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An investigation into computing the digits of pi

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Abstract: *There have been a countless number of methods for computing the value of π , with some dating back to the period of Ancient Babylonians. In this paper, the authors investigate a one-dimensional scenario involving two balls and a wall, to enumerate the digits of pi. Several assumptions follow the scenario, one of which pertains to the perfectly elastic nature of collisions involved. Then, the paper proceeds to analyze the movement of the balls by transforming the movement of the balls into a coordinate plane. This was done through the usage of phase diagrams with regards to energy and momentum. The coordinate planes were expressed as vectors to proceed with the mathematical derivation, in which the small angle approximation for arctan(x) was proven through definite integration of inverse trigonometric functions and other graphical elements.*

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I. Introduction

The number$\pi$is commonly defined as the quotient of a circular object's circumference with its diameter. The value was first computed by Ancient Babylonians, through the calculation of a circle's area using the formula$3r^2$. This resolved pi as a value of 3, which we recognize as the first digit of π today (1).

Following the calculation of the first digit, Egyptians computed the value of π by equating the area of a square of side 8 and the area of a circle of diameter 9. This yielded a value of 3.1605, which was a closer approximation of π (2). One of the most notable attempts to approximate the value of π was Archimedes' method. He measured the perimeter of two polygons: one inscribed within the circle and another circumscribed around the circle. Archimedes concluded that the value of π must be within the range of both values. Through this method, he proved that π was between $3\frac{1}{7}$ and $3\frac{10}{71}$(3).

There were multiple breakthroughs in the approximation of π, such as Gottfried Leibniz's (1646-1716) formula for computing π, and Isaac Newton's (1642-1727) application of an infinite series to calculate 15 digits of π. Despite these breakthroughs, there have been interesting experimental methods to compute digits of π with great accuracy, such as the "Buffon Game"(4).

In a paper written by G.Galperin, the author introduces a peculiar method to calculate the value of π: counting the total number of collisions that occur between two balls and a wall along an axis(5, 6). Although this may not be the most efficient method, it is an intuitive method that is able to enumerate digits of π(6). With regards to the paper of Galperin, this paper will aim to provide a more coherent approach to justify the method shown.

II. Proposed Scenario

In a paper written by Galperin, the author introduces an interesting approach to calculate the digits of π. The one-dimensional billiard scenario involves two balls: one with mass m and another with mass M, where$M \geq m$. The ball of mass M approaches the ball of mass m at rest, resulting in a collision between both balls. The ball of mass$m$, traveling with the velocity vector from mass$M$, approaches a wall located on the left of both balls, colliding and reflecting off it. To iterate the digits of pi in this simulation, the ratio of the masses must be$\frac{M}{m} = 100^n$, where $n \in Z^+$. Additionally, we simplified the scenario such that $m = 1\ kg$,$M = 100^n$. Our focus lies on the total number of collisions, which is the total number of collisions between the two balls and between ball m and the wall. The two balls are considered as point particles and thus only experience translation motion (no circular mechanics involved). With all the conditions set, we can objectify the main argument. Let the mass of ball M increase by a factor of 100^N greater than the mass of the ball m and count the total number of collisions correlated to N. Let N be a fixed positive integer within the equation $M = m * 100^N$, and Π denote the number of collisions. In our proposed scenario,$m = 1\ kg$, therefore this equation can be simplified to $M = 100^N$.For different values of N, the system will output different values for the number Π. Since the value of Π is related to the value of N, we can create $\Pi = \Pi(N)$, a function of the exponent N of the number 100^N.

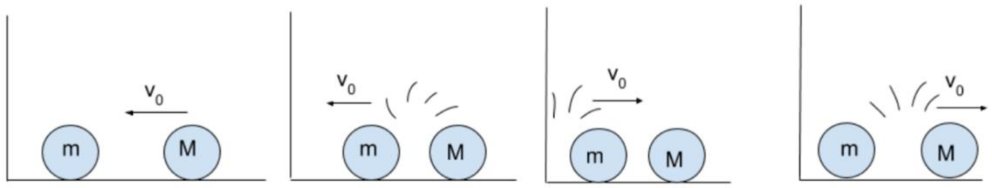

Figure 1

Let us investigate the simplest case $N = 0$, which corresponds to the equality of the masses: $m = M$. Consider a system in which there are two balls and a wall of infinite mass. Initially, ball $m$is stationary and is placed in between the wall and ball$M$. From the right side, ball M moves with a constant velocity of v_0 and collides with ball m. Then, ball m begins to approach the wall and collides with it. Ball m will then change direction and head towards ball m_2, resulting in another collision.

Then, the total number of hits in the system when $m = M$ is 3: two collisions and one reflection. Thus, $\Pi(0) = 3$. Note that 3 is the first digit of π. When the process repeats for $N = 1$ the number of hits, Π, is 31 (two first digits of π) and 314 (three digits of π) for $N = 2$, respectively.

Thus, it is plausible to say that the number of collisions will return $N + 1$ digits of pi when increasing the mass of ball M by a factor of 100: $\Pi(N) = [\pi \times 10^N], where\ N = Z^+$. To examine this scenario in a more accurate way, we have simulated the scenario using a computer simulation. The data collected was recorded in table 1. It can be observed that as the value of n increases by 1, the number of collisions presents another digit of pi, being closer to the literature value.

Table 1-Computer simulation showing the number of collisions when mass M was increased by a factor of 100

n	Mass of M (kg)	Number of collisions
1	100	3
2	10000	31
3	1000000	314
4	100000000	3141
5	10000000000	31415

In the scenario, we assume that the balls are moving on a frictionless surface, such that their velocity vectors are not affected by external factors. Furthermore, we assume all collisions occurring to be perfectly elastic. Elastic collisions are characterized by the conservation of kinetic energy within a system.

Therefore, $\frac{1}{2}mv_1 + \frac{1}{2}Mv_2{}^2 = constant$. Since the masses of both blocks stay constant during the collision, velocities must be conserved (since kinetic energy stays constant). The corollary of the conservation of energy is the conservation of momentum, or $mv_1 + Mv_2 = constant$. Therefore, the proposed scenario obeys the conservation of momentum and conservation of kinetic energy.

The scenario is also conducted under the assumption that the wall has infinite mass. According to the law of conservation of momentum, $m_i v_{1i} + M_i v_i = m_f v_{1f} + M_f v_f$. In order for the ball of mass m to conserve its velocity, the initial velocity prior to the collision with the wall must be equal to the final velocity after colliding with the wall ($v_{1i} = v_{1f}$). Since the initial velocity of the wall is $0\ m/s$, the law of conservation of momentum simplifies to $m_i v_{1i} = m_f v_{1f} + M_f v_f$. To achieve the wanted scenario, the final momentum of the wall must remain at 0, which is only possible when the mass of the wall is maximized (approaching infinity).

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III. Mathematical Proof

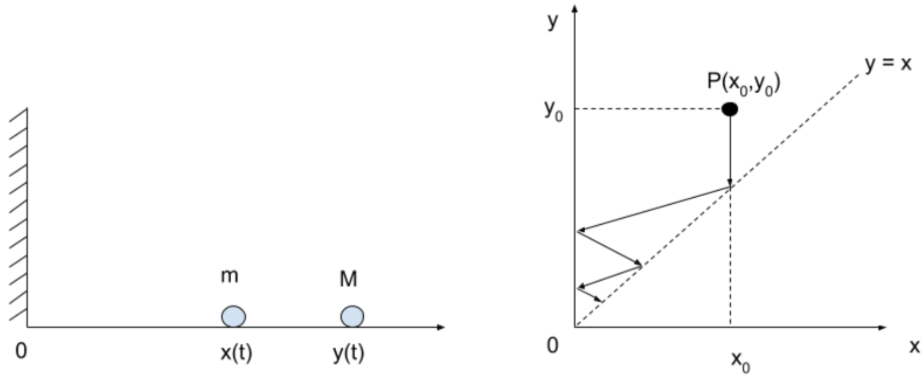


Figure 2

Consider a one-dimensional scenario in which there exists an immovable wall at $x = 0$, ball m at position $x(t)$, ball M at position $y(t)$, and a point $P\big(x(t), y(t)\big)$. By setting the initial time $t = 0$, the movement of the balls over time can be represented by the movement of point P over time. Therefore, when Pintersects the line $y = x$, ball m and M are at the same position, experiencing a collision. Additionally, when Pintersects the y-axis, ball m is at position $x = 0$ (which was previously determined to be the position of the wall), experiencing a collision with the wall. This results in the velocity vector of ball m being opposite in direction and equal in magnitude.Let us define the velocity vector for the rate of change in P with respect to time as

$$\overrightarrow{W} = (\dot{x}(t), \dot{y}(t)) = (u, v)$$

After expressing the movement in a vector equation, we can then look at the movements of the ball with regards to momentum and energy. The law of conservation of momentum holds in the given scenario:

$$mu + Mv = constant, for\ all\ \forall t$$

Given that all collisions are elastic, the conservation of kinetic energy applies. Then, the following holds:

$$\frac{1}{2}mu^2 + \frac{1}{2}Mv^2 = constant, for\ all\ \forall t$$

The corollary to the conservation of kinetic energy is the conservation of energy, which can be represented as $mu^2 + Mv^2 = constant$. When observing the collision between ball m_1 and the wall, we notice that the change in velocity is zero. Let us denote the velocity of ball m_1 before the collision as u_i and the velocity following the collision asu_f. As the law of the conservation of energy states,

$$m{u_i}^2 + Mv^2 = mu_f^2 + Mv^2 \rightarrow {u_i}^2 = {u_f}^2$$

Thus $v_i = -v_f$holds. Then, the velocity vector prior to the collision ($\overrightarrow{W_i}$) and the velocity vector following the collision ($\overrightarrow{W_f}$) are:

$$\overrightarrow{W_i} = (v_i, v), \overrightarrow{W_f} = (-v_f, v)$$

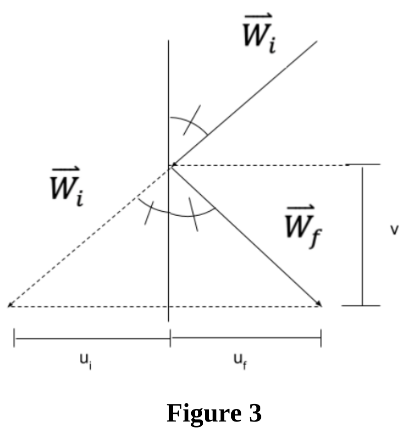


Figure 3

The diagram above depicts an integral relationship between the angle of incidence and the angle of reflection. The law of reflection states that the angle of incidence is equal to the angle of refraction. Let the angle between $Y - axis$ and w_i be the angle of incidence, the angle between$Y - axis$ and w_f be the angle of reflection. Then, the angle of incidence and the angle of reflection between point $P$and the $Y - axis$ are always equal. We can transform the diagram to $X - Y$coordinates.

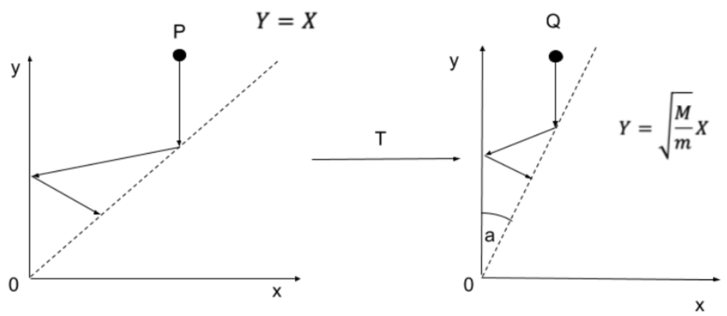


Figure 4

$$T: \mathbb{R}^2 \to \mathbb{R}^2, T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{m} & o \\ 0 & \sqrt{M} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

We now define point $P(x(t), y(t))$ as point $Q(x, y)$ and the velocity vector $\vec{w}(u, v)$ as the vector $\vec{V} = (\sqrt{m}u, \sqrt{M}V)$ on the new coordinate plane. Additionally, the law of reflection follows on the new coordinate plane (the angle created between v_i and the Y-axis is equal to the angle created between V_f and the Y-axis).

$$\vec{w_i}(u_i, v) \xrightarrow{T} \vec{V_i}\ (\sqrt{m}u_i, \sqrt{M}v)$$
$$\vec{w_i}(-u_i, v) \xrightarrow{T} \vec{V_f}\ (-\sqrt{m}u_i, \sqrt{M}v)$$

We will now look into the case where two balls collide with each other, i.e. point$Q$ intersects line $Y = \sqrt{\frac{M}{m}}X$. The magnitude of velocities u and v will change such that

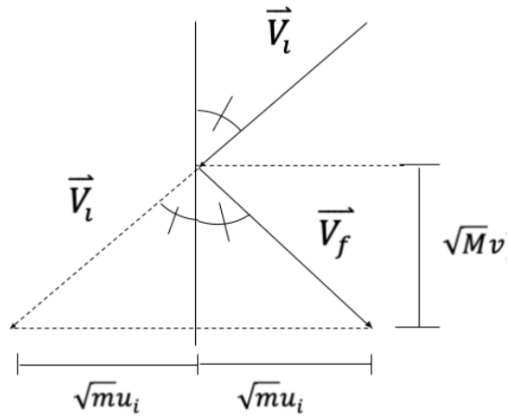


Figure 5

$$\begin{cases} mu + MV = Constant1, & \therefore Momentum\ conservation \\ mu^2 + MV^2 = Constant2, & \therefore Energy\ conservation \end{cases}$$

hold true. We then define vector $\vec{m}$ as $\vec{m} = (\sqrt{m}, \sqrt{M})$ and create a system of equations by substituting it into the velocity vector$\vec{v} = (\sqrt{m}u, \sqrt{M}v)$:

$$\sqrt{m} \cdot \sqrt{m}u + \sqrt{M} \cdot \sqrt{M}v = Constant$$
$$\sqrt{m}u)^2 + (\sqrt{M}v)^2 = Constant$$
$$\vec{m} \cdot \vec{v} = C_1$$
$$|\vec{v}| = C_2$$
$$|\vec{V}| = C_2, (where\ C_1 and\ C_2\ are\ constants)$$

Then, we represent the dot product of two vectors $\vec{m} \cdot \vec{v}$:

$$\vec{m} \cdot \vec{v} = |\vec{m}||\vec{v}|cos\theta = C_1 \text{ , } where\ \theta\ is\ the\ angle\ between\ \vec{m}\ and\ \vec{v}$$

as

$$|\vec{m}| = \sqrt{m+M} \text{ and } |\vec{v}| = C_2,$$

$$cos\theta = \frac{C_1}{C_2\sqrt{m+M}} := C_3, \ (where\ C_3\ is\ a\ constant)$$

Again, the angle of reflection and the angle of incidence are the same.

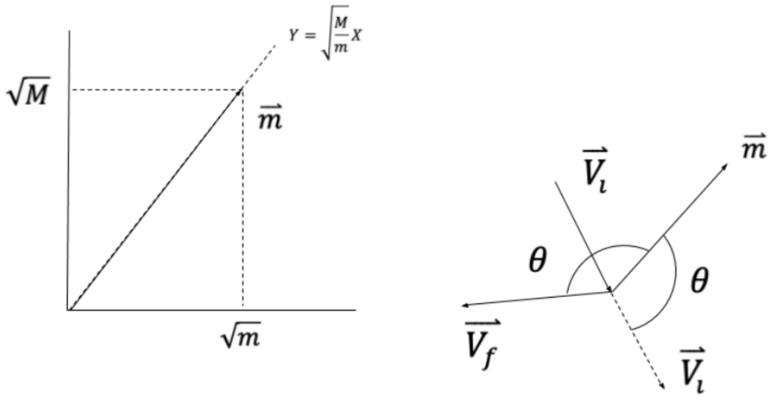


Figure 6

However, the system of equations does not take into account the non-constant value of θ. Following each collision, the value of θ changes. Despite this, the law of reflection still holds, namely the incidence angle and the reflection angle created between Q and $Y = \sqrt{\frac{M}{m}}X$ are the same. In conclusion, the angle of incidence is different for every collision that occurs. However, the angle of reflection for each angle of incidence will always be the same.

We have now proved that the position of point Q will reflect between $Y - Axis$and $Y = \sqrt{\frac{M}{m}}X$. Then, we can now look at the number of collisions as the number of reflections that point Q goes through. Because ball m is at rest, and only ball M is moving, $u_o = 0, v_o < 0$. Thus, the linear trajectory of point Q is always parallel to the line $X = 0$.

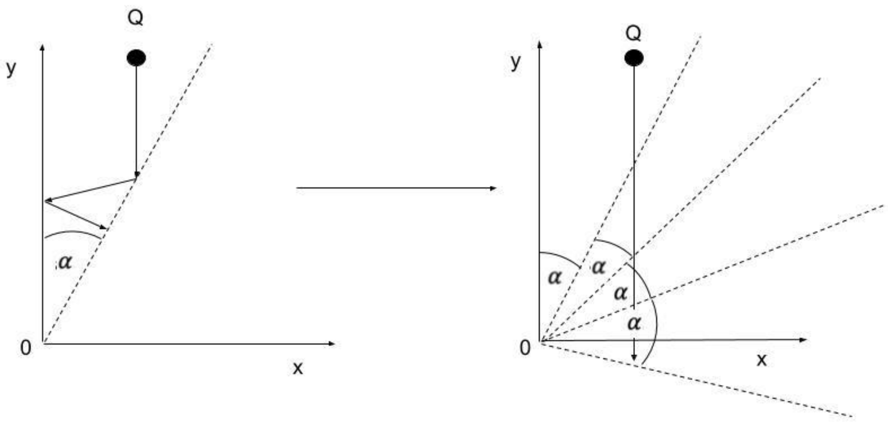


Figure 7

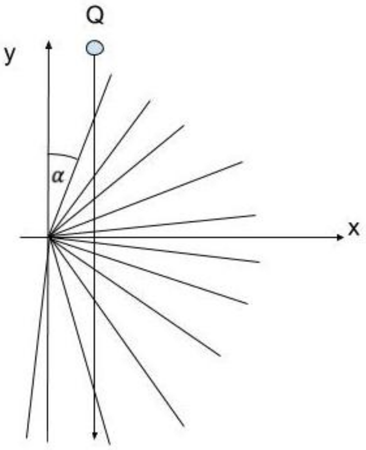


Figure 8

Let the angle between $Y = \sqrt{\frac{M}{m}}X$ and $Y - axis$ be α. Then, the number of collisions between the two balls is $\Pi(\alpha) \cdot \alpha = \pi$.

Knowing that $tan(\alpha) = \sqrt{\frac{m}{M}}$, we can rearrange the equation as follows: $\alpha = arctan(\sqrt{\frac{m}{M}})$.

When substituting this value into the equation above, we obtain $\Pi(\alpha) = \frac{\pi}{\arctan\sqrt{\frac{m}{M}}}$. As previously defined, $\frac{m}{M}$ represents the ratio between m and M, where $\frac{M}{m} = 100^n$.

Thus $\Pi(N) = \frac{\pi}{arctan\ \sqrt{100^{-N}}} = \frac{\pi}{arctan\ 10^{-N}}$.

Let us take the derivative of $arctan(10^{-N})$

$$\frac{d}{dx}arctan(x) = \frac{1}{1+x^2}$$

Then, fixing the value at one point gives an expression for the inverse trigonometric function as a definite integral:

$$arctan(x) = \int_0^x \frac{1}{t^2+1}dt = \int_0^x \frac{dt}{t^2+1}$$

Thus,

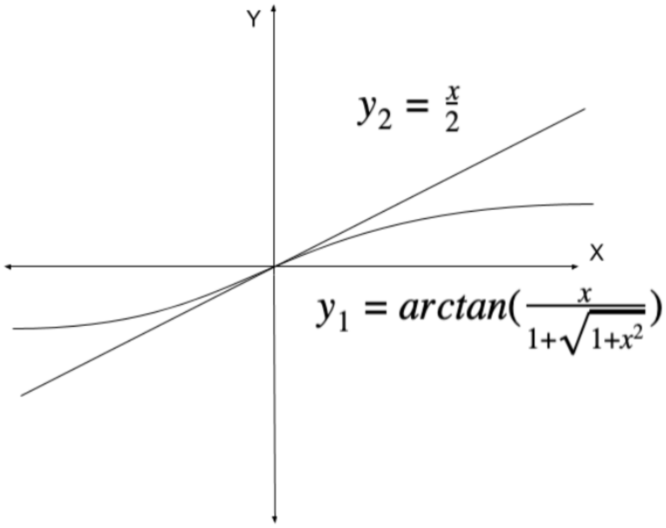
$$arctan(x) = \int_0^x \frac{dt}{t^2+1} \leq \int_0^x 1dt = x$$

$$arctan(x) = x.$$

Another approach involves the half-angle arctangent formula, $arctan(x) = 2arctan(\frac{x}{1+\sqrt{1+x^2}})$. As shown in figure 9, functions $y_1 = arctan(\frac{x}{1+\sqrt{1+x^2}})$ and $y_2 = \frac{x}{2}$. From this, we deduce that near the origin of the graphs, functions gravitate towards one another:

$$\lim_{x\to 0^+} arctan(\frac{x}{1+\sqrt{1+x^2}}) \approx \frac{x}{2}.$$

Therefore $arctan(x) \approx 2 * \frac{x}{2} \approx$



$x.$

Figure 9

Then, $\Pi(N) = \frac{\pi}{arctan\ (10^{-N})} = \frac{\pi}{10^{-N}} = 10^N\pi.$

$$\frac{\Pi(N)}{10^N} = \pi$$

We have now justified that we can obtain the digits of pi from the number of collisions.

IV. Conclusion

We performed a graphical investigation of the billiard scenario, proving why the total number of collisions will enumerate the digits of pi when the ratio of the two masses are in multiples of 100. Studies have investigated a proof of this scenario with the Taylor series, hence the authors of this study explored alternative methods of proving the phenomenon. This includes a graphical analysis of limits and definite integration of inverse trigonometric functions.

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