IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 16, Issue 2 Ser. I (Mar. – Apr. 2020), PP 44-49
www.iosrjournals.org

A New Method for Solving Initial Value Problems

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Abstract:
Background*:Many numerical techniques have been developed for solving first orderordinarydifferentialequationswithan initial condition (initial value problems : IVP)such as Euler'smethod, Huan'smethod, Runge-Kutta'smethod. Thesemethodsare used for engineer, scientistandappliedmathematicianwhoneeds an algorithm to solve IVPeasilyand the mostefficient.*
Materials and Methods*: In this purposed paper a new methodimproves Euler's method and Heun's method by increasing Taylor's series order (usually 1st order) and using total derivative. Some examplesare presented to compare the numerical solutions of Euler's method and Heun's method solutionsby approximation graphs and absolute error graphs.*
Results*: Themost absolute errors ofthe new method are lower than the original methodsand also the approximation graphs of the new method are close to the exact solution.*
Conclusion: *From the results we obtained that the new method can approximates better than original ones and close to the exact solutions very well.*
Key Word*:Ordinary differential equations; Initial value problems; Euler's method; Heun'smethod.*

Date of Submission: 09-03-2020 Date of Acceptance: 23-03-2020

I. Introduction

Many mathematical problems that occur in physical, chemical, biological and engineering can be in form of either ordinary differential equations (ODEs) or partial differential equations (PDEs). However, some problems are complicated to obtain the exact solution for these differential equations especially whenthey are nonlinear equations.Therefore, most researchers use numerical methods to approximate solution to these equations. Amirul (2015) and Gadisa and Garoma (2017) attempted to approximate the solution of ordinary differential equations with initial value problems (IVP) in order to obtain high accuracy using numerical technique, such as Taylor's method, Euler's method, Huan's method and Runge-Kutta's method. Karuzzaman and Nath (2018) work on numerical solutions of IVP by applying of Euler's method and modified Euler's method. Ochoche(2007,2008) determine high order by using several methods; Euler's method, modifier Euler's method and Runge-Kutta's method to solve IVP. Ochoche (2015) discuss on accurate solution of IVP by using Runge-Kutta fourth order method. Amirul (2015) discussed on accurate solution of IVP and also consider on some numerical methods. Moreover, Ogunrinde et al. (2012) studied numerical solution of IVP using a variety of numerical methods.

In this research, we present a numerical technique which modify Euler's method and Huan's method by increasing Taylor's series order (usually 1st order) and using total derivative.

II. Methods

We consider the initial value problem of the first-order ordinary differential equations in the form of,

$$\frac{dy}{dx} = f\left(x, y(x)\right) \tag{1}$$

with initial condition that

$$y(x_0) = y_0. \tag{2}$$

We determine $y(x)$ which is the solution of (1) on a finite interval (x_0, x_n) with an initial point, (x_0, y_0). A continuous approximation to the solution, $y(x)$, will not be obtained directly but approximate solution, y_n, will be generated at various values of n (mesh points) in the interval (x_0, x_n). Numerical techniques provide some approximations to the values of solution series, $\{y_n\}$, $n = 1, 2, 3, \ldots$, corresponding to various selected values of $x_n = x_0 + nh,\ n = 1, 2, 3, \ldots$ where h is called step size. The numerical solutions of (1) given by set of points $\{(x_n, y_n)\}$, $n = 0, 1, 2, \ldots$ are approximates to the corresponding points $\left(x_n, y(x_n)\right)$ on the solution curve.

The simplest method to solve (1) is given by Leonhard Euler(Euler, 1768).This method uses the tangent line of the function f at the beginning point, (x_n, y_n) as shown in Figure no. 1(A), and estimate slope of the left tangent line over the interval $[x_n, x_{n+1}]$. And Euler's method is given by

$$\begin{aligned} y_{n+1} &= y_n + k_1, \\ k_1 &= hf(x_n, y_n) \end{aligned} \tag{3}$$

where $n = 0,1,2,\ldots$. Butcher (1985) showed algorithm of (3) in two steps:

(1) Let's (x_0, y_0) is a condition point and step size, h.
(2) y_{n+1} depends on y_n by computing the function $f(x_n, y_n)$ in the step $x_n = x_0 + nh$, when h has to be chosen to be very small.

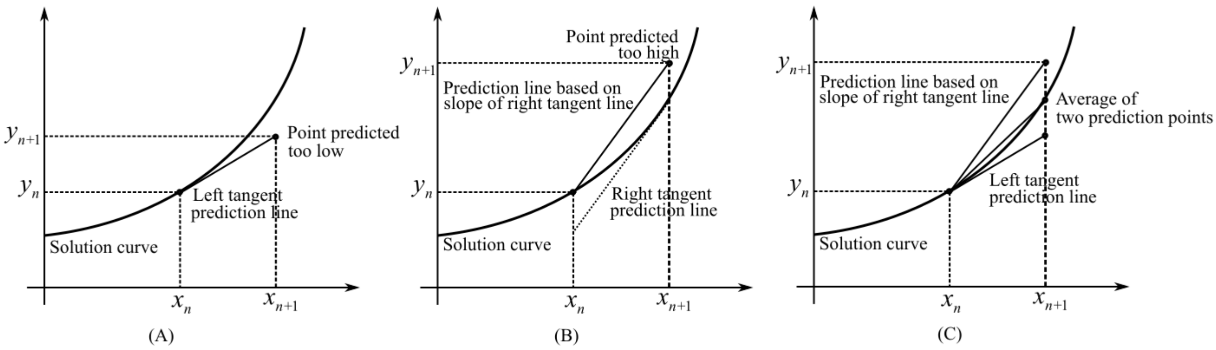

Figure no 1: shown diagram modified of Euler's method (A) by using right tangent prediction line (B) to evaluate
a less error prediction point (C) when compared to the lower order Euler's Method

However, the method of Euler is ideal as an object of theoretical study but low accuracy and not good stability behavior. So, Karl Heunmodified Euler's method by replacing the slope of the tangent line withusing right tangent prediction line at the end point (x_{n+1}, y_{n+1}) in Figure no. 1(B). Then, the slopes of the both tangent lines at the end of the interval, $[x_n, x_{n+1}]$, are averaged shown in Figure no. 1(C), as follow

$$\text{Left slope} = f(x_n, y_n)$$
$$\text{Right slope} = f(x_{n+1}, y_{n+1}) = f\left(x_n + h, y_n + hf(x_n, y_n)\right)$$
$$\text{Avg. Slope} = \tfrac{1}{2}\left(f(x_n, y_n) + f\left(x_n + h, y_n + hf(x_n, y_n)\right)\right).$$

We substitute the averaged slope in the above equation (3) for y_{n+1}, we get Heun's method as follow

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{2}(k_1 + k_2), \\ k_1 &= hf(x_n, y_n), \\ k_2 &= hf\left(x_n + h, y_n + k_1\right). \end{aligned} \tag{4}$$

The accuracy of the Euler method improves only linearly with the step size is very small, whereas the Heun's Method improves accuracy quadratically.

Ochoche(2007,2008) attempt to improve the performance of themodified Euler's method (ME), improved modified Euler's method (IME) and improving the improved modified Euler's method (IIME)by using Taylor series expansion:

$$f(x+m, y+n) = f(x,y) + \frac{\mathrm{D}f(x,y)}{1!} + \frac{\mathrm{D}^2 f(x,y)}{2!} + O(h^3)$$

where total derivative is $\mathrm{D}^n f(x,y) = \left(m\frac{\partial}{\partial x} + n\frac{\partial}{\partial y}\right)^n f(x,y)$ and $n = 1,2$. Ochoche method with 2nd orderperformed to the ME, IME and IIME.

In this proposed paper, we use Ochoche's technique to improve the performance of Euler's method on k_1 in (3).And the new method is given by

$$\begin{aligned} y_{n+1} &= y_n + k_1, \\ k_1 &= hf(x_n, y_n) + \frac{h^2}{2}\left(f_x(x_n, y_n) + f(x_n, y_n) f_y(x_n, y_n) \right). \end{aligned} \tag{5}$$

We also modified Heun's method on k_2 in (4) and the new method is given by

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{2}(k_1 + k_2), \\ k_1 &= hf(x_n, y_n), \\ k_2 &= hf\left(x_n + h, y_n + k_1\right) + \frac{h^2}{2}\left(f_x\left(x_n + h, y_n + k_1\right) + f\left(x_n + h, y_n + k_1\right) f_y\left(x_n + h, y_n + k_1\right)\right), \end{aligned} \tag{6}$$

where $f_x = \frac{\partial}{\partial x} f(x, y)$ and $f_y = \frac{\partial}{\partial y} f(x, y)$.

III. Numerical Results

In this section, we consider three numerical examples which use formula (5) and (6).The numerical solutions are compared with the original ones in (3),(4) and analytical exact solution. Numerical results and comparing graphs are evaluated by MATLAB2017a.

Example 1: consider the IVP $\frac{dy}{dx} = \frac{\cos(x)}{y}$, $y(0) = 2$ and $h = 0.75$ on interval $[0,12]$. The IVP is separable equations, so exact solution of the given problem is $y(x) = \sqrt{2\sin(x) + 4}$. The approximation solutions and absolute errors are obtained as shown in Table no 1.The graphs of the numerical solution and their errors are displayed in Figure no 2.

Table no 1: Exact solution, approximation solution and absolute error of Euler's method, Heun's method and formula (5), (6) in Example 1.

n	x_n	Exact solution, $y(x_n)$	Approximation solution, y_n				Absolute error, $\lvert y(x_n) - y_n \rvert$			
			Euler's method	**Heun's method**	**Eq. (5)**	**Eq. (6)**	**Euler's method**	**Heun's method**	**Eq. (5)**	**Eq. (6)**
0	0.00	2.0000	2.0000	2.0000	2.0000	2.0000	0.0000	0.0000	0.0000	0.0000
1	0.75	2.3159	2.3750	2.3030	2.3486	2.2685	0.0591	0.0128	0.0328	0.0473
2	1.50	2.4485	2.6061	2.4326	2.5123	2.3581	0.1576	0.0159	0.0639	0.0903
3	2.25	2.3571	2.6264	2.3475	2.4497	2.2329	0.2693	0.0096	0.0925	0.1243
4	3.00	2.0694	2.4470	2.0743	2.1847	1.9239	0.3777	0.0049	0.1153	0.1455
5	3.75	1.6902	2.1436	1.7160	1.8113	1.5505	0.4534	0.0258	0.1211	0.1397
6	4.50	1.4300	1.8565	1.4784	1.5142	1.3698	0.4265	0.0484	0.0842	0.0602
7	5.25	1.5107	1.7714	1.5650	1.5433	1.5234	0.2607	0.0543	0.0326	0.0128
8	6.00	1.8550	1.9882	1.8866	1.8945	1.8515	0.1331	0.0315	0.0395	0.0035
9	6.75	2.2136	2.3504	2.2251	2.2771	2.1668	0.1368	0.0115	0.0635	0.0468
10	7.50	2.4240	2.6353	2.4270	2.5153	2.3330	0.2113	0.0030	0.0913	0.0910
11	8.25	2.4177	2.7340	2.4235	2.5384	2.2887	0.3163	0.0058	0.1207	0.1290
12	9.00	2.1964	2.6282	2.2155	2.3459	2.0387	0.4318	0.0191	0.1495	0.1577
13	9.75	1.8333	2.3682	1.8750	2.0040	1.6632	0.5349	0.0417	0.1707	0.1701
14	10.50	1.4969	2.0681	1.5663	1.6594	1.3676	0.5712	0.0694	0.1625	0.1292
15	11.25	1.4368	1.8956	1.5229	1.5459	1.4098	0.4588	0.0861	0.1091	0.0270
16	12.00	1.7108	1.9952	1.7770	1.7964	1.6980	0.2844	0.0662	0.0856	0.0128

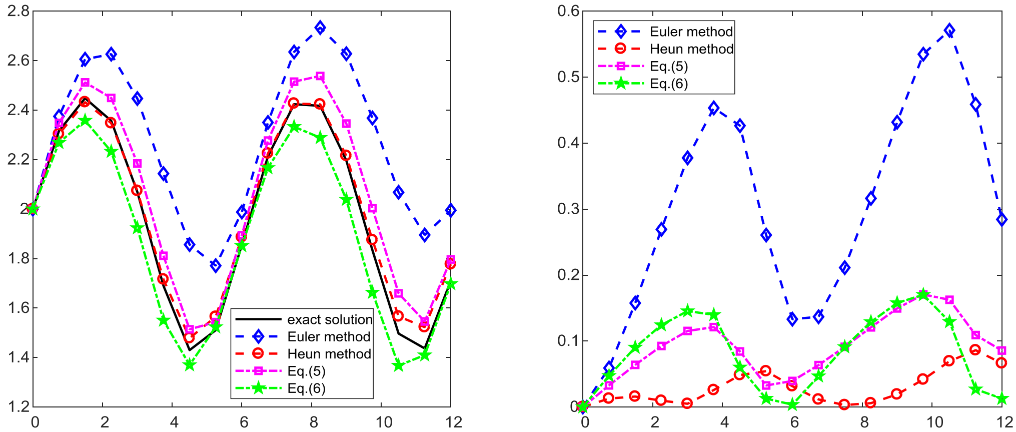

Figure no 2: Comparison on various numerical method with analytical solution (left) and their absolute method errors (right) in Example 1.

Example 2: consider the IVP $\dfrac{dy}{dx} = y\sin(x),\ \ y(0)=1$ and $h=0.75$ on interval $[0,12]$. The IVP is separable equations, so exact solution of the given problem is $y(x)=e^{1-\cos(x)}$. The approximation solutions and absolute errors are obtained as shown in Table no 2. The graphs of the numerical solution and their errors are displayed in Figure no 3.

Table no 2: Exact solution, approximation solution and absolute error of Euler's method, Heun's method and formula (5), (6) in Example 1.

n	x_n	**Exact solution,** $y(x_n)$	**Approximation solution,** y_n				**Absolute error,** $\lvert y(x_n)-y_n \rvert$			
			Euler's method	**Heun's method**	**Eq. (5)**	**Eq. (6)**	**Euler's method**	**Heun's method**	**Eq. (5)**	**Eq. (6)**
0	0.00	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000
1	0.75	1.3078	1.0000	1.2556	1.2109	1.3818	0.3078	0.0521	0.0968	0.0740
2	1.50	2.5326	1.5112	2.2864	2.1356	2.7508	1.0214	0.2463	0.3971	0.2182
3	2.25	5.0946	2.6418	4.3078	4.2133	5.1713	2.4527	0.7868	0.8812	0.0768
4	3.00	7.3155	4.1835	5.9257	6.6518	6.2757	3.1320	1.3898	0.6637	1.0397
5	3.75	6.1753	4.6262	4.8348	5.9947	4.7589	1.5491	1.3405	0.1806	1.4164
6	4.50	3.3562	2.6431	2.7859	2.8004	2.9558	0.7131	0.5702	0.5557	0.4004
7	5.25	1.6289	0.7053	1.5252	1.1873	1.7222	0.9236	0.1037	0.4417	0.0933
8	6.00	1.0406	0.2510	0.9771	0.7354	1.1704	0.7897	0.0635	0.3052	0.1297
9	6.75	1.1129	0.1984	1.0051	0.7424	1.3107	0.9146	0.1079	0.3706	0.1978
10	7.50	1.9220	0.2653	1.6475	1.1645	2.3754	1.6567	0.2745	0.7575	0.4534
11	8.25	3.9978	0.4520	3.1980	2.2850	4.8096	3.5459	0.7998	1.7128	0.8118
12	9.00	6.7607	0.7647	5.1407	4.0905	7.0950	5.9960	1.6200	2.6702	0.3343
13	9.75	7.0117	1.0011	5.1288	4.7152	6.2504	6.0106	1.8829	2.2965	0.7613
14	10.50	4.3734	0.7612	3.2278	2.7443	4.0832	3.6122	1.1456	1.6291	0.2902
15	11.25	2.1134	0.2590	1.7644	1.1064	2.4061	1.8545	0.3490	1.0070	0.2927
16	12.00	1.1690	0.0710	1.0267	0.5806	1.4789	1.0980	0.1422	0.5884	0.3099

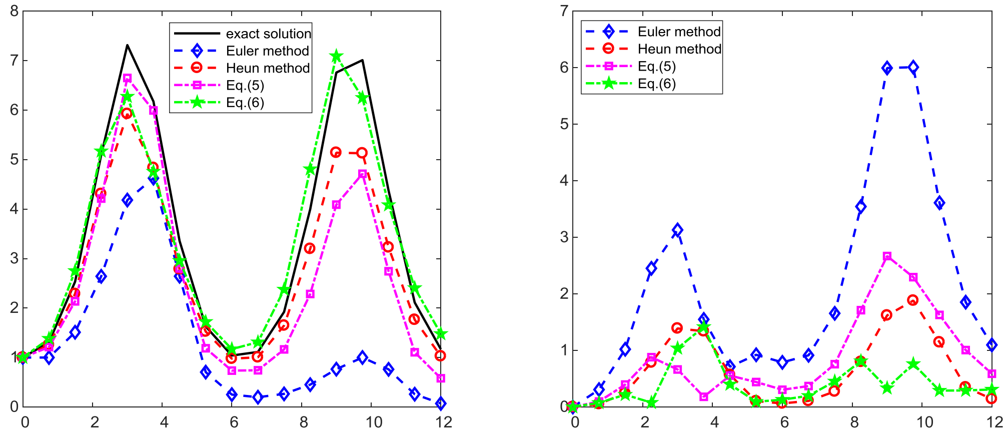

Figure no 3: Comparison on various numerical method with analytical solution (left) and their absolute method errors (right) in Example 2.

Example 3: consider the IVP $\frac{dy}{dx}=\frac{x^{10}}{e^{y}}$, $y(0)=1$ and $h=0.125$ on interval $[0,2]$. The IVP is separable equations, so exact solution of the given problem is $y(x)=\ln(\frac{x^{11}}{11}+e)$. The approximation solutions and absolute errors are obtained as shown in Table no 3. The graphs of the numerical solution and their errors are displayed in Figure no 4.

Table no 3: Exact solution, approximation solution and absolute error of Euler's method, Heun's method and formula (5), (6) in Example 3.

n	x_n	Exact solution, $y(x_n)$	Approximation solution, y_n				Absolute error, $\lvert y(x_n)-y_n \rvert$			
			Euler's method	Heun's method	Eq. (5)	Eq. (6)	Euler's method	Heun's method	Eq. (5)	Eq. (6)
0	0.000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000
1	0.125	1.0000	1.0000	1.0000	1.0000	1.0002	0.0000	0.0000	0.0000	0.0002
2	0.250	1.0000	1.0000	1.0000	1.0004	1.0007	0.0000	0.0000	0.0004	0.0007
3	0.375	1.0000	1.0000	1.0000	1.0013	1.0013	0.0000	0.0000	0.0013	0.0013
4	0.500	1.0000	1.0000	1.0000	1.0027	1.0023	0.0000	0.0000	0.0027	0.0023
5	0.625	1.0002	1.0000	1.0003	1.0045	1.0036	0.0001	0.0001	0.0043	0.0034
6	0.750	1.0014	1.0005	1.0018	1.0072	1.0065	0.0009	0.0003	0.0058	0.0050
7	0.875	1.0077	1.0031	1.0091	1.0124	1.0153	0.0046	0.0014	0.0048	0.0076
8	1.000	1.0329	1.0151	1.0376	1.0275	1.0453	0.0178	0.0047	0.0054	0.0124
9	1.125	1.1153	1.0604	1.1285	1.0756	1.1368	0.0549	0.0133	0.0397	0.0216
10	1.250	1.3288	1.2010	1.3593	1.2166	1.3643	0.1278	0.0305	0.1122	0.0354
11	1.375	1.7471	1.5513	1.7964	1.5576	1.7906	0.1958	0.0493	0.1894	0.0435
12	1.500	2.3591	2.1913	2.4091	2.1712	2.3901	0.1678	0.0500	0.1879	0.0310
13	1.625	3.0766	2.9970	3.1105	2.9527	3.0825	0.0797	0.0338	0.1240	0.0059
14	1.750	3.8194	3.7984	3.8352	3.7474	3.8033	0.0210	0.0158	0.0720	0.0161
15	1.875	4.5461	4.5529	4.5491	4.5024	4.5164	0.0069	0.0031	0.0437	0.0296
16	2.000	5.2412	5.2602	5.2369	5.2120	5.2050	0.0190	0.0043	0.0292	0.0362

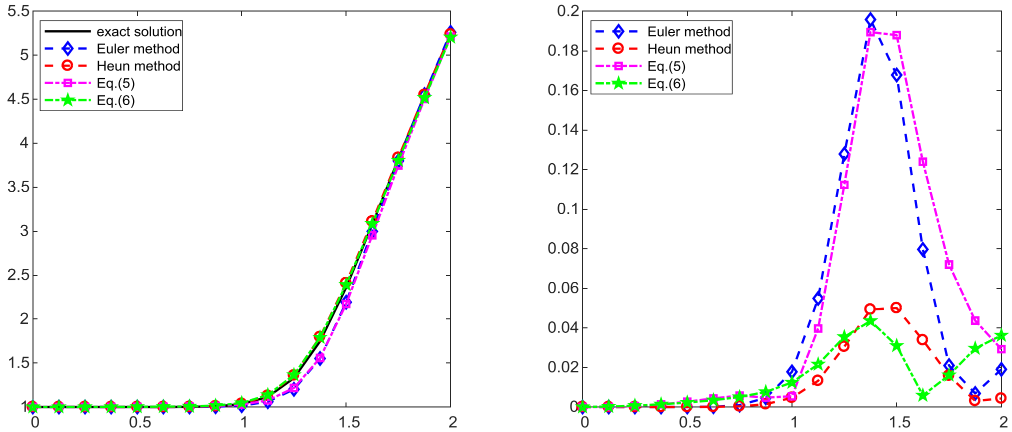

Figure no 4: Comparison on various numerical method with analytical solution (left) and their absolute method errors (right) in Example 3.

IV. Conclusion

In this paper, Euler's method and Heun's method are modified by using the 2nd order total derivative on k_1 and k_2 respectively which become to the new methods in (5) and (6). The new methods are used for solving ordinary differential equations with initial value problems (IVP) shown in Example 1-3 and compared with the original methods and the exact solution. From table no.1-3, we can see the accuracy of the new methods are better than the original ones and close to the exact solution. And also Figure no. 2-4 show that the new methods are powerful and efficient in finding numerical solutions of IVP.

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