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Exact traveling wave solutions for the (2+1)-dimensional Burgers equation using $exp(-\phi(\eta))$ −expansion method

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Abstract
In this work, we investigate the traveling wave solutions to the (2+1)-dimensional Burgers equation using the $exp(-\phi(\eta))$ −expansion method. The traveling wave solutions are expressed in terms of the hyperbolic functions, trigonometric functions, rational function, and exponential functions. The extracted solution plays a significant role in numerous types of scientific investigation such as in nonlinear optics, nuclear physics, magnetic field, and fluid flowetc. This method is one of the powerful methods that appear in recent time in establishing some new exact traveling wave solution to the nonlinear partial differential equations. It is shown that the $exp(-\phi(\eta))$ −expansion method is simple and valuable mathematical instrument for solving nonlinear evolution equation in mathematical physics and engineering.
Keywords:$Exp(-\phi(\eta))$ *−expansion method, the (2+1)-dimensional Burgers equation; traveling wave solutions, solitary wave solution.*

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I. Introduction

There are many physical mechanisms of natural phenomena in this earth is often described by nonlinear evolution equations (NLEEs). It has many wide ranges of application of scientific and engineering fields, as for instance, in fluid mechanics, plasma physics, optical fibers, solid state physics, system identification, and nonlinear opticetc. The numerous applications of analytical solutions to nonlinear partial differential equation indicate that there is a significant demand for better mathematical algorithms with real objects and processes and thus lead to further applications. Therefore, the investigation of the exact traveling wave solutions for nonlinear partial differential equations plays a vital role in the study of nonlinear wave phenomena. In recent decades, there have been significant improvements in the study of exact solutions for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations. As a result, many powerful methods for finding exact solutions of nonlinear partial differential equations, such as,the exp-functions method [1], the modified simple equation method [2], the Jacobi elliptic function expansion method [3-4], the Adomian decomposition method [5], the homogeneous balance method [6-7], the F-expansion method [8], the Backlund transformation method [9], the Darboux transformation method [10], $\exp(-\varphi(\xi))$-expansion method [11], the auxiliary equation method [12], the inverse scattering transform [13], the complex hyperbolic function method [14], the (G'/G)-expansion method [15-17], the novel (G'/G)-expansion method [18-20],the new generalized (G'/G)-expansion method [21-23], the$exp(-\phi(\eta))$ −expansion method(24-27) and so on.

The (2+1)-dimensional Burgers equations is an important class of nonlinear partial differential equation to analyze the basic properties of nonlinear propagation of many physical phenomena, such as amplitude and width of the solitons, solitary wave structure and shock wave structure. A large number of literatures [18, 28, 29], where nonlinear Burgers equation is studied and have demonstrated analytical solutions as well as travelling wave solution using different methods. However, the purpose of this research is to extract the new exact solutions of the (2+1)-dimensional Burgers equations using the $exp(-\phi(\eta))$ −expansion method that appeared in recent time.

II. Description of the $\exp(-\phi(\eta))$ −expansion method

In this section, we illustrate the basic idea of the $\exp(-\phi(\eta))$ −expansion method for obtaining exact solutions of NLPDEs. Consider a general nonlinear partial differential equation of the form
$F(u, u_t, u_x, u_{xx}, u_{tx}, u_{tt} \dots\dots\dots\dots\dots) = 0$……………………(1)

where $u = u(x, t)$ is an unknown function, F is a polynomial in $u(x, t)$ and its derivatives in which highest order derivatives and nonlinear terms are involved and the superscripts stand for the partial derivatives. In the following, we give the main steps of this methods:

Step 1: We combine the real variables x and t by a compound variableη, we suppose that

$$u(x,t) = u(\eta), \quad \eta = x \pm Vt, \dots\dots\dots\dots\dots\dots\dots. (2)$$

where V is the speed of the traveling wave. Substituting Eq. (2) into Eq. (1), we reduce Eq. (1) to the following ordinary differential equation (ODE) for $u = u(\eta)$:

$$\Re\,(u, u', u'', u''', \dots\dots\dots\dots\dots.) = 0 \dots\dots\dots.\dots\dots.. (3)$$

where $\Re$is a function of $u(\eta)$. Here prime denotes the derivative with respect to η.

Step 2: Suppose the traveling wave equation of Eq. (3) can be constructed as a finite series in $\phi(\eta)$as follows:

$$u(\eta) = \sum_{i=0}^{N} A_i \exp(-\phi(\eta))^i, \dots\dots\dots\dots\dots..\dots\dots\dots\dots\dots\dots\dots...(4)$$

where A_i $(0 \le i \le N)$ are constant to be determined later, such that $A_N \ne 0$ and $\phi = \phi(\eta)$ satisfies the following ordinary equation:

$$\phi'(\eta) = exp\big(-\phi(\eta)\big) + \mu\, exp\big(\phi(\eta)\big) + \lambda \dots\dots\dots\dots\dots\dots.\dots. (5)$$

We know that Eq. (5) has been following special solutions:

Family 1: Hyperbolic function solution, when $\mu \ne 0, \quad \lambda^2 - 4\mu > 0$,

$$\phi(\eta) = ln\left(\frac{-\sqrt{(\lambda^2-4\mu)}\, tan\,h\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\eta+E)\right)-\lambda}{2\mu}\right)\dots\dots\dots\dots\dots\dots..(6)$$

Family 2: Trigonometric function solutions, when $\mu \ne 0, \quad \lambda^2 - 4\mu < 0$,

$$\phi(\eta) = ln\left(\frac{\sqrt{(4\mu-\lambda^2)}\;\; tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\eta+E)\right)-\lambda}{2\mu}\right)\dots\dots\dots\dots\dots.\dots\dots..(7)$$

Family 3:Exponential function solutions, when $\mu = 0,\ \lambda \ne 0,\ \lambda^2 - 4\mu > 0$,

$$\phi(\eta) = -ln\left(\frac{\lambda}{exp\,(\lambda(\eta+E))-1}\right)\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.\dots\dots..(8)$$

Family 4: Rational function solutions, when $\mu \ne 0,\ \lambda \ne 0,\ \lambda^2 - 4\mu = 0$,

$$\phi(\eta) = ln\left(-\frac{2(\lambda(\eta+E)+2)}{\lambda^2(\eta+E)}\right)\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.\dots\dots\dots(9)$$

Family 5: When $\mu = 0,\ \lambda = 0,\ \lambda^2 - 4\mu = 0$,

$$\phi(\eta) = ln(\eta + E)\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..\dots(10)$$

Here E is an integrating constant and A_N, V, λ, μ are constants to be determined latter, $A_N \ne 0$. Now the main steps of the $exp\big(-\phi(\eta)\big)$-expansion method is to obtain exact solutions of NLPDEs that can be determined as follows:

Step 3.
i) By considering the homogenous balance between the highest order derivatives for the linear term and the highest nonlinear terms of $u\,(x,t)$ in Eq. (3), we can obtain the positive integer N in Eq. (4).
ii) By substituting Eq. (4) with Eq. (5) into Eq. (3) and collecting all terms with the same powers of $\exp\big(-\phi(\eta)\big)$ together, the left-hand side of Eq. (3) is converted into a polynomial. After setting each coefficient of this polynomial to zero, we obtain a set of algebraic equations in terms of A_N $(N = 0,1,2,3, \dots\dots..n), V,\ \lambda,\ \mu$.
iii) Solving the system of algebraic equations and then substituting the results and the general solution of Eqs. (6) to (10) into Eq. (4) along with Eq. (5), we obtain traveling wave solutions of the nonlinear evolution of Eq. (1).

III. Application of the method

In this section, the method is used to construct some new traveling wave solution of the(2+1) dimensional Burgers equation which is very important nonlinear evolution equations in mathematical physics and engineering.

3.1 The (2+1)-dimensional Burgers equation

In this section, we apply the$exp\big(-\phi(\eta)\big)$-expansion method to construct the exact solution and the solitary wave solution of the (2+1) dimensional Burgers equation [18]. Let us consider the Burgers equation,

$$u_t - uu_x - u_{xx} - u_{yy} = 0 \dots\dots\dots\dots\dots.\dots\dots\dots\dots (11)$$

To look for new traveling wave solution of Eq. (11), we use $u(\eta) = u(x,t),\ \eta = x - Vt$. Then Eq. (11) is reduced to the following nonlinear ordinary differential equation:

$$-Vu' - uu' - 2u'' = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (12)$$

Eq. (12) is integrating, therefore, integrating with respect to η, we get:

$$Vu + \frac{1}{2}u^2 + 2u' + k = 0, \dots\dots\dots\dots (13)$$

wherekis an integration constant which is to be determined.

Taking the homogeneous balance between highest order nonlinear term u^2 and linear term of the highest order u' in equation in Eq. (13), we get $2N = N + 1$ which gives $N = 1$. Therefore, the solution of Eq. (13) is of the form:

$$u(\eta) = a_0 + a_1\, exp(-\phi(\eta)), \dots\dots\dots\dots\dots\dots\dots\dots (14)$$

where a_0, a_1 are constant to be determined such that $A_N \neq 0$, while λ, μ are arbitrary constants. It is easy to see that

$$u'(\eta) = -a_1\, exp(-2\phi(\eta)) - a_1\mu - a_1\lambda\, exp(-\phi(\eta)) \dots\dots\dots (15)$$

$$u^2(\eta) = a_0^2 + 2a_0a_1\, exp(-\phi(\eta)) + a_1^2\, exp(-2\phi(\eta)) \dots\dots\dots\dots (16)$$

Insertingu, u', u^2 into Eq. (13) and then equating the coefficients of like power of these polynomials to zero, we obtain the following a set of algebraic equations:

$$\frac{1}{2}a_1^2 - 2a_1 = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (17)$$

$$Va_1 + a_0a_1 - 2a_1\lambda = 0 = 0 \dots\dots\dots\dots\dots\dots\dots (18)$$

$$Va_0 + \frac{1}{2}a_0^2 - 2a_1\lambda + k = 0 = 0 \dots\dots\dots\dots\dots\dots (19)$$

Solving the above equations 17-19, we get

$k = \frac{1}{2}a_0^2 - 2a_0\lambda + 8\mu,\ \ V = -a_0 + 2\lambda,\ \ a_0 = a_0,\ \ a_1 = 4,$

where λ, μ are arbitrary constants.

Now substituting the values of V, a_0, a_1 into Eq. (14) yields

$$u(\eta) = a_0 + 4\, exp(-\phi(\eta)) \dots\dots\dots\dots\dots\dots\dots\dots\dots (20)$$

where $\eta = x — (-a_0 + 2\lambda)t$

Therefore, substituting Eqs. (6) to (10) into Eq. (20) respectively, we get the following five traveling wave solutions of the (2+1)-dimensional Burgers equation.

When $\mu \neq 0,\quad \lambda^2 - 4\mu > 0,$

$$u_1(\eta) = a_0 - \frac{8\mu}{\sqrt{(\lambda^2 - 4\mu)}\, tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\eta + E)\right) + \lambda}$$

where $\eta = x + (a_0 - 2\lambda)t$and E is an arbitrary constant.

When $\mu \neq 0,\quad \lambda^2 - 4\mu < 0,$

$$u_2(\eta) = a_0 + \frac{8\mu}{\sqrt{(4\mu - \lambda^2)}\, tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\eta + E)\right) - \lambda}$$

Where $\eta = x + (a_0 - 2\lambda)t$ and E is an arbitrary constant.

When $\mu = 0,\ \lambda \neq 0,\ \lambda^2 - 4\mu > 0,$

$$u_3(\eta) = a_0 + \frac{4\lambda}{exp(\lambda(\eta + E)) - 1}$$

Where $\eta = x + (a_0 - 2\lambda)t$ and E is an arbitrary constant.

When $\mu \neq 0,\ \lambda \neq 0,\ \lambda^2 - 4\mu = 0,$

$$u_4(\eta) = a_0 + \frac{2\lambda^2(\eta + E)}{(\lambda(\eta + E)) + 2}$$

Where $\eta = x + (a_0 - 2\lambda)t$ and E is an arbitrary constant.

When $\mu = 0,\ \lambda = 0,\ \lambda^2 - 4\mu = 0,$

$$u_5(\eta) = a_0 + \frac{4}{(\eta + E)}$$

Where $\eta = x + (a_0 - 2\lambda)t$and E is an arbitrary constant.

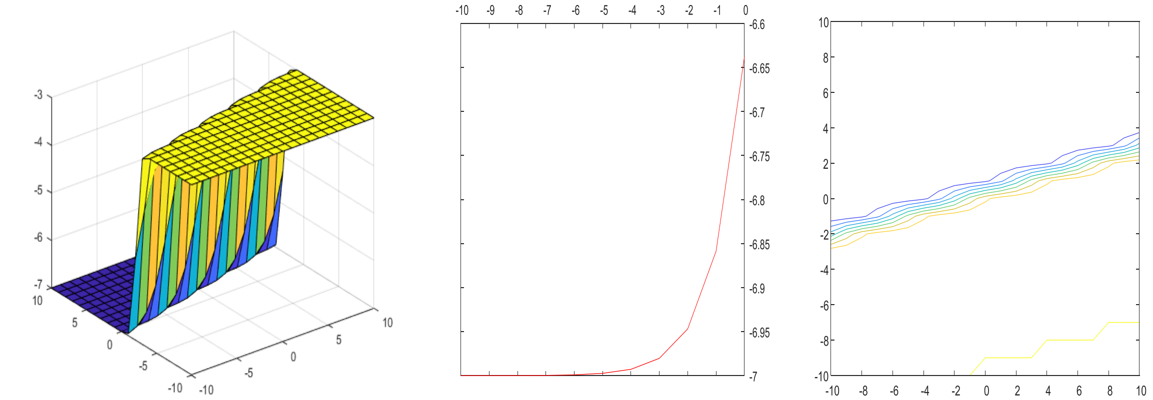
IV. Graphical representation and physical explanations

In this section, we examine the nature of some obtained solutions of equation (11) by selecting particular values of the parameters to visualize the exact solution to the physical phenomena of the (2+1)-dimensional Burgers equation. The obtained solutions of the (2+1)-dimensional equation incorporate four types of explicit solutions namely hyperbolic function, trigonometric function, rational function and exponential function solutions.

4.1 The (2+1)-dimensional Burgers equation

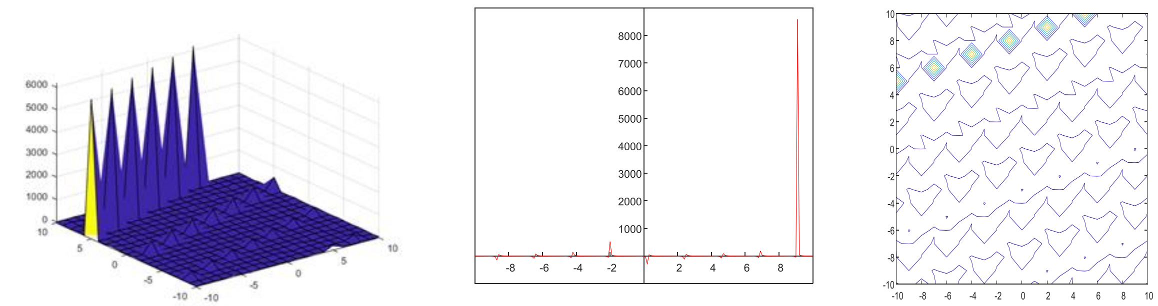
Burgers introduced this equation to capture some of the features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion.The (2+1)-dimensional Burgers equation demonstrates the coupling between dissipation effect of u_{xx}, u_{yy} and the convection process of uu_x. Eq. (11) combines only one of the nonlinear uu_x and dissipation effect of the terms u_{xx}, u_{yy}. We have depicted some graphical representation including 2D, 3D and contour plot graph of the kink soliton solution and wave solutions by substituting the specific values of the unknown constants. From these explicit results we observe that $u_1(\eta)$ represents kink waves which are traveling waves. The kink solutions are approach to a constant at infinity. Fig. 1shows the kink-type solution of $u_1(\eta)$ of the (2+1)-dimensional Burgers equation. Fig. 2 shows the period-type solution of $u_2(\eta)$. Solutions $u_3(\eta)$,$u_4(\eta)$, and $u_5(\eta)$ are the singular kink solution which shown in the figs. 3-5 respectively.

Fig. 1. Graphical representation of the solution in $u_1(\eta)$ and projection at $t = 1$ for the unknown parameters $a_0 = 1,\ \mu = 2,\ \lambda = 3,\ E = 1$within the interval $-10 \leq x, t \leq 10$.



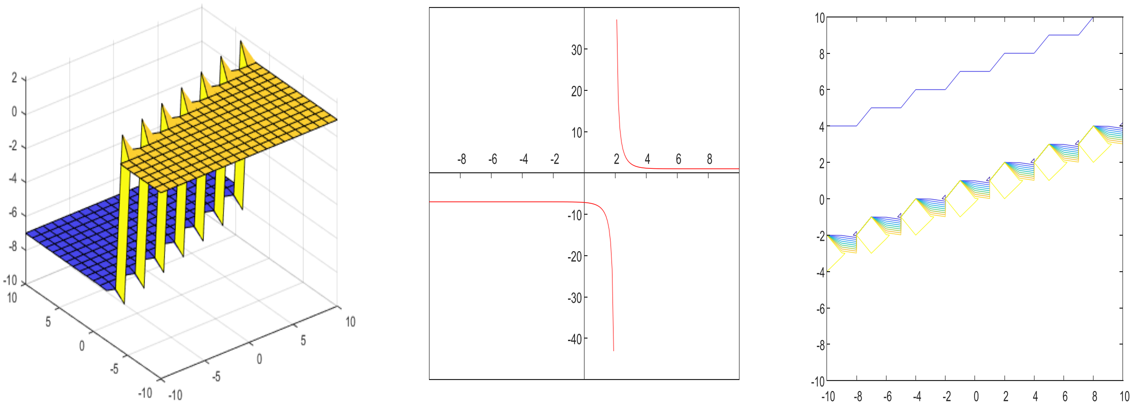
a. 3D graph b. 2D graph c. Contour Plot

Fig. 2. Graphical representation of the solution in $u_2(\eta)$ and projection at $t = 1$ for the unknown parameters $a_0 = 1,\ \mu = 3,\ \lambda = 2,\ E = 1$within the interval $-10 \leq x, t \leq 10$.



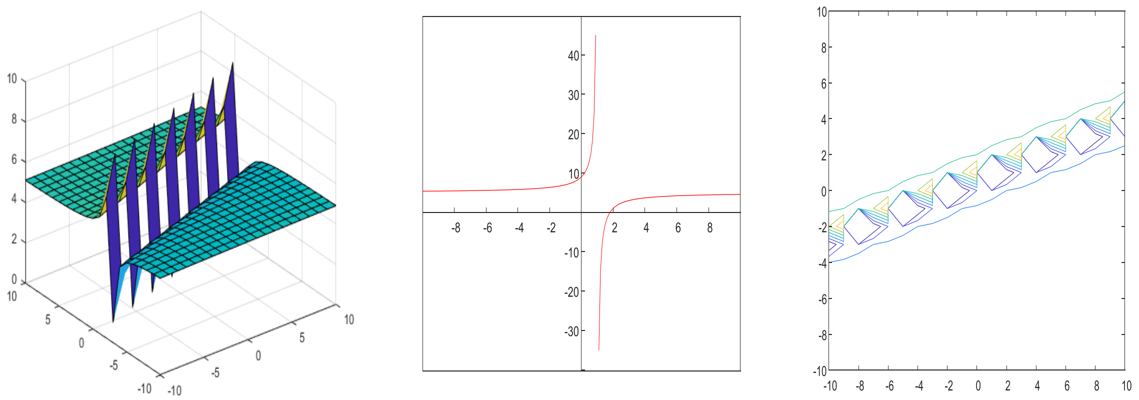
a. 3D graph b. 2D graph c. Contour Plot

Fig. 3. Graphical representation of the solution in $u_3(\eta)$ and projection at $t = 1$ for the unknown parameters $a_0 = 1,\ \mu = 0,\ \lambda = 2,\ E = 1$within the interval $-10 \leq x, t \leq 10$.



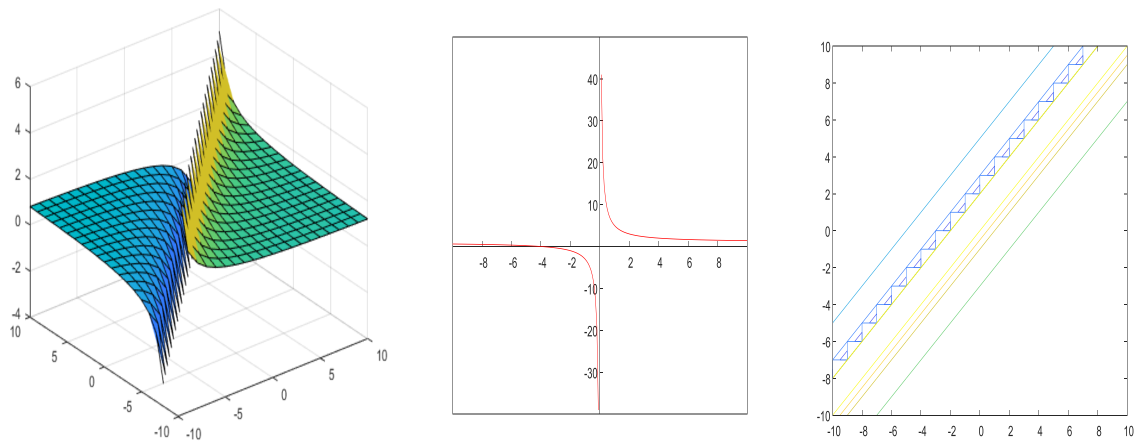
a. 3D graph b. 2D graph c. Contour Plot

Fig. 4. Graphical representation of the solution in $u_4(\eta)$ and projection at $t = 1$ for the unknown parameters $a_0 = 1, \mu = 1, \lambda = 2, E = 1$ within the interval $-10 \leq x, t \leq 10$.



a. 3D graph b. 2D graph c. Contour Plot

Fig. 5. Graphical representation of the solution in $u_5(\eta)$ and projection at $t = 1$ for the unknown parameters $a_0 = 1, \mu = 0, \lambda = 0, E = 1$ within the interval $-10 \leq x, t \leq 10$.



a. 3D graph b. 2D graph c. Contour Plot

V. Conclusion:

In this paper, we have extracted the exact wave solutions for the nonlinear partial differential of (2+1)-dimensional Burgers equations using the $exp(-\phi(\eta))$-expansion method. These exact solutions are composed from the hyperbolic function, trigonometric function, rational function, and exponential function solutions as well, and presented 3D, 2D, contour visualization of some obtained solutions. The obtained solutions can be useful in many circumstances, such as analyze the propagation of gravity waves in ocean, liquid flow, fluid flow in elastic tubes, waves in rivers and lakes in a smaller domain etc. The efficiency of the $exp(-\phi(\eta))$-expansion method is more reliable and easier than the other methods to determine the exact solutions of the (2+1)-dimensional Burgers equations. The method might be fundamental for further research of different nonlinear differential equations in theoretical physics, mathematical physics and other branches of nonlinear science.

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