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A Generalization of Some Krasnosel'skii's Result

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Abstract
In this paper we study some non-compact operator equations for which the existence and uniqueness of a solution is verified. The convergence of the successive approximations to this unique solution is satisfied.
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I. Introduction

Many problems in Applied Mathematics lead to the study of equations of the form

$$x = Ax$$

in (E, P), where E is an ordered Banach space with cone P.
One of the best known approximation procedures consists in given some solution $x*$ which is the limit of the approximating sequence

$$x_n = Ax_{n-1} \quad (n = 1, 2, ...)$$

for a particular (or an arbitrary) initial value x_0.
In this direction, and in the present paper we give a generalization of the result of [4]. Let (E, P) be an ordered Banach space with a normal reproducing cone P and let $A : E \to E$ be an operator. Then in [7] M. A. Krasnosel'skii and P. P. Zabreiko have shown that if there exists a positive linear boundary operator $T : E \to E$ with spectral radius $\sigma(T) < 1$ satisfying

$$-T(x-y) \leq A(x) - A(y) \leq T(x-y), \quad x, y \in E, \quad x \geq y, \qquad (1)$$

then A has a unique fixed point $x*$ in E and for any $x_0 \in E$ if $x_n = Ax_{n-1}(n = 1, 2, 3, ...)$, then $x_n \to x*$ as $n \to \infty$. In the present paper we prove that it's not necessary that the cone P be reproducing and the operator A be uniformly continuous. Hence we avoid these conditions and give a generalization of this important result. The obtained result will be applied in order to search out a fixed point for an operator A which is the sum of two operators : the first of which satisfies only the second inequality of (1) and the second is decreasing. Finally, we discuss the case where the operator A satisfying the second inequality of (1) is increasing and give a generalisation of some Amann's result in [1]. Note that in all the obtained results we do not require the compactness of any considered operator.

II. Main Results

Let $(E, \|.\|_E)$ be a real Banach space and P be a nonempty closed convex set in E.

P is called a cone if it satisfies the following two conditions:

(i) : $x \in P, \lambda \geq 0 \Longrightarrow \lambda x \in P$,

(ii) : $x \in P, -x \in P \Longrightarrow x = \theta$, where θ denotes the zero element in E.

A cone P is said to be generating if $E = P - P$, i.e., every element $x \in E$ can be represented in the form $x = u - v$ where $u, v \in P$.

The cone P defines a linear ordering in E by

$$x \leq y \quad \text{iff} \quad y - x \in P.$$

Let D be a subset of E. An operator $A : D \to E$ is said to be increasing if $x_1 \leq x_2 (x_1, x_2 \in D)$ implies $Ax_1 \leq Ax_2$. A is said to be decreasing if $x_1 \leq x_2 (x_1, x_2 \in D)$ implies $Ax_1 \geq Ax_2$.

The cone P is said to be normal if there exists a constant $N > 0$ such that

$$\theta \leq x \leq y \Longrightarrow \|x\| \leq N\|y\|, \quad x, y \in P.$$

For every $L : E \to E$ a bounded linear operator, define $\sigma(L)$, the spectral radius of L by

$$\sigma(L) = \lim_{n \to +\infty} \|L^n\|^{\frac{1}{n}}.$$

After these preparations we are ready for the statement of our main result :

Theorem 2.1 *Let (E, P) be an ordered Banach space with normal cone P and $A : E \to E$ be a continuous operator such that $A(P) \subset P$. If there exists a positive linear boundary operator $T : E \to E$ with spectral radius $\sigma(T) < 1$ satisfying*

$$-T(x - y) \leq A(x) - A(y) \leq T(x - y), \quad x, y \in E, \quad x \geq y,$$

then A has a unique fixed point $x*$ in P and for any $x_0 \in P$, if $x_n = Ax_{n-1} (n = 1, 2, 3, ...)$, then $x_n \to x*$ as $n \to \infty$.

Set $z_n = A^n(0)$ for $n = 1, 2, ...$, since $z_1 = A(0) \geq 0$ we obtain from inequality (1)

$$-Tz_1 \leq z_2 - z_1 \leq Tz_1,$$

from which it follows that

$$z_2 \geq \frac{1}{2}(z_2 + z_1 - Tz_1), \quad z_1 \geq \frac{1}{2}(z_2 + z_1 - Tz_1).$$

By using inequalities (1) we get

$$-T\Big(\frac{z_2 - z_1 + Tz_1}{2}\Big) \leq z_3 - A\Big(\frac{z_2 + z_1 - Tz_1}{2}\Big) \leq T\Big(\frac{z_2 - z_1 + Tz_1}{2}\Big) \quad (2)$$

and

$$-T\Big(\frac{z_1 - z_2 + Tz_1}{2}\Big) \leq z_2 - A\Big(\frac{z_2 + z_1 - Tz_1}{2}\Big) \leq T\Big(\frac{z_1 - z_2 + Tz_1}{2}\Big). \quad (3)$$

By subtracting (3) for (2), then we have

$$-T^2 z_1 \leq z_3 - z_2 \leq T^2 z_1.$$

By repeating this argument $n-2$ times, we obtain the inequality

$$-T^n z_1 \leq z_{n+1} - z_n \leq T^n z_1.$$

As a consequence of the last inequality we obtain for $n > m \geq 1$

$$-T^m z_1 - T^{m+1} z_1 - \ldots - T^{n-1} z_1 \leq$$

$$\leq z_{m+1} - z_m + z_{m+2} - z_{m+1} + \ldots + z_n - z_{n-1} = z_n - z_m \leq$$

$$\leq T^m z_1 + T^{m+1} z_1 + \ldots + T^{n-1} z_1,$$

from which it follows that

$$-T^m\Big(z_1 + Tz_1 + \ldots + T^{n-1-m} z_1\Big) = z_n - z_m$$
$$\leq T^m\Big(z_1 + Tz_1 + \ldots + T^{n-1-m} z_1\Big).$$

On the other hand, it follows from $r(T) < 1$ and $T(P) \subset P$ that

$$z_1 + Tz_1 + \ldots + T^{n-1-m} z_1 \leq \sum_{i=0}^{+\infty} T^i z_1 = (I - T)^{-1} z_1 = v,$$

therefore

$$-T^m v \le z_n - z_m \le T^m v.$$

It's well known (see [10]) that is follows from $\sigma(T) < 1$ that $T^n v \to 0$ as $n \to \infty$. From this, and from the normality of the cone P, it follows that $z_n - z_m \to 0$ as $n, m \to \infty$, hence $(z_n)_{n=1}^{\infty}$ is a Cauchy sequence. Since E is a Banach space the sequence converges, i.e. there exists a $x* \in E(x* \in P)$ such that $z_n \to x*$ as $n \to \infty$. Here $x*$ is a fixed point of A since A is continuous.
From the above argument it follows that $(A_n(z))_{n=1}^{\infty}$ converges to the unique fixed point independently of the choice of $z \in P$. In fact let $z \in P$, then $z \ge 0$ and by virtue of (1) we have

$$-T(z) \le A(z) - A(0) \le T(z).$$

By using the same argument as above we obtain

$$-T^n(z) \le A^n(z) - A^n(0) \le T^n(z), \quad \forall n = 1, 2, ...$$

From this and from the normality of the cone we assure that $A_n(z) \to x*$ as $n \to \infty$. Similarly we prove that $x*$ is the unique fixed point of A in the cone P, in fact suppose that $x_1 \in P$ is another fixed point of A then we have

$$-T^n(x_1) \le x_1 - A^n(0) \le T^n(x_1), \quad \forall n = 1, 2, ...,$$

from which it follows that $x_1 = x*$.

Remark. 1. Note that our Theorem 2.1 generalizes a result by Krasnosel'skii and Zabreiko (see [7], see also Theorem 3.1.14 in [3]) where the authors take more restrictive assumptions : the cone P is reproducing, the operator A is uniformly continuous and $\|T\| < 1$.
Remark. 2. Theorem 2.1 does not require the compactness of A.
Remark. 3. If the condition (1) is replaced by the following stronger one

$$-T(x-y) \le A(x) - A(y) \le T(x-y), \quad x, y \in E,$$

then A has a unique fixed point $x*$ in E and for any $x_0 \in E$ if $x_n = Ax_{n-1}(n = 1, 2, 3, ...)$, then $x_n \to x*$ as $n \to \infty$. For the proof, it suffices to observe that from the inequality

$$-T(x) \leq A(x) - A(0) \leq T(x),$$

for an arbitrary $x \in E$, follows the inequality

$$-T^n(x) \leq A^n(x) - A^n(0) \leq T^n(x), n = 1, 2, ..$$

In the following, we shall study a fixed point equation of the form

$$x = Mx + Lx$$

where $M, L : E \to E$ are two operators. Let $A = M + L$.

Corollary 2.2 *Suppose that (E, P) is an ordered Banach space with normal cone P and let $A = M + L : E \to E$ be a continuous operator such that $A(P) \subset P$, where $M, L : E \to E$ are two operators verifying the following conditions :*

(i) there exists a positive linear boundary operator T with spectral radius $\sigma(T) < 1$ such that

$$M(x) - M(y) \leq T(x - y), \quad x, y \in E, \quad x \geq y,$$

(ii) L is a decreasing operator,

(iii) the operator $A + M$ is increasing,

then A has a unique fixed point $x*$ in P and for any $x_0 \in P$ if $x_n = Ax_{n-1}(n = 1, 2, 3, ...)$, then $x_n \to x*$ as $n \to \infty$.

We are going to see that A satisfies all conditions of Theorem 2.1. Indeed, since $A - M = L$ is a decreasing operator, then for $x, y \in E, \quad x \geq y$ we have

$$A(x) - M(x) \leq A(y) - M(y),$$

hence

$$A(x) - A(y) \leq M(x) - M(y) \leq T(x - y).$$

On the other hand, since $A + M$ is an increasing operator, then for any $x, y \in E, \quad x \geq y$, we have

$$A(y) + M(y) \leq A(x) + M(x),$$

from which it follows that

$$-T(x-y) \leq -(M(x)-M(y)) \leq A(x)-A(y).$$

Consequently

$$-T(x-y) \leq A(x)-A(y) \leq T(x-y), \quad x,y \in E, \quad x \geq y,$$

with this the operator A satisfies the condition (1) of Theorem 2.1. This completes the proof of the theorem. **Remark. 4.** It should be remarked above that if we take $M = T$ in Corollary 2.3 then the condition (iii) is satisfied if L has the Frechet derivatives $L'(x)$ at every point x of the space E which satisfies the inequality $-L'(x) \leq 2T$.
Remark. 5. Note that in Corollary 2.3 we do not require the compactness of any operator M, L and T.

Latter, suppose that $A : P \to P$ is an increasing operator satisfying the second inequality in condition (1), that is

$$A(x)-A(y) \leq T(x-y), \quad x,y \in E, \quad x \geq y,$$

then it follows from the inequality $A(x) \leq Tx + A(0)$ for every $x \in P$ that $A(v_0) \leq v_0$ where $v_0 = (I-T)^{-1}A(0) = \sum_{n=0}^{\infty} T^n A(0)$. In order to be convinced of this, it suffices to observe that

$$\begin{aligned} A(\sum_{n=0}^{\infty} T^n A(0)) &\leq T(\sum_{n=0}^{\infty} T^n A(0)) + A(0) \\ &= \sum_{n=0}^{\infty} T^n A(0). \end{aligned}$$

From which it follows that A leaves the interval $[0, v_0]$ invariant. Hence by using Theorem 4.1 of Krasnosel'skii in [9], it suffices that anyone of the following conditions be satisfied for the existence on $[0, v_0]$ of at least one fixed point for the map A.
(a) The cone P is strongly minihedral;
(b) The cone P is regular, the map A is continuous;
(c) The cone P is normal, the map A is completely continuous;
(d) The cone P is normal, the space E is weakly complete, the unit sphere in E is weakly compact, the map A is weakly continuous.
Also, it not hard to see that with the fulfillment of the condition (b) or condition (c) or condition (d) the fixed point $\bar{x}$ of A can be obtained as the limit of the sequence

$$x_n = A(x_{n-1}) \quad n = 1, 2, ...$$

where $x_0 = 0$, that is , $\bar{x}$ can be computed by an iterative method. (for condition (d) x_n converges weakly to $\bar{x}$).
Also, it's not difficult to see that if the cone P is normal then the obtained fixed point $\bar{x}$ is unique. In fact suppose that $\bar{y}$ is another fixed point of A in the cone P. Since A is increasing we have $\bar{x} \leq \bar{y}$ (if condition (a) is satisfied we get $\bar{y} \leq \bar{x}$ see the proof of Theorem 4.1 in [9]), from which it follows that $\bar{y} - \bar{x} \leq T(\bar{y} - \bar{x})$. An easy induction argument shows that $0 \leq \bar{y} - \bar{x} \leq T^n(\bar{y} - \bar{x})$ for any positive integer n. Since $\sigma(T) < 1$ we have $T^n(\bar{y} - \bar{x}) \to 0$, then from the normality of the cone P we have $\bar{y} = \bar{x}$. Therefore we have shown the following statement

Theorem 2.3 *Suppose that (E, P) is an ordered Banach space with normal cone P and let $A : P \to P$ be an increasing operator satisfying anyone of the above conditions $(a)-(d)$. If there exists a positive linear boundary operator*

$T : E \to E$ with spectral radius $\sigma(T) < 1$ satisfying

$$A(x) - A(y) \leq T(x - y), \quad x, y \in E, \quad x \geq y, \tag{4}$$

then A has a unique fixed point $x*$ in P and with the fulfillment of the conditions (b) or (c) or (d) for any $x_0 \in P$, if $x_n = Ax_{n-1}(n = 1, 2, 3, ...)$, then $x_n \to x*$ as $n \to \infty$.

Remark. 6. Suppose in addition that $A : P \to P$ is a right differentiable operator where its right derivatives A'_+ satisfies the inequality $0 \leq A'_+(x) \leq T$ for any $x \in P$ (as in Theorem 8.2 in [1]), then it follows from the inequality $(T - A)'_+(x) = T - A'_+(x) \geq 0$ that $T - A$ is an increasing operator. This implies that A satisfies the condition (4). On the other hand it follows from the inequality $0 \leq A'_+(x)$ that A is increasing on the cone P. Therefore, the above Theorem 2.4 generalizes Theorem 8.2 given by Amann in [1] where more restrictive conditions are supposed in T : T is strongly positive and compact.
Remark. 7. Note that from the fact that A leaves the interval $[0, v_0]$ invariant we can also apply the result of [2] by D. Guo and V. Lakshmikantham to prove the existence of a minimal and a maximal fixed point of A in $[0, v_0]$.

III. Conclusion

In this paper we have generalized and improved some well-known results by Amann and Krasnosel'skii. Here we note that the present results can be developed in order to generalize another corresponding results in the literature.

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