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Application of the Laplace Transform to solve mixed problems associated with a Partial Differential Equation

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Abstract: *This work presents the application of the Laplace transform to solve mixed problems associated with Equations in Partial Derivatives (EDP). Laplace transform is defined, the condition of existence of Laplace transform is established, also is presented a table of transformation of some functions, linear property and the derivative transform that is very fundamental for the theme that we propose to develop . Finally, a practical example of how the Laplace transform is applied to a mixed problem associated with EDP.*

Keywords: *Laplace transform. Mixed problems. Applications*

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I. Introduction:

The work deals with the application of Laplace transform to solve mixed problems associated with an EDP. The main objective of this work is to: Apply the Laplace transforms to the solution of Partial Differential Equations (EDP) with contour conditions (mixed problems).

For this we will try to explain theoretically an example, to know a concrete way of applying the Laplace transform to solve mixed problems associated with EDP.

II. Development:

2.1. - Laplace Transforms

Definition

Be f: $[0,\infty[\to R$,a function defined for t ≥ 0, then the function F defined by:

$$F(s)=\int_0^{+\infty} e^{-st}f(t)dt=\lim_{b\to+\infty}\int_0^b e^{-st}f(t)dt$$

It is called Laplace transform of F, whenever the limit exists.

Symbolically the Laplace transform of F is denoted by:

$L\{f(t)\}$, this means$L\{f(t)\}=\int_0^{+\infty} e^{-st}f(t)dt=F(s)$

Example: Calculate L {f (t)}, where: a) f (t) = 1, L {f (t)}, where: b) f (t) = t

Solution: a)

$$L\{f(t)\}=\int_0^{+\infty}e^{-st}f(t)dt=\int_0^{+\infty} e^{-st}dt=\lim_{b\to+\infty}\int_0^b e^{-st}dt=\lim_{b\to+\infty}\left[-\frac{1}{s}e^{-st}\right]_0^b=\lim_{b\to+\infty}\left[-\frac{1}{s}e^{-sb}+\frac{1}{s}\right]=\frac{1}{s}\ ,s>0$$

b)

$$L\{f(t)\}=\int_0^{+\infty}e^{-st}f(t)dt=\int_0^{+\infty} e^{-st}tdt=\lim_{b\to+\infty}\int_0^b e^{-st}tdt=\lim_{b\to+\infty}\left[-\frac{t}{s}e^{-st}-\frac{e^{-st}}{s^2}\right]_0^b$$

$$=\lim_{b\to+\infty}\left[\left(-\frac{be^{-sb}}{s}-\frac{e^{-sb}}{s^2}\right)+\frac{1}{s^2}\right]=\frac{1}{s^2}\ \ ,s>0$$

2.2. - Sufficient conditions for the existence of L {f (t)}

The sufficient conditions guaranteeing the existence of L {f (t)} are that f (t) is continuous by branch or sectional continuum for t> 0 and further that it is of exponential order for $t>T$.

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Definition: A function is said to be sectionally continuous or continuous by branch in a closed interval [a, b] if it is possible to divide the interval into a finite number of subintervals such that the function is continuous in each of them and has a limit to the left and to the right.
Definition: The function f: [0, + ∞) → R, is of exponential order if there are
Constants c> 0 and α such that $|f(t)| \leq ce^{\alpha t}, \forall t \geq 0$
Ejemplo:
$f(t) = e^{at}cosbt$. It is of exponential order
Solution:
As$|cosbt| \leq 1,\ \forall t \geq 0\ then e^{at}|cosbt| \leq e^{at}, \forall t \geq 0$from where
$|e^{at}cosbt| \leq e^{at} \rightarrow |f(t)| \leq e^{at}$therefore$f(t) = e^{at}cosbt$. is of exponential order.

Note:
If f: $[0, +\infty[$ → R is a sectionally continuous function in the interval$[0, +\infty[$, then:

i) The function f is of exponential order whenever there exists α∈R such that:

$$\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}} = 0$$

ii) The function f is of exponential orderse:

$$\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}} = \infty$$

2.3.-Laplace transform table of some functions:

	Functions	**Transform**		**Functions**	**Transform**
1	**1**	$\frac{1}{s}$	**6**	$senh\omega.t$	$\frac{\omega}{s^2-\omega^2}$
2	t^n	$\frac{n!}{s^{n+1}}$	**7**	$\cosh\omega.t$	$\frac{s}{s^2-\omega^2}$
3	$e^{\lambda.t}$	$\frac{1}{s-\lambda}$	**8**	$e^{\lambda.t}.sen\omega.t$	$\frac{\omega}{(s-\lambda)^2+\omega^2}$
4	$sen\omega.t$	$\frac{\omega}{s^2+\omega^2}$	**9**	$e^{\lambda.t}.\cos\omega.t$	$\frac{s-\lambda}{(s-\lambda)^2+\omega^2}$
5	$\cos\omega.t$	$\frac{s}{s^2+\omega^2}$			

2.4. - Linear property:
If the functions $f(t)$ and g(t)have Laplace transforms and α, β are any two constants, then:
$L[\alpha.f(t)+\beta.g(t)] = \alpha.L[f(t)]+\beta.L[g(t)]$

2.5. - Derivative transformation:
If the function has continuous derivatives until the order n in the interval $(0;\ \ \infty)$, and the function $f^{(n)}(t)$ has Laplace transform, then:

$$L\left[f^{(n)}(t)\right] = s^n.L[f(t)] - s^{n-1}.f(0) - s^{n-2}.f'(0) - \dots\ \ - f^{(n-1)}(0).$$

2.6. - Application of Laplace Transform to solve mixed problem.
The equations of linear partial derivatives with initial conditions can be solved by means of the Laplace transform.
We can have situations when solving an EDP in which boundary conditions and initial conditions appear. Such problems will be termed mixed problems.

Suppose we want to find the solution of Equations.

$$\frac{\partial^2 u}{\partial t^2} = a^2 . \frac{\partial^2 u}{\partial x^2} \qquad t > 0, \qquad 0 < x < l \tag{1}$$

With the boundary conditions

$$u(0,t) = \psi_1(t), \qquad u(l,t) = \psi_2(t), \qquad t \geq 0 \tag{2}$$

And initial conditions

$$u(x,0) = \varphi_1(x), \quad \frac{\partial u}{\partial t}(x,0) = \varphi_2(x), \qquad 0 \leq x \leq l. \tag{3}$$

$0 \leq t < +\infty$

Apply to your transformed Laplace solution with respect to variable $t.$

Be $V(x,s) = L[u(x,t)] = \int_0^{+\infty} u(x,t).e^{-st} dt$

$$L\left[\frac{\partial u}{\partial t}\right] = s.V(x,s) - u(x,0) = s.V(x,s) - \varphi_1(x)$$

$$L\left[\frac{\partial^2 u}{\partial t^2}\right] = s^2.V(x,s) - s.u(x,0) - \frac{\partial u(x,0)}{\partial t} = s^2.V(x,s) - s.\varphi_1(x) - \varphi_2(x)$$

On the other hand, let's call

$$V(0,s) = L[u(0,t)] = \int_0^{+\infty} u(0,t).e^{-st} dt = M_1(s)$$

$$V(l,s) = L[u(l,t)] = \int_0^{+\infty} u(l,t).e^{-st} dt = M_2(s)$$

Considering the equation in V

$\frac{\partial^2 V}{\partial t^2} = a^2 \frac{\partial^2 V}{\partial x^2}$ $(*)$and let's do

$\frac{\partial^2 v}{\partial t^2} = L\left[\frac{\partial^2 u}{\partial t^2}\right] = s^2 V - s\varphi_1(x) - \varphi_2(x)$and replacing in (*) we have:

$a^2 \frac{\partial^2 v}{\partial x^2} = s^2 V - s\varphi_1(x) - \varphi_2(x)$, multiplying by $\frac{1}{a^2}$

$$a^2 \frac{\partial^2 V}{\partial x^2} = s^2 V - s\varphi_1(x) - \varphi_2(x) \,|. \frac{1}{a^2}$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{s^2 V}{a^2} - \frac{s}{a^2}\varphi_1(x) - \frac{1}{a^2}\varphi_2(x)$$

$$\frac{\partial^2 v}{\partial x^2} - \frac{s^2 V}{a^2} = -\frac{s}{a^2}\varphi_1(x) - \frac{1}{a^2}\varphi_2(x)$$

In this way we arrive at the following problem:

Find the solution of the equation

$$\frac{d^{''} V}{dx^2} - \frac{s^2 V}{a^2} = -\frac{s}{a^2}\varphi_1(x) - \frac{1}{a^2}\varphi_2(x) \tag{4}$$

which satisfies the boundary conditions

$$V(0,s) = M_1(s) \qquad V(l,s) = M_2(s) \tag{5}$$

Suppose it is the solution to the problem(4) – (5). Then

$u(x,t) = L^{-1}[V(x,s)]$will be the solution of the initial problem (1) – (3).

2.5. – Example:

A long l of the rope is attached to the ends $x=0$ and $x=l$.
The initial deviation of the rope is determined by the formula

$u(x,0)=A.sen\dfrac{\pi x}{l}, \quad A=const.$ Initial velocities are absent. Find solution $u(x,t)$ of the rope to $t>0.$

Solution:
The problem is to find the solution of the equation

$$\frac{\partial^2 u}{\partial t^2}=a^2.\frac{\partial^2 u}{\partial x^2} \qquad t>0, \qquad 0<x<l \tag{6}$$

For initial conditions

$$u(x,0)=A.sen\frac{\pi x}{l}, \quad \frac{\partial u}{\partial t}(x,0)=0 \qquad 0\le x\le l. \tag{7}$$

And border conditions

$$u(0,t)=0, \qquad u(l,t)=0, \qquad t\ge 0 \tag{8}$$

Let us apply the Laplace transform with respect t

Be $V(x,s)=L[u(x,t)]=\int_0^{+\infty} u(x,t).e^{-st}dt$

$$L\left[\frac{\partial u}{\partial t}\right]=s.V(x,s)-u(x,0)=s.V(x,s)-A.sen\frac{\pi .x}{l}$$

$$L\left[\frac{\partial^2 u}{\partial t^2}\right]=s^2.V(x,s)-s.u(x,0)-\frac{\partial u(x,0)}{\partial t}=s^2.V(x,s)-s.A.sen\frac{\pi .x}{l}.$$

Border conditions (8) give us

$$V(0,s)=L[u(0,t)]=\int_0^{+\infty} u(0,t).e^{-st}dt=0$$

$$V(l,s)=L[u(l,t)]=\int_0^{+\infty} u(l,t).e^{-st}dt=0.$$

Then we arrive at a differential linear equation non-homogeneous
with constant coefficients:

$$\frac{d''V}{dx^2}-\frac{s^2}{a^2}.V=-\frac{A.s}{a^2}.sen\frac{\pi .x}{l}. \tag{9}$$

which satisfy the boundary conditions

$$V(0,s)=0 \qquad V(l,s)=0 \tag{10}$$

Solving the equation(9) $V''-\frac{s^2}{a^2}V=-\frac{AS}{a^2}sen\frac{\pi x}{l}$ which has as characteristic equation:

$$m^2-\frac{s^2}{a^2}=0 \Rightarrow m=\pm\frac{s}{a}$$

Which gives us the complementary solution:

$$V(x,s)=c_1e^{\frac{sx}{a}}+c_2e^{-\frac{sx}{a}}$$

And it is proposed as a private solution : $V_p=\alpha.Cos\frac{\pi x}{l}+\lambda.sen\frac{\pi x}{l}$ whose derivatives are:

$$V'=-\frac{\alpha\pi}{l}sen\frac{\pi x}{l}+\frac{\lambda\pi}{l}cos\frac{\pi x}{l}$$

$V'' = -\frac{\alpha\pi^2}{l^2}cos\frac{\pi x}{l} - \frac{\lambda\pi^2}{l^2}sen\frac{\pi x}{l}$

replacing V, V'' in (9) we have:

$$\boxed{V'' - \frac{s^2}{a^2}V = -\frac{AS}{a^2}sen\frac{\pi x}{l}}$$

$-\frac{\alpha\pi^2}{l^2}cos\frac{\pi x}{l} - \frac{\lambda\pi^2}{l^2}sen\frac{\pi x}{l} - \frac{s^2}{a^2}$

So:

$$\begin{cases} -\frac{\lambda\pi^2}{l^2} - \frac{s^2\lambda}{a^2} = -\frac{A.s}{a^2} \\ -\frac{\alpha\pi^2}{l^2} - \frac{s^2\alpha}{a^2} = 0 \end{cases} \Rightarrow \begin{cases} \lambda = \frac{A.s}{s^2 + \frac{a^2\pi^2}{l^2}} \\ \alpha = 0 \end{cases}$$

Then$V_p = \frac{A.s}{s^2+\frac{a^2\pi^2}{l^2}}sen\frac{\pi x}{l}$and the general solution of equation (9) it is:

$$V(x,s) = C_1.e^{\frac{s.x}{a}} + C_2.e^{\frac{-s.x}{a}} + \frac{A.s}{s^2 + \frac{a^2.\pi^2}{l^2}}.sen\frac{\pi.x}{l}$$

applying the boundary conditions(10)$V(0,s) = 0$, $V(l,s) = 0$ we have $C_1 = C_2 = 0$,

$V(x,s) = \frac{A.s}{s^2 + \frac{a^2.\pi^2}{l^2}}.sen\frac{\pi.x}{l}$, Applying the inverse transform using property 5 of the transformation table mentioned above:

$L[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \Rightarrow L^{-1}\left[\frac{s}{s^2+\omega^2}\right] = \cos(\omega t)$this way we have:

$u(x,t) = L^{-1}[V(x,s)] = L^{-1}\left[\frac{A.s}{s^2 + \frac{a^2.\pi^2}{l^2}}.sen\frac{\pi.x}{l}\right] = A.\cos\frac{\pi.a.t}{l}.sen\frac{\pi.x}{l}$ which is the solution to the problem(6) – (8).

3. - Conclusions:

- In this work we can see the usefulness of the Laplace Transforms in solving mixed problems associated to an EDP, which facilitates the calculations. in this case we only see the solution of the hyperbolic equation with the boundary conditions, in Partial Differential Equations these equations are solved using a variable separation method that requires knowledge about the Fourier series.

- Given that a partial differential equation (EDP) is a powerful tool to investigate problems in science and engineering, trying to solve problems of this nature through the use of Laplace transform is a difficult task.

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