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Rayleigh and Burr Probability Distributions Alternative to Weibull: Application to Wind Data

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Abstract: *It is true that several probability distributions exist for modeling lifetime data; however, some of these lifetime data do not follow any of the existing and well known standard probability distributions (models) or at least are inappropriately described by them. Therefore, in this paper, we derived maximum likelihood estimate of the parameters of both Rayleigh and Burr distributions and compared their performances with Weibull distribution in order to find an alternative to the Weibull computation. Random samples of different sizes with different shape parameter settings were drawn from the Weibull distribution and the parameters are estimated using both Rayleigh, Burr with Weibull serving as reference. The estimate of the parameters of the considered distributions alongside with the model selection criteria (AIC and BIC) for the simulated data as well as the wind data were tabulated and presented in graphs for the comparison of the model selection criteria under different sample and parameter settings were displayed. Based on our findings with respect to the model selection criteria, we concluded that two parameters Burr XII can be used as an alternative that best described the considered Weibull distribution and wind data.*

Key words*: Weibull distribution, Rayleigh distribution, Burr distribution, wind data*

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I. Introduction

Inferential statistics is the branch of statistics which is concerned with using concept of probability to deal with uncertainty in decision making. It refers to drawing conclusion about the unknown population characteristics on the basis of information on the sample characteristics. Arun et al. (2017) has derived the probability density function of the size p-dimensional Rayleigh distribution and presented its properties. They discussed it's suitability as a survival model by obtaining its survival and hazard functions. They also discussed Bayesian estimation of the parameter of the size based p-dimensional Rayleigh distribution, the Bayes estimators were obtained by taking quisi prior and the loss functions used are squared error and precautionary loss functions. In a similar study, Faton and Ibrahim (2015) studied a three parameters life model, called the Waibull Rayleigh distribution; they obtained the mathematical properties of this distribution and some structural properties. The method of maximum likelihood and the least squares were used in obtaining the model parameters. The Fisher's information matrix for the distribution were derived and finally applied to real data for illustrating its performance. Saima et al.(2016) also presented a paper titled generalized Rayleigh distribution; they obtained Bayesian estimation of the shape parameter for the two parameters generalized Rayleigh distribution using single and double priors. R software was used to conduct a simulation study in order to compare the different priors. However, Mkolesia et al. (2016) presented a technique for estimating the scale parameter for Rayleigh distribution through minimizing a goal function using differential method. They proposed difference least square method (DLSM) and compare the performance of the proposed method with maximum likelihood method graphically using Monte Carlo simulation. Several classical distributions have been widely used over the past decades for modelling lifetime data in many areas such as reliability, engineering, economics, biological studies, environmental actuarial, environmental and medical sciences, demography, and insurance. However, in many applied areas such as lifetime analysis, finance, and insurance, there is a clear need for extended forms of these distributions. This is because there still remain many important problems where the real data does not follow any of the classical or standard probability models. For that reason, numerous methods for generating new families of distributions have been considered (Bourguignon et al., 2014). To handle this, there is a strong need to propose useful models for the better study of the real-life marvel. Introducing new probability models or their classes is an old practice and has ever been considered as valuable as many other practical problems in statistics. According to Tahir and Cordeiro (2016), the idea simply started with defining different mathematical functional forms, and then adding of location, scale or shape parameter(s).

This study therefore determines the distribution alternatives to Weibull distribution at various sample sizes and parameter. It is true that several probability distributions exist for modeling lifetime data; however, some of these lifetime data do not follow any of the existing and well known standard probability distributions (models) or at least are inappropriately described by them. This, therefore, creates room for developing new distributions or finding alternative among the existing one, which could better describe some of these phenomena and therefore provide greater flexibility and wider acceptability in the modeling of lifetime data. This paper considered performance of Rayleigh and Burr XII distribution and compared them with Weibull in order to deal with such requirements.

1.1 Rayleigh distribution

Rayleigh distribution (RD) is considered to be a very useful life distribution. Rayleigh distribution is an important distribution in statistics and operational research. It is applied in several areas such as health, agricultural, biology and other sciences. One major application of this distribution is used in analyzing wind data. (Afaq, 2015).A continuous random variable Y is said to have Rayleigh distribution with parameter δ if its probability density function (pdf)given by;

$f(y,\delta)=\frac{y}{\delta^2}e^{-\frac{y^2}{2\delta^2}}$ For $y\geq 0$ (Mkolesia, 2016) (1)

Where δ is the scale parameter of the distribution. The cumulative distribution function (cdf) is given by;

$F(y,\delta)=p(Y\leq y)=1-e^{-\frac{y^2}{2\delta^2}}$ for $0<y\leq\infty$ (2)

1.2 Burr XII Distribution

The burr XII distribution is also a continuous probability distribution often used by many researchers to model a wide variety of lifetime data including crop price, household income, risk insurance and travel time.

A continuous random variable Y is said to have Burr XII distribution if its probability density function(pdf)can be expressed as; $f(y,\alpha,\beta)=\frac{\alpha\beta\, y^{\alpha-1}}{(1+y^{\alpha})^{\beta+1}},\ y>0$ (3)

Where $\alpha>0$>0 and $\beta>0$ are the shape parameters

If we put $\alpha=1$ in equation (3), then the density function will become unimodal. (Muhammad and Muhammad, 2014)

The cumulative distribution function (cdf) for the Burr XII distribution is given as; $F(y,\alpha,\beta)=p(Y\leq y)=1-(1+y^{\alpha})^{-\beta}\ \ \alpha>0,\beta>0$ (4)

II. Methodology

Definition (Likelihood Function)

Let $Y_1,Y_2,\dots,Y_n$ independent, identically distributed (iid) random sample of a random variable Y with pdf given by $f(y/\delta)$, then the likelihood function $L(\delta{:}\,y)$ of $Y_1,Y_2,\dots,Y_n$ is the joint density function when regarded as a function of the parameter. That is

$$L(\delta{:}\,y)=\Pi_{i=1}^{n}f(y_i,\delta) \qquad (5)$$

It is more convenient to use the log likelihood.

$$l(\delta{:}\,y)=lnL(\delta,y) \qquad (6)$$

The estimate of the parameter can be obtained by taking the partial derivative of the log likelihood function with respect to the parameter and equating to zero, that is

$$\frac{\partial y}{\partial\delta}\,lnL(\delta,y)=0 \qquad (7)$$

2.1 Maximum likelihood for Rayleigh distribution

Let $Y_1,Y_2,\dots,Y_n$ be a random sample of size n from a Rayleigh distribution with a pdf given by (1)) the likelihood function $L(\delta{:}\,y)$ of this sample is given as

$$L(\delta{:}\,y)=\Pi_{i=1}^{n}f(y_i,\delta)=\Pi_{i=1}^{n}\frac{y_i}{\delta^2}e^{-\frac{y_i^{\,2}}{2\delta^2}}$$

$L(\delta{:}\,y)=\sum_{i=1}^{n}(y_i)\frac{1}{\delta^{2n}}e^{-\frac{1}{2}\sum_{i=1}^{n}(\frac{y_i}{\delta})^2}$(8)

Taking the log of the likelihood function gives

$$l(\delta,\gamma)=\ln\left(\Sigma_{i=1}^{n}(y_i)\frac{1}{\delta^{2n}}e^{-1/2\sum_{i=1}^{n}(\frac{y_i}{\delta})^2}\right)$$

$=ln\sum_{i=1}^{n}(y_i)-2nln\delta-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{y_i}{\delta}\right)^2$ (9)

To maximize equation(9), we take it partial derivative with respect to δ and equate to zero

$$\frac{\partial l}{\partial \delta} = -\frac{2n}{\delta} + \frac{\sum_{i=1}^{n} {y_i}^2}{\delta^3} = 0 \qquad (10)$$

Simplifying equation (10) gives

$$\hat{\delta} = \sqrt{\frac{\sum_{i=1}^{n} {y_i}^2}{2n}} \qquad (11)$$

2.2 Maximum likelihood estimation for Burr XII distribution

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from Burr XII distribution with $pdf f(y_i, \alpha, \beta)$, the likelihood function is given by;

$$L(y{:}\, \alpha, \beta) = \Pi_{i=1}^{n} \alpha\beta {y_i}^{\alpha-1}(1 + {y_i}^{\alpha})^{-(\beta+1)} \qquad (12)$$

Taking the log of the likelihood function (12) yields

$$l(\alpha, \beta) = nln\alpha + nln\beta + (\alpha - 1)ln \sum_{i=1}^{n}(y_i) - (\beta + 1)ln \sum_{i=1}^{n}(1 + {y_i}^{\alpha}) \qquad (13)$$

Now, differentiating (13) with respect to α and β yields

$$\frac{\partial l(\alpha,\beta)}{\partial \alpha} = \frac{n}{\alpha} + ln \sum_{i=1}^{n}(y_i) - (\beta + 1)\left(\sum_{i=1}^{n}\left(\frac{{y_i}^{\alpha}}{1 + {y_i}^{\alpha}}\right)lny_i\right) = 0 \quad (14)$$

$$\frac{\partial l(\alpha,\beta)}{\partial \beta} = \frac{n}{\beta} - ln \sum_{i=1}^{n}(1 + {y_i}^{\alpha}) = 0 \qquad (15)$$

Solving (15) we get

$$\widehat{\beta} = \frac{n}{ln \sum_{i=1}^{n}(1 + {y_i}^{\alpha})} \qquad (16)$$

Estimate for α can be obtained by applying numerical methods such as Newton Raphson iteration. Fatma (2018).

III. Analysis

A Monte Carlo simulation study was extensively carried out in order to estimate the parameters and compare the distributions (Rayleigh and Burr) and to see whether the two can be used as an alternative to the Weibull distribution. Random samples of size 20, 30, 40, 50 and 60 with different shape parameter settings (0.2, 0.4, 0.6 and 0.8) from the Weibull distribution were chosen, and the parameters are re estimated using both Rayleigh and Burr with Weibull which serves as a reference point. The results were discussed and tabulated in the tables 3.1 to 3.4.

3.1 Simulation Results

The random observations obtained from the simulations through Weibull distribution with the specified parameter at different sample sizes fitted to Weibull, Raleigh and Bur distributions are presented in table 3.1-3.4.

Table 3.1: Parameter Estimates for The Distributions with Fixed Parameters; Weibull (0.2, 2), Rayleigh (2) and Burr (3, 4)

Sample n	Distributions	Parameter(s)	MLE	AIC	BIC	Skewness	Kurtosis	Mean
20	Weibull	Scale	5.5187	73.2615	75.2530	3.8502	18.5624	117.466
		Shape	0.1894					
	Rayleigh	Scale	2.0381	332.0621	334.6673			
	Burr	Shape1	1.2670	77.5259	79.5173			
		Shape2	0.2275					
30	Weibull	Scale	1.6917	33.5351	36.3375	5.4615	32.8786	756.4419
		Shape	0.1659					
	Rayleigh	Scale	2754.754	1183.819	1185.22			
	Burr	Shape1	1.5211	34.8658	37.6682			
		Shape2	0.2201					
40	Weibull	Scale	0.6083	-48.0204	-44.6426	5.9148	38.9855	627.5097
		Shape	0.1540					
	Rayleigh	Scale	2361.843	1645.636	1647.325			
	Burr	Shape1	1.7599	-47.8414	-44.4636			
		Shape2	0.2027					
50	Weibull	Scale	0.9421	22.0302	25.8542	6.9585	51.8980	56.2487
		Shape	0.1982					
	Rayleigh	Scale	2361.843	1645.636	1647.325			
	Burr	Shape1	1.6440	26.9878	30.8118			
		Shape2	0.2478					
60	Weibull	Scale	1.1582	45.7321	49.9207	6.1044	44.1526	55.5744
		Shape	0.1982					

	Rayleigh	Scale	2361.843	1645.636	1647.325			
	Burr	Shape1	1.5868	52.2085	56.3972			
		Shape2	0.2519					

The table3.1 above shows the estimates of the parameters obtained for the three distributions (Weibull Rayleigh and Burr), the values of the model selection criteria (AIC and BIC), kurtosis and skewness and means for the various samples considered. It can be observed that the Weibulland the Burr distributions significantlyfit the simulated data better than Rayleigh with minimum values of AIC and BIC. The AIC and BIC of the Weibull(0.2,2) and two parameters Burr (3,4) were found to be smaller compared to that of Rayleigh distribution, and so, the Burr distribution which has AIC and BIC closer to Weibull distribution could be considered as an alternative to the Weibull distribution.

Table 3.2: Parameter Estimates for The Distributions with Fixed Parameters; Weibull (0.4,2), Rayleigh (2) and Burr(3,4)

Sample N	Distributions	Parameter(s)	MLE	AIC	BIC	Skewness	Kurtosis	Mean
20	Weibull	Scale	2.5257	87.0585	89.0499	4.2735	21.6993	7.1263
		Shape	0.4581					
	Rayleigh	Scale	2361.843	1645.636	1647.325			
	Burr	Shape1	1.1167	86.6461	88.6375			
		Shape2	0.7030					
30	Weibull	Scale	1.8836	107.536	110.3384	4.2995	23.5196	4.4834
		Shape	0.4549					
	Rayleigh	Scale	2361.843	1645.636	1647.325			
	Burr	Shape1	1.2789	110.9067	113.7091			
		Shape2	0.6076					
40	Weibull	Scale	1.1519	83.6790	87.0568	2.4655	8.28116	5.9962
		Shape	0.3164					
	Rayleigh	Scale	2361.843	1645.636	1647.325			
	Burr	Shape1	1.5672	88.0098	91.3876			
		Shape2	0.4032					
50	Weibull	Scale	1.8715	176.7372	180.5612	2.9208	11.8167	3.8346
		Shape	0.4792					
	Rayleigh	Scale	5.6744	508.5054	510.4174			
	Burr	Shape1	1.2714	183.1925	187.0166			
		Shape2	0.6269					
60	Weibull	Scale	1.3092	147.5916	151.7803	3.9527	21.0707	5.6526
		Shape	0.3435					
	Rayleigh	Scale	5.6744	508.5054	510.4174			
	Burr	Shape1	1.5044	154.9037	159.0924			
		Shape2	0.4369					

Table 3.2 above indicates the estimates of the parameters obtained for the three distributions considered (Weibull, Rayleigh and Burr), the values of the model selection criteria (AIC and BIC), skewness and kurtosis and means for the various samples considered. It can be observed that the Weibull and the Burr distributions significantly fit the simulated data well. The AIC and BIC values for the Weibull and the Burr for the different samples were found to be smaller comparedto the Rayleigh distribution. In this case, the skewness and Kurtosis values were observed to be asthe samples increases, the mean values moderately closer to the mean in table 5 which is the reference point.

Table 3.3: Parameter Estimates for The Distributions with Fixed Parameters; Weibull (0.6,2), Rayleigh (2) and Burr(3,4)

Sample N	Distributions	Parameter(s)	MLE	AIC	BIC	Skewness	Kurtosis	Mean
20	Weibull	Scale	2.8052	89.1178	91.10927	1.3530	4.8849	4.1045
		Shape	0.5682					
	Rayleigh	Scale	4.4096	162.8778	163.8735			
	Burr	Shape1	1.0438	94.1454	96.136			
		Shape2	0.7017					
30	Weibull	Scale	2.3127	122.3959	125.1983	2.6456	10.6363	3.2497
		Shape	0.6480					
	Rayleigh	Scale	4.3533	239.1236	240.5248			
	Burr	Shape1	1.0039	123.0828	125.8852			
		Shape2	1.0082					

40	Weibull	Scale	1.4512	125.5619	128.9397	2.862577	11.1411	2.1839
		Shape	0.6199					
	Rayleigh	Scale	3.1220	304.0384	305.7272			
	Burr	Shape1	1.3425	124.8001	128.1778			
		Shape2	0.8931					
50	Weibull	Scale	1.8559	176.1368	179.9608	2.3005	10.2243	2.59875
		Shape	0.6053					
	Rayleigh	Scale	3.0072	361.4853	363.3973			
	Burr	Shape1	1.2079	186.4495	190.2736			
		Shape2	0.7547					
60	Weibull	Scale	2.3414	240.6694	244.8581	2.597933	11.5048	3.2730
		Shape	0.6306					
	Rayleigh	Scale	4.0944	468.5323	470.6267			
	Burr	Shape1	1.0456	248.8649	253.0536			
		Shape2	0.8711					

From above Table3.3, the Weibull distribution having the smallest AIC and BIC values is the best to fit the simulated data followed by Burr distribution with closer values of AIC and BIC. The Burr could be considered as an alternative to the Weibull distribution. The Rayleigh distribution performs poorer with the largest AIC and BIC values. It could also be observed that with the shape increased to 0.6, the data is becoming less skewed and the kurtosis is moderately good. The means, at the different sample sizes approximates the real data in Table.5.

Table 3.4: parameter estimates for The Distributions with Fixed Parameters; Weibull (0.8,2), Rayleigh (2) and Burr(3,4)

Sample n	Distributions	Parameter(s)	MLE	AIC	BIC	Skewness	Kurtosis	Mean
20	Weibull	Scale	2.7267	87.2398	89.2313	1.3343	3.6105	3.0656
		Shape	0.8112					
	Rayleigh	Scale	3.2847	124.8178	125.8135			
	Burr	Shape1	0.8157	89.1372	91.1287			
		Shape2	1.2518					
30	Weibull	Scale	1.9868	104.3376	107.1399	1.857961	6.8893	1.9604
		Shape	1.0332					
	Rayleigh	Scale	1.9367	133.6112	135.0124			
	Burr	Shape1	0.8858	108.7516	111.5539			
		Shape2	1.5432					
40	Weibull	Scale	1.6244	131.9051	131.9051	2.1318	6.7180	2.3841
		Shape	0.5962					
	Rayleigh	Scale	2.9944	298.6593	300.3482			
	Burr	Shape1	1.2780	137.935	141.3128			
		Shape2	0.7770					
50	Weibull	Scale	0.0871	180.7285	184.5525	1.7042	5.5463	2.2673
		Shape	0.7899					
	Rayleigh	Scale	2.5129	289.5759	291.4879			
	Burr	Shape1	1.0317	184.5263	188.3504			
		Shape2	1.1476					
60	Weibull	Scale	2.2513	228.9974	233.1861	1.8414	6.9474	2.4624
		Shape	0.8407					
	Rayleigh	Scale	2.6181	338.358	340.4523			
	Burr	Shape1	0.9281	238.4283	242.617			
		Shape2	1.1916					

It is clear from the table.4 above that the Weibull and the Burr distributions significantly fit the simulated data better than Rayleigh. Weibull distribution fit the data very well having the smallest AIC and BIC values followed by Burr distribution with closest AIC and BIC values. The Rayleigh distribution indicates poorest fit compared to the Weibull and Burr distributions. The skewness and kurtosis at the different sample sizes indicates less significant compared to the previous shape parameter settings. The mean values at the different sample sizes also shows less significant compared to the mean in Table.5.

3.3 Fitting and Analyzing the Real Data

Table 3.5: Parameter Estimates for the Distributions with Fixed Parameters; Rayleigh (2) and Burr (3, 4) for the Wind Data.

Distributions	Parameters	MLE	AIC	BIC	Skewness	Kurtosis	Mean	KST
Burr	Shape1	0.2157	468.6974	471.5802	0.2250	2.4025	3.4583	0.3897
	Shape2	3.9307						
Rayleigh	Scale	2.6105	643.8048	649.5704				0.1931
Weibull	Scale	3.8724	446.8891	452.6547				0.0774
	shape	2.8765						

The table 3.5above represents the estimates of the parameters obtained for the distributions considered, comparison of the model selection criteria, skewness, kurtosis and mean for the real wind data. It is obvious that all the three distributions significantly fit the wind data. The values of AIC and BIC for the two parameters Weibull and two parameters Burr distributions were found to be smaller; this shows that these two distributions fit the wind data very well. We can say that the Burr distribution can be used as an alternative to the Weibull distribution instead of Rayleigh distribution. For the skewness and the kurtosis, the values indicate insignificant. In terms of Kolmogorov Smirnoff statistics also, the best performance gives the Weibull distribution which has the smallest value of 0.0774 followed by the Burr with value 0.1931 and the Rayleigh distribution with greater value of 0.3897.

IV. Conclusion

Based on the results of the analysis of both simulated and the real wind data, tables and graphs shown and with respect to the models selection criteria, the Burr distribution competes well with Weibull compared to the Raleigh distributions. The Burr distribution can therefore be used as an alternative distribution that best describe the data from the Weibull family with higher shape parameter and sample sizes, so also the wind data. It can also be observed from the table that the values for skewness and the kurtosis were increasing as the sample sizes increases while the shape remained constant at lower shape parameter but were relatively decreasing as sample sizes and the shape parameter were increased.

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