IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 16, Issue 3 Ser. II (May – June 2020), PP 13-18
www.iosrjournals.org

Method for solving the Goldbach binary problem

Vadim N. Romanov
Doctor of Technical Sciences, Professor St.-Petersburg, Russia
vromanvpi@mail.ru

Abstract
The paper proposes a method for proving the Goldbach binary conjecture, based on the properties of representations of even numbers and the rules of formal logic. Numerical evaluations confirming the correctness of the method are carried out. The connection between the Goldbach conjecture and the Legendre hypothesis is considered.
Keywords and phrases: *number theory, prime numbers, Goldbach's binary hypothesis.*
Classification code: 11A41 – Primes; 11D85 – Representation problems.

Date of Submission: 06-05-2020 Date of Acceptance: 19-05-2020

I. Introduction. General relations

One of the unresolved problems of the number theory is the proof of the Goldbach binary conjecture. Experimental studies and calculations confirm its validity for large numbers [1 – 3]. In this paper, we consider a method for proving a hypothesis based on the properties of representations of even numbers and the rules of formal logic. As is known, every natural number admits a trivial representation as a sum of units. Combining (grouping) units in different ways, we obtain all the representations [4, 5]. Consider the even numbers and their representations as a sum of two odd summands. For an arbitrary even integer p the trivial representation has the form $p = 1 + 1 + ... + 1$ (p units). If p is divisible by 4, then its representation as a sum of two odd numbers can be written as a chain of equalities

$$p = [1 + (p-1)] = [3 + (p-3)] = ... = [(p/2-1) + (p/2+1)], \quad (1)$$

where $p/2$ is the center of representation (even number). The terms in square brackets, we call the conjugate numbers. The first term is to the left of the center and the second term is to the right of the center. If p is not divisible by 4, its representation has the form

$$p = [1 + (p-1)] = [3 + (p-3)] = ... = [(p/2-2) + (p/2+2)] = [p/2 + p/2], \quad (2)$$

where $p/2$ is the center (odd number). For number $p + 2$ the representation is written in the form

$$p + 2 = [1 + (p+1)] = [3 + (p-1)] = ... = [(p/2-1) + (p/2+3)] = [(p/2+1) + (p/2+1)], \quad (3)$$

where $(p/2 + 1)$ is the center (odd number). The number of pairs depends on the parity of center. These regularities are of a general nature. If we move from the number p, which is divisible by 4, to the next even number $p + 2$, which is not divisible by 4, then the even center of the representation becomes odd ($p/2 \rightarrow p/2 + 1$), and the number of pairs of conjugate numbers increases by 1 ($p/4 \rightarrow p/4 + 1$). If we move from the number $p + 2$ to the next even number $p + 4$, which is divisible by 4, the odd center becomes even ($p/2 + 1 \rightarrow p/2 + 2$), and the number of pairs of conjugate numbers does not change ($p/4 + 1 \rightarrow p/4 + 1$). Acceptable combinations of ends in conjugate numbers and their sequence are completely determined by the end of even number. For even numbers ending in 0 acceptable combinations of ends and their sequence are (towards the center) 1-9; 3-7; 5-5; 7-3; 9-1 and then the ends are repeated (first end corresponds with numbers to the left of the center, and the second end corresponds with numbers to the right of the center). If we consider only prime conjugate numbers, the combination 5-5 can be eliminated. For even numbers ending in 2 the pairs of ends and their sequence (towards the center) are 1-1; 3-9; 5(number)-7; 7-5; 9-3 and then the ends are repeated. For even numbers ending in 4 the pairs of ends and their sequence are 1-3, 3-1, 5(number)-9, 7-7, 9-5, 1-3 etc. If we consider only prime numbers, the pair 9-5 can be eliminated. For even numbers ending in 6 the pairs of ends and their sequence have the form 1-5 (can be eliminated), 3-3, 5(number)-1, 7-9, 9-7, 1-5 etc. For even numbers ending in 8 the pairs of ends and their sequence have the form 1-7, 3-5 (can be excluded), 5(number)-3, 7-1, 9-9, 1-7 etc. Consider what it will be, when p increases. With increasing of even number p the number of conjugate pairs and the number of primes included in the representation of even number increase. From (1) it can be seen that initial prime numbers 3, 5, 7, 11, 13, 17, 19, 23 etc. are included in all representations for sufficiently large p and their number increases with increasing p. There is an asymmetry in the distribution of prime numbers to the left and to the right of the center, which depends on two processes. The number of primes to the left of the center rises (not decreases), as they transfer from the right to the left. Therefore the number of primes on the right of the center can decrease (does not increase), but not much. It is compensated by appearance of new primes and does not vanish when p increases. With increasing p between these two processes is the dynamic equilibrium that determines an asymmetry of the distribution of prime numbers and depends on order of magnitude of number p. Prime numbers regularly appear both on the left and on the right of the center. The appearance of pairs of prime conjugate numbers depends on the total number of primes less than p and on the irregularity of their distribution on the left and on the right of the center. Since prime numbers are infinitely many, pairs of prime conjugate numbers appear regularly and cannot completely vanish with increasing p. We will prove that otherwise, after certain p would not be prime numbers, which is impossible.

II. Proof of Goldbach's binary conjecture

We prove the binary conjecture (hypothesis) by induction. We are going to prove the statement: every even integer not less 4 admits a representation as a sum of two primes. The validity of this statement can be checked for any finite even number and, more important for us, for any interval of even numbers when it is doubled. Consider the set of even integers. Let p is such a number. Suppose that for numbers n in the interval $p_1 \le n \le p$, where $p = 2p_1$, the binary hypothesis is valid. We will prove the induction transition not for one value of n, but for the interval of values. We prove that the binary hypothesis holds for interval $(p+2) \le n_1 \le p_2$, where $p_2 = 2(p + 2)$. The choice of the interval is determined by the fact that the ratio of the even number to the center of the representation is equal to the ratio of the end to the beginning of the interval; the following relation holds $p/p_1 = p_2/(p + 2) = n/(n/2) = n_1/(n_1/2) = 2$. Therefore, there is vertical-horizontal symmetry in the representation of even numbers. Suppose the contrary, that is, in the interval $(p+2) \le n_1 \le p_2$ the hypothesis is not satisfied for any even number. Here, a general-negative judgment corresponds in scope to a general-affirmative judgment. We consider the interval $(p+2) \le n_1 \le p_2$ as a whole and all n_1 in this interval as equal (equivalent); therefore, we reject a partial-negative as well as a partial-affirmative judgment. Consider a sequence of even numbers with step 2. It is enough to consider the representation of even numbers in the interval [p, p + 10]. The choice of interval is due to the fact that the numbers p and p +10 have the identical ends, and then the ends are repeated. Let us start with the number p + 2. Compare representations of numbers p and p + 2. In our case from (1), (3) it follows that the center in the representation of p is equal to p/2 (even number); the center in the representation of p + 2 is p/2 + 1 (odd number). The number of conjugate pairs in the representation of p is equal to p/4 and in the representation of p + 2 it is equal to p/4 + 1 (see above). The number of primes is not reduced after transition from p to p + 2. To the left of the center in the representation of p + 2 there is one new number $p/2+1$. Number $p/2+1$ appears because of transition from the right of the center in the representation of p to the left of the center in the representation of p + 2 and due to change in the parity of center. The other numbers are the same as in the representation of p. Number $p/2+1$ can be prime or composite. In the representation of p + 2 to the right of the center there is one new number p + 1. The other numbers are the same as in the representation of p; they change position (increase) by 2 relative to the numbers in the representation of p. Find out whether the number (p + 1) is prime. It cannot be prime number. Otherwise the binary hypothesis will be true for all numbers $p+2m$ (m = 1, 2, 3 etc.) in representation of which pairs $(2m-1, p+1)$ are the pairs of prime conjugate numbers, which contradicts our assumption. Note that depending on the end of an even number p, the number p + 1 can be a multiple of 3 or 5, i.e. obviously not prime number, but we do not consider these cases. We write the representation for the next number p + 4

$$p+4 = [1+(p+3)] = [3+(p+1)] = ... = [(p/2-1)+(p/2+5)] = [(p/2+1)+(p/2+3)] \quad (4)$$

From comparison (3) and (4), it follows that the center in the representation of p + 4 is p/2 + 2 (even number); the number of conjugate pairs is equal to p/4 +1, i.e. does not change. In the representation of p + 4 to the right of the center there is one new number p + 3. It cannot be prime number. Otherwise, the binary hypothesis will be true for all numbers $p+2m$ in representation of which pairs $(2m-3, p+3)$ are the pairs of prime conjugate numbers, which contradicts our assumption. Consider the representation for the next number p + 6

$$p+6 = [1+(p+5)] = [3+(p+3)] = ... = [(p/2+1)+(p/2+5)] = [(p/2+3)+(p/2+3)] \quad (5)$$

From comparison (5) and (4), it follows that the center in the representation of $p + 6$ is $p/2 + 3$ (odd number); the number of conjugate pairs is equal to $p/4 + 2$. In the representation of $p + 6$ to the right of the center there is one new number $p + 5$. It cannot be prime number. Otherwise, the binary hypothesis will be true for all numbers $p + 2m$ in representation of which pairs $(2m - 5, p + 5)$ are the pairs of prime conjugate numbers, which contradicts our assumption. Consider the representation for the next number $p + 8$

$$p + 8 = [1 + (p + 7)] = [3 + (p + 5)] = ... = [(p/2 + 1) + (p/2 + 7)] = [(p/2 + 3) + (p/2 + 5)] \quad (6)$$

From comparison (6) and (5), it follows that the center in the representation of $p + 8$ is $p/2 + 4$ (even number); the number of conjugate pairs is equal to $p/4 + 2$. In the representation of $p + 8$ to the right of the center there is one new number $p + 7$. It cannot be prime number. Otherwise, the binary hypothesis will be true for all numbers $p + 2m$ in representation of which pairs $(2m - 7, p + 7)$ are the pairs of prime conjugate numbers, which contradicts our assumption. Consider the representation for the next number $p + 10$

$$p + 10 = [1 + (p + 9)] = [3 + (p + 7)] = ... = [(p/2 + 3) + (p/2 + 7)] = [(p/2 + 5) + (p/2 + 5)] \quad (7)$$

From comparison (7) and (6), it follows that the center in the representation of $p + 10$ is $p/2 + 5$ (odd number); the number of conjugate pairs is equal to $p/4 + 3$. In the representation of $p + 10$ to the right of the center, there is one new number $p + 9$. It cannot be prime number. Otherwise, the binary hypothesis will be true for all numbers $p + 2m$ in representation of which pairs $(2m - 9, p + 9)$ are the pairs of prime conjugate numbers, which contradicts our assumption. This analysis is valid *mutatis mutandis* for arbitrary two adjacent even numbers from the considered interval $(p + 2) \le n_1 \le p_2$. Therefore, it follows that new prime numbers on the right of the center in the representation of the number n_1 do not appear with increasing n_1. Now consider the distribution of prime numbers in the representation of the number p. Since, by assumption, for number $n = p$ the binary hypothesis is valid, there is its representation as a sum of two prime conjugate numbers. We designate them as k and $p - k$, so that $p = k + (p - k)$. From our assumption that the binary hypothesis is not valid in the interval $(p + 2) \le n_1 \le p_2$, it follows that k is the largest prime number on the left of the center in the representation of p, and $(p - k)$ is the largest prime number on the right of the center. Therefore, the prime numbers on the left of the center are located in the interval from 1 to k, and the prime numbers on the right of the center are located from $(p - k)$ to $p/2$. So, the pair $(k, p - k)$ is the only one pair of prime conjugate numbers in the representation of p. Determine where the pair $(k, p - k)$ is located. The relationship $k > p/4$ should be satisfied, then $(p - k) < 3p/4$. It is easy to verify that if $k \le p/4$, then by successively decreasing the number n, we get the number $n_0 = p/2$ for which the binary hypothesis will not hold, since in the representation of this number there will not be prime numbers on the right of the center, which contradicts our assumption. Indeed. If p is divisible by 4, but $p/2$ is not divisible by 4, then the representation for number $p/2$ has the form

$$p/2 = [1 + (p/2 - 1)] = [3 + (p/2 - 3)] = ... = [(p/4 - 1) + (p/4 + 1)] = [p/4, p/4] \quad (8)$$

If, moreover, $p/2$ is divisible by 4, then the representation has the form

$$p/2 = [1 + (p/2 - 1)] = [3 + (p/2 - 3)] = ... = [p/4 - 1, p/4 + 1], \quad (8a)$$

where pair $(p/4, p/4)$ is excluded, since it is an even number. Comparing (8) with (1) or with (2), it is easy to verify the validity of our estimates. Determine the position of the number k more accurately. Note that if the number p decreases to $p/2$, the numbers in the interval from $p/4$ to $p/2$ on the left of the center in the representation of the number p will be located on the right of the center in the representation of the number $p/2$. In order not to violate the binary hypothesis for $n < p$, namely for $p_1 \le n \le p$ the following relations should be satisfied

$$k > p/4, \quad (9)$$

$$(p - k) < 3p/4, \quad (9a)$$

$$(p/2 - k) < p/4. \quad (9b)$$

In addition, numbers k, $(p - k)$, $(p/2 - k)$ should be prime numbers; pairs $(k, p - k)$ and $(p/2 - k, k)$ should be pairs of prime conjugate numbers. Note that in the representation of an even number $p/2$, the number k is located on the right of the center, and the number $(p/2 - k)$ is located on the left of the center. We now consider the distribution of prime numbers in the representation of even numbers $n_1 > p$. We will consistently increase the number n_1 and put $n_1 = p + 2$, $n_1 = p + 4$ and so on. After a certain number of steps, we get the number N_0, in the representation of which there will be no prime numbers on the right of the center. The number N_0 is located in the interval $p < N_0 \leq p + p/2$; its exact value depends on the position of the pair $(k, p - k)$.Taking into account the relation (9a) it is easy to verify that for number p which is divisible by 4: $N_0 \leq p + p/2$. Equality is achieved if the prime number $(p – k)$ is prime number closest to the number $3p/4$. So, for all even numbers in the interval $N_0 \leq n_1 \leq 2(p + 2)$ there is no one prime number on the right of the center of the representation. In addition, for even numbers in the interval $(p+2) \leq n_1 \leq 2(p+2)$, there are no new primes on the right of the center of the representation. We have obtained a contradiction, since the number of primes should increase with increasing n_1. Therefore, our assumption is not true, and prime numbers, as well as pairs of prime conjugate numbers, are regularly formed with increasing n_1. From a general-negative judgment follows a general-affirmative judgment by virtue of the previous analysis. Partial-affirmative judgment must be rejected, since all n_1 in the considered interval are equal (equivalent). Thus, the induction transition, and hence the binary hypothesis, is proved.

III. Numerical estimates

Now we confirm the validity of our method of proof by direct calculations. Consider some properties of numbers that allow us to determine the number of composite numbers that do not exceed a given even number. We use algebraic method. Each odd number k generates a sequence (arithmetic progression) of the form

$$P_k = k + 2kl_k \,, \qquad (10)$$

where $l_k = 1, 2, 3$, etc. From (10) it is clear that if k is a prime number, then it appears as a divisor of a composite number, first with a factor 3, then with a factor 5, then 7, etc.; the repetition period is $2k$. So, if k increases, the period of its repetition increases. The number k we call generator, and P_k we call the sequence generated by the number k. Therefore, for the number 3, the period is 6, for number 5 −10, etc.; if k is a composite number, then the sequence generated by it is already contained in the sequences of the preceding primes. From (10) it is clear that if the order of magnitude of primes increases, then the repetition period increases. In general, for an arbitrary even n, if $k < n/2$, then the period does not exceed n. Consequently, in conjugate pairs representing a number n, odd numbers k on the left of the representation center can be repeated as part of composite numbers, but numbers $(n – k)$ cannot be repeated. A prime number k can occur as a composite number with a multiplier of 3 on the left of the center in conjugate pairs if $k < n/6$, and on the right of the center if $n/6 < k < n/3$ (these estimates determine the upper limit of such numbers). As is known, for an arbitrary even n. the number of generators is determined by the relation

$$l_n \leq [\sqrt{n}], \qquad (11)$$

where $[\sqrt{n}]$ is the integer part of the number $\sqrt{n}$. When forming composite numbers, the terms of the sequence generated by a prime number k should be taken into account up to the value determined by the ratio

$$L_{nk} = [n/k] \qquad (12)$$

So, for $n = 100$ and $k = 3$, when forming composite numbers, all odd numbers up to [100/3] = 33 should be taken into account, i.e. 16 numbers; for $k = 5$ − odd numbers up to [100/5] = 20 not divisible by 3, namely, 5 · 5, 5 · 7, ..., 5 · 19, i.e. 6 numbers; for $k = 7$ − odd numbers up to [100/7] = 14, not divisible by 3 and 5, namely, 7 · 7, 7 · 11, 7 · 13, i.e. 3 numbers. We have a total 25 composite numbers for number 100. We write the general relations. From (10) it follows that the number of contributions of the terms of the sequence generated by the prime number k for the number n is determined by the relation

$$l_{nk} = [(n-k)/2k] \,. \qquad (13)$$

In this case, it is necessary to take into account the duplication of contributions at $k > 3$. If k is composite number, the number of contributions is equal to

$$l_{nk} = [(n-k)/2k] + 1 \,. \qquad (13a)$$

All these contributions are already contained in sequences of primes, less than k. Therefore, they do not need to be considered when we calculate the number of composite numbers. The total number of composite numbers for even number n is equal to

$$m(n) = \sum_k [(n-k)/2k], \qquad (14)$$

where the summation is over all generators of number n, and the upper limit of summation is determined by (12). For $k = 3$, all odd numbers up to $[n/3]$ are taken into account, for $k = 5$ − primes up to $[n/5]$, more than or equal to 5, for $k = 7$ − primes up to $[n/7]$, more than or equal to 7, and so on. Relations (10) − (14) allow us to determine whether a certain odd number ends in 1, 3, 7 or 9 is composite. A number is composite if it is contained in at least one of the sequences formed by its generators; otherwise, this number is prime number. Now we estimate how the set of composite numbers changes as the even number increases from $n/2$ to n. For convenience, we assume that n is divisible by 4, so as not to use the integer part of the number. Designate $m(n)$ − the number of odd composite numbers less than even number n; $m_R(n)$ − the number of odd composite numbers on the right of the center in representation of n, $m_L(n)$ − the number of odd composite numbers on the left of the center in representation of n; $p(n)$ − the number of prime numbers less than n; $p_R(n)$ − the number of prime numbers on the right of the center in representation of n; $p_L(n)$ − the number of prime numbers on the left of the center in representation of n. We have the following obvious relations: $m(n) + p(n) = n/2$, $m_R(n) + p_R(n) = n/4$, $m_L(n) + p_L(n) = n/4$, $m_R(n) + m_L(n) = m$, $p_R(n) + p_L(n) = p$. In addition, we have $m_R(n) > m_L(n)$, $p_R(n) < p_L(n)$. Determine how these values change when even number increases from $n/2$ to n. As the representation center is displaced from $n/4$ to $n/2$, we have $m_L(n) = m_L(n/2) + m_R(n/2) = m(n/2)$. The change of the number of composite numbers $m(n)$ occurs due to two factors: the increase of the limits of change in (12) from $[n/2k]$ to $[n/k]$ and the increase of the number of generators in (11) from $[\sqrt{n/2}]$ to $[\sqrt{n}]$. The first factor gives the increase of about two times, and the second – by $([\sqrt{n}]-[\sqrt{n/2}])/2$ or, with respect to $[\sqrt{n/2}]$, 0.2 times, i.e. a total 2.2 times (in the calculations, the sign of integer part of number was omitted). Of course, this is an approximate estimate, but it gives the correct order of magnitudes. When we evaluate the first factor, we do not take into account that for numbers $k > 3$ the ratio increases due to the exclusion of repeated numbers, and when we estimate the second factor, we do not take into account that for numbers $k > 3$ the ratio decreases due to the exclusion of repeated products of new generators with each other. The results were compared with tabular data for n from 200 to 6000, which confirmed the correctness of the estimates; for $n <$ 200 the evaluation is underestimated due to the first factor. We have $m(n) = 2{,}2m(n/2)$. Since $m_L(n) = m(n/2)$, we obtain $p_L - p_R = m_R - m_L = 0{,}2m(n/2) = 0{,}09m(n)$; so the difference between the number of primes on the left and on the right of the center does not increase much. We write the estimates in a more convenient way

$$p_R / p_L = \frac{n/2 - 1{,}1m(n)}{n/2 - 0{,}9m(n)} \qquad (15)$$

From (15) we obtain

$$p(n) = p_R + p_L = \frac{p_L(n - 2m(n))}{n/2 - 0{,}9m(n)}. \qquad (16)$$

It follows from (16) that) $p(n) > p(n/2)$. Thus, the total number of primes increases with n, which confirms the proposed method for proving the hypothesis.

IV. Connection of the binary hypothesis with the Legendre's conjecture

Legendre's conjecture consists of the assertion that for any natural n in the interval between n^2 and $(n + 1)^2$ there is always a prime number. It is easy to see that when n changes, these intervals cover the positive part of the numerical axis, and the values of adjacent intervals relate to each other as odd numbers; so, we have $[(n + 1)^2 - n^2]/ [n^2 - (n - 1)^2] = (2n + 1)/(2n - 1)$. Let us prove that the validity of this conjecture follows from the validity of Goldbach's binary conjecture. The proof is carried out by induction. It is easy to verify that for $n = 1$ the Legendre conjecture is true, since the interval (1, 4) contains prime numbers 2 and 3. Suppose that for $n = p$ the hypothesis is true, i.e. the interval $(p^2, (p + 1)^2)$ contains a prime number. We put $n = p + 1$; we prove that the interval $((p + 1)^2, (p + 2)^2)$ contains a prime number. Suppose the contrary. For definiteness, let p be an odd number, then $(p + 1)$ is an even number, and $(p + 2)$ is an odd number. Consider the interval $(p^2, (p + 2)^2)$. We extend the interval by decreasing the left boundary by 2: $(p^2 - 2, (p + 2)^2)$. The center of this new interval is $(p + 1)^2 = (p^2 - 2 + (p + 2)^2)/2$. However, at the same time $(p + 1)^2$ is the center of the representation when the number $2(p + 1)^2$ is represented as the sum of two odd numbers. We get that there is a prime number on the left of this center, since for $n = p$ the Legendre conjecture is true, *a fortiori* it is true for the extended interval, but there are no prime numbers on the right of the center, that is, in the interval $((p + 1)^2, (p + 2)^2)$, since we assumed that the Legendre conjecture is not satisfied for $n = p + 1$. Then for the number $2(p + 1)^2$ there is no representation as a sum of two prime conjugate numbers, which contradicts Goldbach's binary hypothesis, the validity of which is established. Hence, there is a prime number in the interval $((p + 1)^2, (p + 2)^2)$. The case when p is even, $p + 1$ is odd and $p + 2$ is an even number is considered similarly. The interval is given in the form $(p^2 - 1, (p + 2)^2 - 1)$. Since the numbers p^2 and $(p + 2)^2$ are both even numbers, shifting the boundaries of the interval to the left by 1 does not affect the final result. The center of this interval is again $(p + 1)^2$, and at the same time it is the center of the representation when the number $2(p + 1)^2$ is represented as a sum of two odd numbers. Further arguments are the same as in the first case for odd p and $p + 2$.

V. Conclusion

Thus, the obtained proof of the binary hypothesis is based on the induction method and uses the properties of the representation of even numbers as the sum of two odd numbers, as well as the rules of formal logic. The validity of the Goldbach binary hypothesis implies the validity of the Legendre hypothesis.

References

[1]. Deshouillers J.–M., Riele H.J.J.te., Sauoter Y. New experimental results concerning the Goldbach conjecture. Amsterdam: Stichting math. centrum, 1998.

[2]. Granville A., Lune J. van de., Riele H.J.J.te. Checking the Goldbach conjecture on a vector computer. Amsterdam: Stichting math. centrum, 1988.

[3]. Jutila M. On the least Golbach's number in an arithmetical progression with a prime difference. Turku, 1968.

[4]. Romanov V.N. Representation of integer positive number as a sum of natural summands. IJERA, V.7, Issue 6 (Part 4), June 2017, pp. 01 – 08.

[5]. Romanov V.N. Study of fundamental problems of number theory. Saint-Petersburg, Publishing house "Asterion", 2015 (in Russian).

Vadim N. Romanov. "Method for solving the Goldbach binary problem." *IOSR Journal of Mathematics (IOSR-JM)*, 16(3), (2020): pp. 13-18.