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Numerical Approximation of Non-Nodal Solutions of Partial Differential Equations with boundary Conditions

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Abstract: *This work focuses on the application of Newton's divided difference interpolation formula to obtain numerical solutions to certain parabolic, hyperbolic and elliptic partial differential equations at non-nodal points in a discretized domain. The finite difference method (FDM) is first applied to solve the initial boundary value problems (IBVPs) to obtain non-nodal point solutions. Interpolating polynomials $P_{i,j}(s)$ are constructed using the nodal points solutions, which are later used to obtain the non-nodal point solutions. The non-nodal point solutions obtained are found to be accurate to one decimal place for the parabolic PDE, two decimal places for the hyperbolic PDE and exact for the elliptic PDE. The numerical results obtained show that this procedure is significantly efficient and accurate. A general interpolating polynomial is also obtained for computation at any other time level t greater than zero.*

Keywords:*non-nodal, partial differential equations, boundary conditions, finite difference method, numerical approximation*

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I. Introduction

The importance of numerical methods in general and finite difference methods (FDM) in particular in solving problems resulting into partial differential equations (IBVPs) cannot be over emphasized. The method focuses on the discretization of a problem domain as a rectangular network of meshes, and numerical solutions $\psi_{i,j}$ at points (s_i, t_j) obtained by replacing the derivatives involved in the PDE by using appropriate difference equations.The implementation of this beautiful method however, has a major set-back, as it fails to be sufficient for use in obtaining numerical solutions at non-nodal points along grid lines as may be required. Thus, this research hopes to address this problem. It is aimed at obtaining numerical solutions $\psi_{i.j}$ to partial differential equations in a discretized domain at non-nodal points.

The introduction of finite difference methods (FDM) has helped tremendously in solving both linear and non-linear differential equations (ordinary and partial) that represents several physical systems, especially those for which analytic solutions cannot readily be obtained, and so the impact of this subject matter cannot be over emphasized. The FDM is extended towards its new stochastic versions for some real systems with random parameters solved before using thetraditional FDM in deterministic cases. One of such extension methods is thegeneralized perturbation-based stochastic technique, where the Taylor seriesexpansions of all random quantities lead to a system of equilibrium equations of ascending order [8].

We shall in this research, however, focus mainly on the numerical solutions of parabolic, hyperbolic and elliptic partial differential equations typifying heat, vibration (wave) and steady state problems. We shall also limit this study to linear partial differential equations.

In this work, we obtain the numerical solutions $\psi_{i,j}$ of parabolic, hyperbolic and elliptic partial differential equations at non-nodal points in discretized domains by employing the technique of NEWTON'S divided difference interpolation, while comparing such numerical solutions obtained with the exact solutions to estimate the absolute error. Furthermore, a general interpolating polynomial for the implementation of the procedure above is obtained.

II. Materials and Methods

Due to the fact that not much work has been done in respect to obtaining numerical solutions to PDEs at non-nodal points in a discretized region, this study outlines the procedure employed in solving the partial differential equations discussed by finite difference methods and also analyses the application of NEWTON'S divided difference interpolation method in obtaining such numerical solutions at the non-nodal points.

2.1 Computational procedure for solving parabolic partial differential equations

From the initial value $\psi(s,0) = p(s)$ of the parabolic PDE, we shall obtain all nodal solutions on the initial time level (t=0). With the boundary values $\psi(0,t) = g(t),\ \psi(l,t) = h(t),\ 0 < t$, the grid point values will be obtained at all time levels along the line s=0 and the line s=l. If the values of λ and h are not given, then they are chosen, which eventually results to the time step length k. using the explicit method, the solutions at all interior grid points are obtained on level 1 with t=0. These computations are repeated for the other required levels using the already computed preceding values [1].

2.2 Computational procedure for solving hyperbolic partial differential equations

$\psi(0,t) = g(t)$ and $\psi(l,t) = h(t), 0 < t$, the boundary conditions, at all time levels will give the solutions along s=0 and s=l. a value for h and k if not given is chosen and this gives the value of the grid parameter λ. $h = k$, if $\lambda = 1$ and $\alpha = 1$. $\psi(s,0) = p(s)$ the initial condition, will at the initial level t=0, provide solutions at all grid points. The solutions required on the level t=k will then be gotten using the initial value $\psi_t(s,0) = g(s)$ and the central difference approximation as follows:

$$\frac{\partial\psi}{\partial t}(s,0) \approx \frac{(\psi_{i,1} - \psi_{i,-1})}{2k} = g(s_i) \tag{1}$$

This however brings in the external point $\psi_{i,-1}$, which when solved results to:

$$\psi_{i,-1} = \psi_{i,1} - 2kg(s_i) \tag{2}$$

Using equation (20), on time level t=k, (j=0), we obtain:

$$2\psi_{i,0} - 2\lambda^2\psi_{i,0} + \lambda^2\psi_{i+1,0} + \lambda^2\psi_{i-1,0} + 2kg(s_i) = \psi_{i,1} \tag{3}$$

If $\frac{\partial\psi}{\partial t}(s,0) = 0$, (i.e $g(s_1) = 0$) is the prescribed initial condition, then equation (2) reduces to $\psi_{i,-1} = \psi\psi_{i,1}$, and the formula (3) becomes:

$$\psi_{i,0} - \lambda^2\psi_{i,0} + \frac{\lambda^2}{2}\psi_{i+1,0} + \frac{\lambda^2}{2}\psi_{i-1,0} = \psi_{i,1} \tag{4}$$

This method will become:

$$\frac{1}{2}\psi_{i+`1,0} + \frac{1}{2}\psi_{i-1,0} = \psi_{i,1} \tag{5}$$

and thus, nodal solutions on the first level (j=1) can be computed. For $j \geq 1$, the two equations

$$\psi_{i,j+1} = 2\psi_{i,j} - 2\lambda^2\psi_{i,j} + \lambda^2\psi_{i+1,j} + \lambda^2\psi_{i-1,j} - \psi_{i,j-1}$$

and

$$\psi_{i,j+1} = \psi_{i-1,j} + \psi_{i+1,j} - \psi_{i,j+1}$$

will be used. These computations for the required number of steps will be repeated [2].

2.3 Computational procedure for elliptic partial differential equations

Here, we partition *[a,b] as h=(b-a)/n*, with width *h*, and also partition [c,d] as *k=(d-c)/m*, with width *k*. further, we set $s_i = ih + a$, for values of *i* ranging from *0* to *n* and $y_j = jk + c$, for values of *j* ranging from *0* to *m*. On the region $\gamma = \{(s,y)|s\epsilon(a,b), y\epsilon(c,d)\}$, we consider the boundary conditions $\psi(s_0, y_j) = h(s_0, y_j)$ and $\psi(s_n, y_j) = h(s_n, y_j)$. Also $\psi(s_i, y_0) = h(s_i, y_0)$ and $\psi(s_i, y_m) = h(s_i, y_m)$ and apply the finite difference equation (30) in obtaining the numerical approximations $\psi_{i,j}$.

2.4 Application of the NEWTON'S divided difference interpolation

After solving the partial differential equations, and the nodal points values obtained by the finite difference methods described above, we shall then, first consider the nodal point values on time level*j=1* that have been obtained by the finite difference method. With these values, we construct a difference table, from which a unique polynomial of degree $\leq$*n-1* will be obtained. With this interpolating polynomial, we can then predict the solutions at other points that are non-nodal on this first time level, by substituting the desired non-nodal value(s) to obtain results [5,6,7].

The numerical results so obtained will then be compared with the analytic solutions, and the absolute errors involved in the computation calculated. This procedure will be repeated at the next time level, and can be done at any desired time level where nodal values have been previously obtained. This will be done without recourse to the solutions obtained at previous time levels. The procedures described so far shall be applied in solving the following numerical problems:

2.5 Numerical problem 1

We apply the Schmidt method to solve the heat conducting equation

$$\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial s^2},\ 0 \le s \le l \tag{6}$$

with the conditions $\psi(s,0) = \sin(\pi s)$, $\psi(0,t) = \psi(1,t) = 0$. Taking $h = \frac{1}{3}$,we compute for two time steps, with $\lambda = \frac{1}{2}$. Given that the analytic solution is $\psi(s,t) = E(-\pi^2 t)\sin(\pi s)$, we use NEWTON'S divided difference interpolation to compute numerical results at the non-nodal points *s=0.5* and *s=0.7* and compare with the exact solution.

2.6 Numerical problem 2

By applying the explicit method, we solve the hyperbolic equation

$$\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial s^2};\ 0 \le s \le 1 \tag{7}$$

with $\psi(s,0) = \sin(\pi s)$, $\psi_t(s,0)$, $s\epsilon[0,l]$, $\psi(0,t) = \psi(1,t), 0 < t$. Taking h=0.25 and k=0.125,we compute for two time steps, given that the exact solution of the PDE is $\psi(s,t) = \cos(\pi t)\sin(\pi s)$. We also use the NEWTON'S divided difference interpolation to compare results with the exact solution.

2.7 Numerical problem 3

We solve the Laplace equation

$$\psi_{ss} + \psi_{yy} = 0 \tag{8}$$

with the boundary values

$$\psi(s,0) = 0, \psi(0,y) = 0, \psi(s,0.5) = 200s, \psi(0.5,y) = 200y \tag{9}$$

Taking n=m=4, we use the NEWTON'S divided difference interpolation to compute numerical results at the non-nodal points s=0.1 and s=0.3, given that the exact solution is $\psi(s,y) = 400sy$. We then compare results with the exact solution.

III. Results and Discussion

In the previous section, we outlined the numerical procedures involved in solving the parabolic, hyperbolic and elliptic PDEs by the finite difference method. We also outlined how the NEWTON'S divided difference interpolation technique can be applied to obtain solutions to these partial differential equations at the discretized domains that are non-nodal. Here, we solve the three numerical problems outlined previously, one for each type of PDE, by the already described methods. The numerical problems considered are those with known exact (analytic) solutions, so that numerical solutions obtained can be easily compared with exact solutions and the absolute errors calculated. Conclusively, a general interpolating polynomial will be obtained for the implementation of the process at any required time level.

3.1 Results for problem 1

The Schmidt formula with $\lambda = \frac{1}{2}$ is expressed as:

$$\psi_{i,j+1} = \frac{1}{2}\psi_{i-1,j} + \frac{1}{2}\psi_{i+1,j} \tag{10}$$

Since $h = \frac{1}{3}$, it implies that there are four nodes on each mesh line, and so solutions at two interior points are required (see figure 1). From $\psi(s,0) = \sin(\pi s)$, the initial value, we have that $\psi(0,0) = \sin(0) = 0$; $\psi\left(\frac{1}{2},0\right) = \sin\left(\frac{\pi}{3}\right) = 0.866025$; $\psi\left(\frac{2}{3},0\right) = sin\left(\frac{2\pi}{3}\right) = 0.866025$; $\psi(1,0) = \sin(\pi s) = 0$.From $\psi(0,t) = \psi(1,t)$, the boundary conditions, we have that $\psi_{0,0} = 0$; $\psi_{0,1} = 0$; $\psi_{0,2} = 0$; $\psi_{1,0} = 0$; $\psi_{1,1} = 0$; $\psi_{1,2} = 0$.

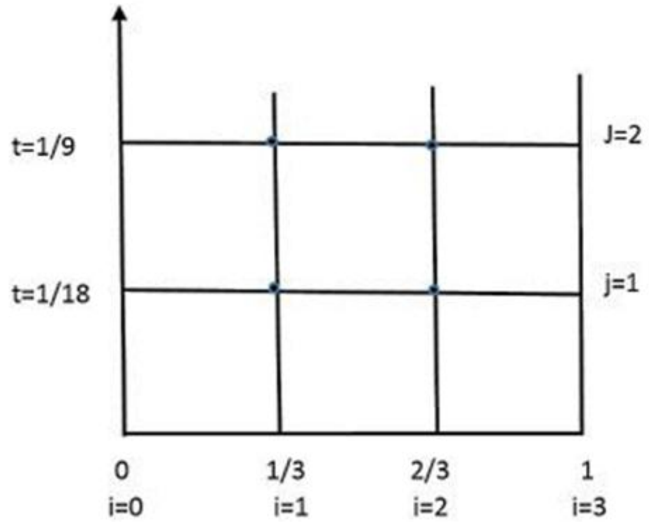

FIG. 1: Problem 1 solution domain

If $\lambda = \frac{1}{2}$ and $h = \frac{1}{3}$, then $k = \lambda h^2 = \frac{1}{18}$. Therefore, computing for two time steps would mean computing up to time step $t = \frac{1}{9}$. The recurrence equation therefore will be

$$\psi_{i,j+1} = \frac{1}{2}\psi_{i-1,j} + \frac{1}{2}\psi_{i+1,j}, \; j = 0,1; \;\; i = 1,2.$$

(11)

Setting j=0, leads to the recurrence equation

$$\psi_{i,1} = \frac{1}{2}\psi_{i-1,0} + \frac{1}{2}\psi_{i+1,0}.$$

(12)

Then, for i=1, we have $\psi_{1,1} = \frac{1}{2}\psi_{0,0} + \frac{1}{2}\psi_{2,0} = \frac{1}{2}0 + \frac{1}{2}0.8660 = 0.433013.$

For i=2, we have $\psi_{2,1} = \frac{1}{2}\psi_{1,0} + \frac{1}{2}\psi_{3,0} = \frac{1}{2}0.866025 + \frac{1}{2}0 = 0.433013.$

Setting j=1,we have the recurrence equation as

$$\psi_{i,2} = \frac{1}{2}\psi_{i-1,1} + \frac{1}{2}\psi_{i+1,1}$$

(13)

Then, for i=1, we have $\psi_{i,2} = \frac{1}{2}(\psi_{0,1} + \psi_{2,1}) = \frac{1}{2}(0 + 0.433013) = 0.216507$

For i=2, we have $\psi_{2,2} = \frac{1}{2}(\psi_{1,1} + \psi_{3,1}) = \frac{1}{2}(0.433013 + 0) = 0.216507$

We now apply the NEWTON'S divided difference interpolation to compute the values at the required non-nodal points. From the first time level j=1, we obtained nodal point values by the finite difference method as shown on table 1a, the values $\psi_{i,1}$ representing the nodal temperatures obtained from the finite difference method. The divided difference table for the values on time level j=1 is shown on table 1b.

Now, let the interpolating polynomial be denoted as $P_{i,j}(s)$ where i denotes the degree of the interpolating polynomial and j the computed time level, for j=0,1,2,3,…. From the NEWTON'S divided difference table (see table 1b), we have the interpolating polynomial of degree two for time level j=1, i.e at $\mathrm{t} = \frac{1}{18}$ as

$$P_{2,1}(s) = \psi(s_0) + (s - s_0)\psi[s_0, s_1] + (s - s_0)(s - s_1)\psi[s_0, s_1, s_2]$$

(14)

Substituting values from the table, we then have

$$P_{2,1}(s) = s(1.29939) + s\left(s - \frac{1}{3}\right)(-1.9485585) = 1.299039s - 1.9485585s\left(s - \frac{1}{3}\right).$$

From the interpolating polynomial (14), we compute the value at the non-nodal point s=0.2 as follows:

$$P_{2,1}(0.2) = (0.2)(1.29939) + (0.2)\left(0.2 - \frac{1}{3}\right)(-1.9485585) = 0.2598078 + 0.05196156 = 0.31176936.$$

Computing the exact solution $\psi(s,t)$ at the point $\psi(s,t) = \psi(0.2, 1/18)$, we obtain

$$\psi\left(0.2, \frac{1}{18}\right) = \exp(-\pi^2 t)\sin(\pi s) = \exp\left(-\frac{\pi^2}{18}\right)\sin(0.2\pi) = 0.577924896 \times 0.587785252 = 0.33969571$$

The absolute error is $|0.33969571 - 0.311769361| = 0.0279266349.$

Similarly, at the non-nodal point s=0.7, we have the following:

$$P_{2,1}(0.7) = 1.299039(0.7) - 1.9485585(0.7)\left(0.7 - \frac{1}{3}\right) = 0.9093273 - 0.50130015 = 0.409197285$$

Computing the exact solution $\psi(s,t)$ at $\psi(s,t) = \psi 0.7, 1/18$, we obtain

$\psi(s,t) = \exp(-\pi^2 t)\sin(\pi s) = \exp\left(-\frac{\pi^2}{18}\right)\sin(0.7\pi) = 0.577924896 \times 0.809016994 = 0.46755106.$

The absolute error is $|0.467551062 - 0.409197285| = 0.056353777.$

Again, on the second time level, i.e, atj $= 2$ $(t = 1/9)$, we obtained nodal point values as shown on table 1c, the divided difference table for the values on time level j=2 being shown on table 1d. From the divided difference table, a degree two interpolating polynomial for time level j=2, i.e, at t=1/9 is obtained as follows:

$P_{2,2}(s) = \psi(s_0) + (s - s_0)\psi[s_0, s_1] + (s - s_0)(s - s_1)\psi[s_0, s_1, s_2] = s(0.649521) + s(s - 1/3) \times (-0.9742815) = 0.649521s - 0.9742815s(s - 1/3).$

From the interpolating polynomial above, we compute the value at the non-nodal point s=0.2 (at t=1/9) as follows:

$P_{2,2}(0.2) = (0.2)(0.649521) - (0.2)\left(0.2 - \frac{1}{3}\right)(0.9742815) = 0.1299042 + 0.02598084 = 0.15588504.$

Computing the exact solution $\psi(s,t)$ at the point $\psi(s,t) = \psi(0.2, 1/9)$, we obtain

$$\psi\left(0.2, \frac{1}{9}\right) = \exp(-\pi^2 t)\sin(\pi s) = \exp\left(-\frac{\pi^2}{9}\right)\sin(0.2\pi) = 0.333997186 \times 0.587785252 = 0.19631862$$

The absolute error is $|0.19631862 - 0.15588504| = 0.04043358.$ Similarly, at the non-nodal point s=0.7, we have the following:

$P_{2,2}(0.7) = 0.64952(0.7) - 0.9742815(0.7)\left(0.7 - \frac{1}{3}\right) = 0.454664 - 0.250065585 = 0.204598415.$

Computing the exact solution $\psi(s,t)$ at $\psi(s,t) = \psi(0.7, \frac{1}{9})$, we obtain

$\psi(s,t) = \exp(-\pi^2 t)\sin(\pi s) = \exp\left(-\frac{\pi^2}{9}\right)\sin(0.7\pi) = 0.333997186 \times 0.809016994 = 0.270209249.$

The absolute error is $|0.270209249 - 0.2045998415| = 0.065610834.$

TABLE 1a: Nodal point values for level j=1

s_i	0	1/3	2/3	1
$\psi_{i,1}$	0	0.433013	0.433013	0

TABLE 1b: Divided difference table for level j=1

I	s_i	$\psi_{i,1}$	First divided difference	Second divided difference	Third divided difference
0	0	0			
			1.299039		
1	1/3	0.433013		-1.9485585	
			0		0
2	2/3	0.433013		-1.9485585	
			-1.299039		
3	1	0			

TABLE 1c: Nodal point values for level j=2

s_i	0	1/3	2/3	1
$\psi_{i,2}$	0	0.216507	0.216507	0

TABLE 1d: Divided difference table for level j=2

I	s_i	$\psi_{i,1}$	First divided difference	Second divided difference	Third divided difference
0	0	0			
			0.649521		
1	1/3	0.216507		-0.9742815	
			0		0
2	2/3	0.216507		-0.9742815	
			-0.649521		
3	1	0			

3.2 Results for problem 2

The explicit formula is given by

$$2\psi_{i,j} - 2\lambda^2\psi_{i,j} + \lambda^2\psi_{i+1,j} + \lambda^2\psi_{i-1,j} - \psi_{i,j-1} = \psi_{i,j+1}$$

(15)

Since $\alpha = 1$ and $h = 1/4$, it implies that there are five nodes on each time level and so solutions at three interior grid points will be needed (see figure 2). $\psi(s,0) = \sin(\pi s)$being the initial value yields $\psi(0,0) =$

$\sin(0) = 0$; $\psi\left(\frac{1}{4,0}\right) = \sin\left(\frac{\pi}{4}\right) = 0.70711$; $\psi\left(\frac{1}{2,0}\right) = \sin\left(\frac{\pi}{2}\right) = 1$; $\psi\left(\frac{3}{4,0}\right) = \sin\left(\frac{3\pi}{4}\right) = 0.70711$; $\psi(1,0) = \sin(\pi) = 0$.

The initial value $\frac{\partial\psi}{\partial t}(s,0) = 0$ yields $\psi_{i,-1} = \psi_{i,1} * \psi(0,t) = \psi(1,t) = 0$. The boundary conditions, for all values of j yields $\psi_{0,j} = \psi_{4,j} = 0$. The grid parameter $\lambda = \frac{k}{h} = \frac{4}{8} = 1/2$, and so the recurrence equation becomes

$$\psi_{i,j+1} = \frac{1}{4}\psi_{i+1} + \frac{1}{4}\psi_{i-1,j} - \psi_{i,j-1} + 2\left(1-\frac{1}{4}\right)\psi_{i,j} = 0.25\psi_{i-1,j} + 1.5\psi_{i,j} + 0.25\psi_{i+1,j} - \psi_{i,j-1} \quad (16)$$

We require computations to be done up to time level t=1/4 (i.e 2 time step). For j=0, we have that $\psi_{i,-1} = \psi_{i,1}$[3] resulting to

$$\psi_{i,1} = \left(1-\frac{1}{4}\right)\psi_{i,0} + \frac{1}{8}\left(\psi_{i+1,0} + \psi_{i-1,0}\right) = 0.75\psi_{i,0} + 0.125\psi_{i+1,0} + 0.125\psi_{i-1,0}.$$
(17)

Then, for i=1, we obtain

$\psi_{1,1} = 0.75\psi_{1,0} + 0.125\psi_{2,0} + 0.125\psi_{0,0} = 0.75(0.70711) + 0.125 \times 1 + (0.125 \times 0) = 0.65533.$

For i=2, we obtain

$\psi_{2,1} = 0.75\psi_{2,0} + 0.125\psi_{3,0} + 0.125\psi_{1,0} = 0.75 \times 1 + 0.125 \times 1 + 0.125(0.70711) = 0.92678.$

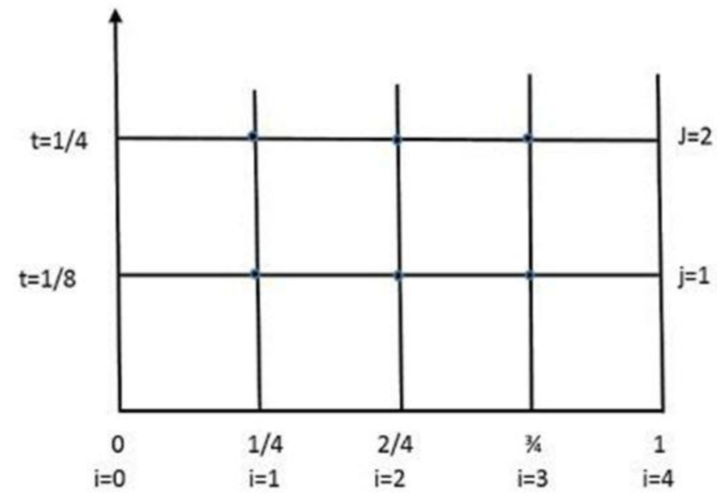

FIG. 2: Problem 2 solution domain

For i=3, we obtain

$$\psi_{3,1} = 0.75\psi_{3,0} + 0.125\psi_{4,0} + 0.125\psi_{2,0} = 0.75(0.70711) + 0.125 \times 0 + 0.125 \times 1 = 0.65533$$

At level j=1, we use the recurrent relation (16), with j=1, to obtain

$$\psi_{i,2} = 1.5\psi_{i,1} + 0.25\psi_{i+1,1} + 0.25\psi_{i-1,1} - \psi_{i,0}.$$

Then, for i=1, we obtain

$\psi_{1,2} = \frac{15}{10}\psi_{1,1} + \frac{25}{100}\psi_{2,1} + \frac{25}{100}\psi_{0,1} + \psi_{1,0} = \frac{15}{10}(0.65533) + \frac{25}{100} \times 0.92678 + (0.125 \times 0) - 0.70711 = 0.50758.$

For i=2, we obtain

$\psi_{2,2} = 1.5\psi_{2,1} + 0.25\psi_{3,1} + 0.25\psi_{1,1} + 0.25\psi_{2,0} = 1.5(092678) + 0.25 \times 0.65533 + (0.25 \times$ *0.65533−1.0=071784.*

For i=3, we obtain

$$\psi_{3,2} = 1.5\psi_{3,1} + 0.25\psi_{4,1} + 0.25\psi_{2,1} + 0.25\psi_{3,0}$$
$$= 1.5(0.65533) + 0.25 \times 0.92678 + (0.25 \times 0) - 0.70711 = 0.50758.$$

We shall now apply the NEWTON'S divided difference interpolation technique to calculate solutions that are non-nodal points. From the first time level j=1, the nodal point values shown on table 2a are obtained. The divided difference table for the values on this level is shown on table 2b. we see from the table that the interpolating is a degree four polynomial of the form

$P_{4,1}(s) = \psi(s_0) + (s-s_0) \times \psi[s_0,s_1] + (s-s_0) \times (s-s_1) \times \psi[s_0,s_1,s_2] + (s-s_0) \times (s-s_1) \times (s-s_2) \times \psi[s_0,s_1,s_2,s_3] + (s-s_0) \times (s-s_1) \times (s-s_2) \times (s-s_3) \times \psi[s_0,s_1,s_2,s_3,s_4].$

This implies that

$P_{4,1}(s) = 2.62132s - 3.07104s(s-0.25) - 1.69621s(s-0.25)(s-0.5) + 3.38831s(s-0.25) \times (s-0.5) \times (s-0.75)$.

We shall interpolate at the points s=0.2 and s=0.7 (t=1/8, j=1). At s=0.2, we have

$P_{4,1}(0.2) = 2.62132(0.2) - 3.07104(0.2)(0.2-0.25) - 1.69621(0.2)(0.2-0.25)(0.2-0.5) + .38831(0.2)(0.2-0.25)(0.2-0.5(0.2-0.75))$. (18)

This implies that

$P_{4,1}(0.2) = 0.524264 + 0.0307104 - 0.00508863 - 0.0055907115 = 0.544295058$.

The exact solution $\psi(s,t)$ at the point $\psi(s,t) = (0.2,1/8)$ is

$\psi(s,t) = \cos(\pi t)\sin(\pi s) = \cos\left(\frac{\pi}{8}\right)\sin(0.2\pi) = 0.543042764$.

The absolute error is $|0.544295058 - 0.543042764| = 0.00125229$.

At s=0.7, we have

$P_{4,1}(0.7) = 2.62132(0.7) - 3.07104(0.7)(0.7-0.25) - 1.69621(0.7)(0.7-0.25)(0.7-0.5) + 3.38831(0.7)(0.7-0.25)(0.7-0.5)(0.7-0.75))$.

This implies that

$P_{4,1}(0.7) = 1.834924 - 0.9673776 - 0.10686123 - 0.010673176 = 0.750011994$.

The exact solution $\psi(s,t)$ at the point $\psi(s,t) = (0.7,1/8)$ will be

$\psi(s,t) = \cos(\pi t)\sin(\pi s) = \cos\left(\frac{\pi}{8}\right)\sin(0.7\pi) = 0.747434242$.

The absolute error is $|0.750011994 - 0.747434242| = 0.00257775$.

Similarly, the data values at time level j=2 (t=1/4) are shown on table 2c, and the table of divided differences for the values at this time level is shown on table 2d. We see from the table that the interpolating polynomial is a degree 4 polynomial of the form

$$P_{4,2}(s) = \psi(s_0) + (s-s_0) \times \psi[s_0,s_1] + (s-s_0) \times (s-s_1) \times \psi[s_0,s_1,s_2] + (s-s_0) \times (s-s_1) \times (s-s_2) \times \psi[s_0,s_1,s_2,s_3] + (s-s_0) \times (s-s_1) \times (s-s_2) \times (s-s_3) \times \psi[s_0,s_1,s_2,s_3,s_4]$$

This implies that

$P_{4,2}(s) = 2.03032s - 2.45856s\left(s - \frac{25}{100}\right) - 0.994133s\left(s - \frac{5}{10}\right) + 4.4468263s(s - 25/100)(s - 5/10)(s - 75/100)$.

We shall now interpolate at the points s=0.2 and s=0.7 (t=1/4, j=2). At s=0.2, we have

$P_{4,2}(0.2) = 2.03032(0.2) - 2.45856(0.2)(0.2-0.25) - 0.994133(0.2)(0.2-0.25)(0.2-0.5) + 4.4468263(0.2)(0.2-0.25)(0.2-0.5)(0.2-0.75)$. (19)

This implies that

$P_{4,2}(0.2) = 0.406064 + 0.0245856 - 0.002982399 - 0.00733726 = 0.420300337$.

The analytic solution $\psi(s,t)$ at the point $\psi(s,t) = \psi(0.2,1/4)$ is

$\psi(s,t) = \cos(\pi t)\sin(\pi s) = \cos\left(\frac{\pi}{4}\right)\sin(0.2\pi) = 0.415626937$.

The absolute error is $|0.415626937 - 0.420300337| = 0.004673399223$.

At s=0.7, we have

$P_{4,2}(0.7) = 2.03032(0.7) - 2.45856(0.7)(0.7-0.25) - 0.994133(0.7)(0.7-0.25)(0.7-0.5) + 4.4468263(0.7)(0.7-0.25)(0.7-0.5)(0.7-0.75)$. (20)

TABLE 2aNodal point values for level j=1

s_i	0	0.25	0.5	0.75	1
$\psi_{i,1}$	0	0.65533	0.92678	0.65533	0

TABLE 2b Divided difference table for level j=1

i	s_i	$\psi_{i,1}$	First divided difference	Second divided difference	Third divided difference	Fourth divided difference
0	0	0				
			2.62132			
1	0.25	0.65533		-3.07104		
			1.0858		-1.69621	
2	0.5	0.92678		-4.3432		3.38831
			-1.0858		1.69621	
3	0.75	0.65533		-3.07104		
			-2.62132			
4	1	0				

TABLE 2c Nodal point values for level j=2

s_i	0	0.25	0.5	0.75	1
$\psi_{i,1}$	0	0.5078	0.70784	0.50758	0

TABLE 2d Divided difference table for level j=2

i	s_i	$\psi_{i,1}$	First divided difference	Second divided difference	Third divided difference	Fourth divided difference
0	0	0				
			2.03032			
1	0.25	0.5078		-2.45856		
			0.80104		-0.994133	
2	0.5	0.70784		-3.20416		4.4468253
			-0.80104		3.4526933	
3	0.75	0.50758		-0.61464		
			-2.03032			
4	1	0				

This implies that
$P_{4,2}(0.7) = 1.421224 - 0.7744464 - 0.062630379 - 0.014007502 = 0.570139718.$
The analytic solution $\psi(s,t)$ at the point $\psi(s,t) = \psi(0.7,1/4)$ will be
$\psi(s,t) = \cos(\pi t)\sin(\pi s) = \cos\left(\frac{\pi}{4}\right)\sin(0.7\pi) = 0.572061402.$
The absolute error is $|0.572061402 - 0.570139718| = 0.001921684818.$

3.3 Results for problem 3

Setting $\lambda = 1$ and $f_{i,j} = 0$ into the recurrence equation

$$\psi_{i+1,j} + \lambda^2\psi_{i,j+1} - 2 + 2\lambda^2\psi_{i,j} + \lambda^2\psi_{i,j-1} + \psi_{i-1,j} = h^2 f_{i,j} \quad , \qquad (21)$$

we obtain

$$4\psi_{i,j} - \psi_{i-1,j} - \psi_{i+1,j} - \psi_{i,j+1} - \psi_{i,j-1} = 0; \;\; i = 1,2,3; \;\; j = 1,2,3 \qquad (22)$$

Using the transformation$\psi_{i,j} = P_i$, with $l = i + (m-1-j)(n-1)$, we obtain the nodal notations for the interior nodal points of the required region (see figure 3) as $\psi_{1,3} = P_1,\ \psi_{2,3} = P_2,\ \psi_{3,3} = P_3, \psi_{1,2} = P_4,$ $\psi_{2,2} = P_5,\ \psi_{3,2} = P_6,\ \psi_{1,1} = P_7,\ \psi_{2,1} = P_8,\ \psi_{3,1} = P_9.$
The boundary condition $\psi(s,0) = 0$ yields the conditions $\psi(0,0) = 0,\ \psi(1,0) = 0,\ \psi(2,0) = 0,\ \psi(3,0) = 0,$ $\psi(4,0) = 0$. For the boundary condition $\psi(0,y) = 0$, we obtain $\psi(0,1) = 0,\ \psi(0,2) = 0,\ \psi(0,3) = 0,$ $\psi(0,4) = 0$. The boundary condition $\psi(0.5,y) = 200y$ yields $\psi(4,0) = 0,\ \psi(4,1) = 25,\ \psi(4,2) = 50,$ $\psi(4,3) = 75,\ \psi(4,4) = 100$. From the boundary condition $\psi(s,0.5) = 20s$, we obtain
$\psi(1,4) = 25; \quad \psi(2,4) = 50; \quad \psi(3,4) = 75.$

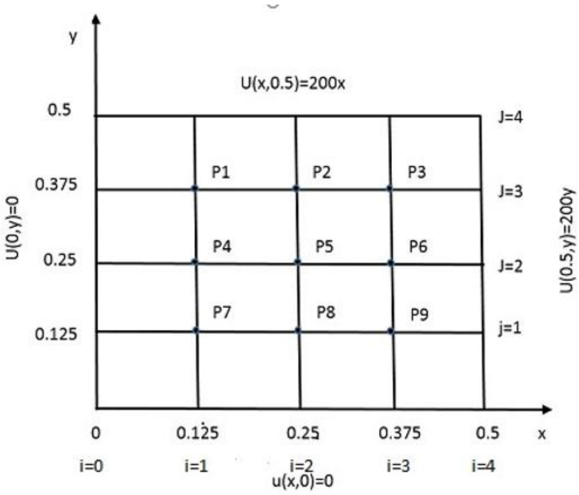

Figure 3. Problem 3 solution domain

The linear equations associated with the interior nodes of the region are as follows:
For $P_1 = \psi_{1,3}$, we obtain $4p\psi_{1,3} - \psi_{2,3} - \psi_{0,3} - \psi_{1,2} - \psi_{1,4} = 0,$

$$4P_1 - P_2 - P_4 = 25. \qquad (23)$$

For $P_2 = \psi_{2,3}$, we obtain $4p\psi_{2,3} - \psi_{1,3} - \psi_{3,3} - \psi_{2,2} - \psi_{2,4} = 0$

$4P_2 - P_3 - P_1 - P_5 = 50.$

(24)

For $P_3 = \psi_{3,3}$, we obtain $4p\psi_{3,3} - \psi_{2,3} - \psi_{4,3} - \psi_{3,2} - \psi_{3,4} = 0$

$4P_3 - P_2 - P_6 = 150.$

(25)

For $P_4 = \psi_{1,2}$, we obtain $4p\psi_{1,2} - \psi_{2,2} - \psi_{0,2} - \psi_{1,1} - \psi_{1,3} = 0$

$4P_4 - P_5 - P_7 - P_1 = 0.$

(26)

For $P_5 = \psi_{2,2}$, we obtain $4p\psi_{2,2} - \psi_{3,2} - \psi_{1,2} - \psi_{2,1} - \psi_{2,3} = 0$

$4P_5 - P_6 - P_4 - P_8 - P_2 = 0.$

(27)

For $P_6 = \psi_{3,2}$, we obtain $4p\psi_{3,2} - \psi_{4,2} - \psi_{2,2} - \psi_{3,1} - \psi_{3,3} = 0$

$4P_6 - P_5 - P_9 - P_3 = 50.$

(28)

For $P_7 = \psi_{1,1}$, we obtain $4p\psi_{1,1} - \psi_{2,1} - \psi_{0,1} - \psi_{1,0} - \psi_{1,2} = 0$

$4P_7 - P_8 - P_4 = 0.$

(29)

For $P_8 = \psi_{2,1}$, we obtain $4p\psi_{2,1} - \psi_{3,1} - \psi_{1,1} - \psi_{2,0} - \psi_{2,2} = 0$

$4P_8 - P_9 - P_7 - P_5 = 0.$

(30)

For $P_9 = \psi_{3,1}$, we obtain $4p\psi_{3,1} - \psi_{4,1} - \psi_{2,1} - \psi_{3,0} - \psi_{3,2} = 0$

$4P_9 - P_8 - P_6 = 25$

(31)

With the equations thus obtained, we have the following linear system:

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} p1 \\ p2 \\ p3 \\ p4 \\ p5 \\ p6 \\ p7 \\ p8 \\ p9 \end{bmatrix} = \begin{bmatrix} 25 \\ 50 \\ 150 \\ 0 \\ 0 \\ 50 \\ 0 \\ 0 \\ 25 \end{bmatrix}$$

(32)

In solving equation (32), which is a 9×9 matrix system, we use the matlab code $A\hat{b}$, where A is the coefficient matrix, and b the constant vector [4]. The matrix is diagonally dominant. We obtain the solutions to this matrix system as shown on table 3a. The nodal point values at level j=1 are obtained as shown on table 3b. The divided difference table associated with the values at this point is shown on table 3c. We see that the nodal values on this level (j=1) is represented by a first degree polynomial of the form

$P_{1,1}(s) = \psi(s_0) + (s - s_0)\psi[s_0, s_1].$

(33)

This implies that

$P_{1,1}(s) = 50s.$

(34)

Interpolating at the point s=0.1, we obtain $P_{1,1}(0.1) = 50(0.1) = 5.$ The analytic value $\psi(s, y)$ at the point $\psi(s, y) = \psi(0.1, 0.125)$ i.e on level j=1 (t=0.125) is obtained as $\psi(s, y) = 400sy = 400(0.1)(0.125) = 5.$
The absolute error is $|5 - 5| = 0.$

Interpolating at the point s=0.3, we obtain $P_{1,1}(0.3) = 50(0.3) = 15.$ The analytic value $\psi(s, y)$ at the point $\psi(s, y) = \psi(0.3, 0.125)$, i.e on level j=1 (t=0.125) is obtained as $\psi(s, y) = 400sy = 400(0.3)(0.125) = 15.$
The absolute error is $|15 - 15| = 0.$

The nodal values at level j=2 are shown on table 3d. The divided difference table associated with the values at this level is shown on table 3e. We see that the nodal point values on this level (j=2) is represented by a first degree polynomial of the form

$P_{1,2}(s) = \psi(s_0) + (s - s_0) \times \psi[s_0, s_1]$.
(35)

This implies that

$P_{1,2}(s) = 100s$
(36)

Interpolating at the point s=0.1, we obtain $P_{1,2}(0.1) = 100(0.1) = 10$. The analytic value $\psi(s,y)$ at the point $\psi(s,y) = \psi(0.1,0.25)$ i.e on level j=1 (t=0.125) is obtained as $\psi(s,y) = 400sy = 400(0.1)(0.25) = 10$. The absolute error is $|10 - 10| = 0$.

TABLE 3a All nodal point values for the system (32)

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
18.75	37.50	56.25	12.50	25.00	37.50	6.25	12.50	18.75

TABLE 3b Nodal point values for level j=1

s_i	0	0.125	0.250	0.375	0.5
$\psi_{i,1}$	0	6.25	12.50	18.75	25.00

TABLE 3c Divided difference table for level j=1

I	s_i	$\psi_{i,1}$	First divided difference	Second divided difference
0	0	0		
			50	
1	0.125	6.25		0
			50	
2	0.25	12.50		0
			50	
3	0.375	18.75		0
			50	
4	0.5	25.00		

TABLE 3d Nodal point values for level j=2

s_i	0	0.125	0.250	0.375	0.5
$\psi_{i,1}$	0	12.50	25.00	37.50	50.00

TABLE 3e Divided difference table for level j=2

I	s_i	$\psi_{i,1}$	First divided difference	Second divided difference
0	0	0		
			100	
1	0.125	12.50		0
			100	
2	0.25	25.00		0
			100	
3	0.375	37.50		0
			100	
4	0.5	50.00		

Interpolating at the point s=0.3, we obtain $P_{1,2}(0.3) = 100(0.3) = 30$. The analytic value $\psi(s,y)$ at the point $\psi(s,y) = \psi(0.3,0.25)$ i.e on level j=1 (t=0.125) is obtained as $\psi(s,y) = 400sy = 400(0.3)(0.25) = 30$. The absolute error is: $|30 - 30| = 0$.

3.4 General interpolating polynomial

The general interpolating for the implementation of the above process is given as follows: suppose on t-level, j, we need to interpolate at $s_i, i = 0,1,\ldots,m, t = t_j$. Then the interpolating polynomial for this level, written as $P_m(s)$ is

$P_j^m(s) = \sum_{i=0}^{m} l_i(s)\psi(s_i)$,
(37)

where $\psi(s_0) \equiv \psi(s_i, t_j)$ and $l_i(s)$ are Lagrange basis functions, each of degree m. Thus, to find $\psi(\alpha, t_j)$, we substitute into

$P_j^m(\alpha) = \sum_{i=0}^{m} l_i(\alpha)\psi(s_i)$, $s_i \neq \alpha$ for all i. (39)

IV. Conclusion

In this research, the numerical solutions $\psi_{i,j}$ to certain parabolic, hyperbolic and elliptic partial differential equations with boundary conditions were investigated. The test problems considered in this work were those with known exact solutions. For the parabolic partial differential equation it was observed that the computations of the non-nodal solutions were accurate to one decimal place, while for the hyperbolic partial differential equation, it was observed that the computations of the non-nodal solutions were accurate to two decimal places. However, the computations for the elliptic partial differential equation were exact. The exactness of this solution (elliptic) was due to the linearity of the analytic solution of the elliptic problem. A general interpolating polynomial was obtained, which can be used for the computation of non-nodal point values at any level t>0, once the nodal point values have been obtained by the finite difference methods. We also constructed and developed polynomials $P_{i,j}(s)$ using the NEWTON'S divided difference interpolation procedure for each time level t=jk for the purpose of obtaining the non-nodal point values. Our numerical results show that this procedure is significantly efficient and accurate.

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