hence, the sequence

$$\left(\frac{1}{\lambda\mu} \sum_{m=s_i}^{s_i+m_i-1} \sum_{n=t_i}^{t_i+n_i-1} (Ax)_{pq} \right)$$

doesn't converge uniformly in $s,t \in \mathbb{N}\ as\ m,n \to \infty$. This means that $Ax \notin \mathcal{F}_{[\lambda\mu]}$, which is a contradiction. So, (5) holds. In the same way, we get the necessity of (4). On the other hand, for the necessity of the condition (3) it is enough to take the sequence $e_{jk} = 1$ for each j, k.

This completes the proof of the theorem.

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Zakawat U. Siddiqui. "On F_([λ,μ])-Regular Four-Dimensional Matrices for [λ,μ]-Almost Convergence of Double Sequences." *IOSR Journal of Mathematics (IOSR-JM)*, 16(3), (2020): pp. 19-24.