IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 16, Issue 2 Ser. I (Mar. – Apr. 2020), PP 31-43
www.iosrjournals.org

Modifications of Adomian Decomposition Method to solve Singular Two-point Boundary Value Problems

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Abstract
Singular two-point boundary value problems arise in different areas of applied sciences such as engineering, physics and thermal management. Numerous methods like DTM, HAM and MVIM have been applied to determine solutions of these problems that require additional computational work since all boundary conditions are not included in the canonical form. This research investigated solutions for the problems in a direct way both numerically and analytically using the modifications of the decomposition method.
Symbolic programming was employed to handle linear and nonlinear STPBVPs both analytically and numerically. Examples were solved and analyzed using tables and figures for better elaborations where appreciable agreement between the approximate and exact solutions was observed. All the computations were performed using MATHEMATICA and MATLAB.
Key Words: *MATHEMATICA, MATLAB, STPBVPs, ADM*

Date of Submission: 02-03-2020 Date of Acceptance: 17-03-2020

I. Introduction

Background of the study
A boundary value problem specifies values or equations for solution components at more than one point in the range of the independent variable x.
The commonly existing problems in various fields of studies mentioned earlier lead to singular BVPs of the form:
$$u''(x) + p(x)u'(x) + q(x)f\big(u(x)\big) = r(x), x \in (a, b) \tag{1.1}$$
subject to a boundary condition with at least one of the functions $p(x), q(x)$ and $r(x)$ have a singular point. For example, when $p(x) = r(x) = 0, q(x) = -x^{-1/2}$ and $f\big(u(x)\big) = [u(x)]^{3/2}$, (1.1) is known as the Thomas-Fermi equation given by the singular equation $u'' = x^{-1/2}u^{3/2}$ which arises in the study of electrical potential in an atom.

The Adomian decomposition method (ADM)is a well-known systematic method for practical solution of linear or nonlinear and deterministic or stochastic operator equations, including ODEs. The method is a powerful technique, which provides efficient algorithms for analytic approximate solutions and numeric simulations for real-world applications in the applied sciences and engineering. It permits to solve both nonlinear IVPs and BVPs without restrictive assumptions such as required by linearization, perturbation, discretization, guessing the initial term or a set of basis functions, and so forth. The accuracy of the analytic approximate solutions obtained can be verified by direct substitution. A key notion is the Adomian polynomials, which are tailored to the particular nonlinearity to solve nonlinear operator equations. The decomposition method has been used extensively to solve effectively a class of linear and nonlinear ordinary and partial differential equations. However, a little attention was devoted for its application in solving the singular two-point boundary value problems (STPBVPs). This research treated some classes of singular second-order two-point boundary value problems both analytically and numerically using the ADM and the modifications Improved Adomian Decomposition Method (IADM) and MADM focusing on the Dirchlet and mixed boundary conditions; and applied the symbolic softwares MATLAB and MATHEMATICA to facilitate computing.

II. Materials and Methods

Sources in the web and libraries were used to collect all the pieces of information about the singular two-point boundary value problems together with the methods and recorded subsequently. Specifically,

- relevant journals and books were addressed to gather information about STPBVPs and the methods to treat the problems.
- Identifying the second order singular linear and nonlinear two-point BVPs the collected information was arranged keeping coherence for analysis.

- symbolic softwares, MATLAB and MATHEMATICA were applied suitably to ease the computations by the methods and graphs were plotted using the programs.
- It was studied from October, 2017 to July, 2018.

III. Singular Two-Point BVPs using the Stated Methods

3.1 General Description of the Adomian Decomposition Method

In reviewing the basic methodology involved, consider a general differential equation in an operator form:

$$Lu + Ru + Nu = r \tag{3.1}$$

where L is an operator representing the linear portion which is easily invertible, N is the nonlinear operator representing the nonlinear term and$R$ is a linear operator for the remainder of the linear portion.

The Adomian decomposition method introduces the solution $u(x)$ and the nonlinear function Nu by the infinite series as:

$$u(x) = \sum_{i=0}^{\infty} u_n(x) \tag{3.2}$$

and

$$Nu = \sum_{i=0}^{\infty} A_n(u_0, u_1, u_2, \dots, u_n) \tag{3.3}$$

where, $A_n$are the Adomian polynomials thatcan bedetermined byAdomian formula:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}, n = 0,1,2, \dots \tag{3.4}$$

So for $n = 0$, (3.4) reduces to

$$A_0 = N(u_0)$$

For $n = 1$, it reduces to

$$A_1 = \frac{d}{d\lambda}[N(u_0 + \lambda u_1)]_{\lambda=0}$$

$$\Rightarrow A_1 = u_1 N'(u_0)$$

.
.
.

It can be observed that A_0depends only on u_0, A_1depends only on u_0 and u_1, A_2 depends only on u_0, u_1 and u_2 and so on.

Optionally, a simple way of computing Adomian polynomials of any type of nonlinearity is presented by applying the decomposition of positive integers n as a subscript of the variable u for nonlinear terms through the use of MATHEMATICA. MATHEMATICA exploits general symbolic programming for generating Adomian polynomials.

Now, substituting (3.2) and (3.3) into (3.1), one can get:

$$\sum_{i=0}^{\infty} u_n(x) = g(x) - L^{-1}\sum_{i=0}^{\infty} Ru_n - L^{-1}\sum_{i=0}^{\infty} A_n \tag{3.5}$$

The recursive relationship is found to be

$$u_0 = g(x)$$
$$u_1 = -L^{-1}Ru_0 - L^{-1}A_0$$
$$u_2 = -L^{-1}Ru_1 - L^{-1}A_1$$

.
.
.

$$u_i = -L^{-1}Ru_{i-1} - L^{-1}A_{i-1}$$

for $i = 1,2,3, \dots$. So, having determined the components $u_n, n \geq 0$ the solution $u$in a series form follows immediately by (3.2).

So, u_1 will be a polynomial. The same procedure holds to obtain u_k as a polynomial.

The classical ADM is very powerful in treating nonlinear BVPs, though this is one of the qualities of the method over some other methods it has its own short comings like its failure to treat some nonlinear singular boundary value problems. The following subsection addresses some merits and demerits of the method.

3.1.1 Advantages and disadvantages of the ADM

Many authors found that the ADM requires less computational work than traditional approaches. In addition, it includes the ability to solve nonlinear problems without linearization, the wide applicability to several types of problems and scientific fields, and the development of a reliable, analytic solution. This method does not linearize the problem nor use assumptions of weak nonlinearity and therefore can handle nonlinearities which are quite general and generates solutions that may be more realistic than those achieved by simplifying the model to achieve conditions required for other techniques. Moreover, the ADM is analytic, requiring neither linearization nor perturbation, and continuous with no resort to discretization.

The ADM does have some disadvantages, however. To begin with, the method gives a series solution which must be truncated for practical applications. ADM requires the use of Adomian polynomials for nonlinear terms, and this needs more work. In addition, the rate and region of convergence are potential shortcomings. Although the series can be rapidly convergent in a very small region, it has very slow convergence rate in wider region and the truncated series solution is an inaccurate solution in that region, which will restrict the application area of the method.

3.1.2 MADM

Since the introduction of the method in early 1980's, ADM has led to several modifications made by various researchers in an attempt to improve the accuracy and expand the application of the original method. As pointed out above, the rate of convergence of the series solutions is one of the potential shortcomings of the decomposition method. To improve on this, the authors tried to introduce different modifications of the method. To begin with, based on Wazwaz (1999) the standard ADM is modified in such a way that the function g in (3.5) can bedivided into two parts as follows to increase rate of convergence of the series solution and minimize the size of computations. This modification is applicable irrespective of the types of the BVPs under consideration.

$$g = g_0 + g_1 \tag{3.6}$$

Accordingly, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the parts $g_0$be assigned to the component u_0 , whereas the remaining part g_1 be combined with other terms in the recursive relation to define u_1 to get the recursive relation:

$$\begin{aligned} u_0 &= g_0 \\ u_1 &= g_1 - L^{-1}Ru_0 - L^{-1}A_0 \\ u_{n+1} &= -L^{-1}(Ru_n) - L^{-1}(A_n), n \geq 1 \end{aligned} \tag{3.7}$$

Although this variation in the formation of u_0 and u_1 is slight, however it plays a major role in accelerating the convergence of the solution and in minimizing the size of calculations. In many cases the modified scheme avoids unnecessary computations, especially in calculation of the Adomian polynomials. In other words, sometimes there is no need to evaluate the so-called Adomian polynomials required for nonlinear operators or if needed to evaluate these polynomials the computation will be reduced very considerably by using the modified recursive scheme. There are two important remarks related to the modified method. First, by proper selection of the functions g_0 and g_1, the exact solution u may be obtained by using very few iterations, and sometimes by evaluating only two components. The success of this modification depends only on the choice of g_0 and g_1, and this can be made through trials, that are the only criteria which can be applied so far. Second, if g consists of one term only, the scheme (3.7) should be employed in this case.

Another modification of the standard ADM which alleviates the deficiency of treating some singular boundary value problems like, BVPs subjected to a mixed boundary conditions is MADM presented by Hasan and Zhu (2009). In fact, it is a slight refinement to the original ADM; it only modifies the involved differential operator. Generally, MADM by the authors mentioned proposes the differential and inverse operators:

$$L = x^{-1}\frac{d^{n-1}}{dx^{n-1}}x^{n-k}\frac{d}{dx}x^{k-n+1}\frac{d}{dx}(.) \tag{3.8}$$

and

$$L^{-1}(.) = \underbrace{\int_b^x x^{n-k-1}\int_0^x x^{k-n}\int_0^x \dots \int_0^x}_{n-1 \text{ times}} x(.)dx \dots dx \tag{3.9}$$

for treatment of $n+1$order boundary value problem of the form:

$$u^{(n+1)} + \frac{k}{x}u^{(n)} + Nu = f \qquad n = 0,1,2,\dots; k = 0,1, or\ 2 \tag{3.10}$$

Kim and Chun (2010) came up with another modification of the standard ADM to solve singular $n+1$order boundary value problems. This scheme is designed in such a way that BVPs with singular nature can easily be treated. In this research the operators are used to treat singular second-order BVPs with mixed boundary conditions. Generally, to see what MADM by the authors mentioned look like, consider the singular boundary value problem of $n+1$ order ordinary differential equation (3.10) given the following way:

$$u^{(n+1)} + \frac{k}{x} u^{(n)} + Nu = f(x) \tag{3.11}$$

$$u(0) = a_0, u'(0) = a_1, \dots, u^{r-1}(0) = a_{r-1},$$
$$u(b) = c_0, u'(b) = c_1, \dots, u^{n-r}(b) = c_{n-r} \tag{3.12}$$

Where N is nonlinear differential operator of order less than n , $f(x)$ is a given function, $a_0, a_1, \dots, a_{r-1}, c_0, c_1, \dots, c_{n-r}, b$ are given constants, where $k \le r \le n, r \ge 1$.
Now (3.11) can be re written in the form

$$x^{-2} \frac{d^{n-1}}{dx^{n-1}} [x^2 u'' + (k - 2n + 2) x u'] + Nu = f \tag{3.13}$$

Or equivalently,

$$x^{-2} \frac{d^{n-1}}{dx^{n-1}} \left[x^{2n-k} \frac{d}{dx} \left(x^{k-2n+2} \frac{du}{dx} \right) \right] + Nu = f \tag{3.14}$$

(3.13) can be written in the operator form

$$L_2 L_1 u = f(x) - Nu \tag{3.15}$$

Where, the differential operator L employs the first two derivatives

$$L_1 = x^{2n-k} \frac{d}{dx} \left(x^{k-2n+2} \frac{d}{dx} \right) \tag{3.16}$$

$$L_2 = x^{-2} \frac{d^{n-1}}{dx^{n-1}} \tag{3.17}$$

in order to overcome the singularity behavior at $x = 0$.
In view of (3.16) and (3.17), the inverse operators L_1^{-1} and L_2^{-1} are the integral operators defined by

$$L_1^{-1} = \int_0^x x^{2n-k-2} \int_b^x x^{k-2n} (.) dx dx, \tag{3.18}$$

$$L_2^{-1} = \int_0^x \dots \int_0^x x^2 (.) dx \dots dx, \tag{3.19}$$

$n - 1$ time
By applying L_2^{-1} on (3.15), one can have

$$L_1 u = \Psi_1(x) + L_2^{-1} f(x) - L_2^{-1} Nu \tag{3.20}$$

such that

$$L_2 \Psi_1(x) = 0 \tag{3.21}$$

By applying L_1^{-1} on (3.20), one can have

$$u(x) = \Psi_2(x) + L_1^{-1} \Psi_1(x) + L_1^{-1} L_2^{-1} f(x) - L_1^{-1} L_2^{-1} Nu \tag{3.22}$$

such that

$$L_1 \Psi_2(x) = 0 \tag{3.23}$$

The standard ADM introduces the solution $u(x)$ and the nonlinear function Nu by infinite series given by (3.2)and (3.3) where the Adomian polynomials are determined by the formula at (3.5). Substituting (3.2) and (3.3) in to (3.22) gives

$$\sum_{n=0}^{\infty} u_n = \Psi_2(x) + L_1^{-1} \Psi_1(x) + L_1^{-1} L_2^{-1} f(x) - L_1^{-1} L_2^{-1} \sum_{n=0}^{\infty} A_n \tag{3.24}$$

Identifying $u_0 = \Psi_2(x) + L_1^{-1} \Psi_1(x) + L_1^{-1} L_2^{-1} f(x)$, the Adomian method admits the use of the recursive relation
$u_0 = \Psi_2(x) + L_1^{-1} \Psi_1(x) + L_1^{-1} L_2^{-1} f(x)$

$$u_{n+1} = -L^{-1} A_n \tag{3.25}$$

which gives

$$\begin{aligned} u_0 &= \Psi(x) + L^{-1} f(x) \\ u_1 &= -L^{-1} A_0 \\ u_2 &= -L^{-1} A_1 \\ u_3 &= -L^{-1} A_2 \\ &\vdots \end{aligned} \tag{3.26}$$

This leads to the complete determination of the components u_n of $u(x)$. The series solution $u(x)$ defined by (3.2) follows immediately.

3.1.3 IADM

Ebaid (2010) made improvements of operators developed earlier by Lesnic (2001) for the purpose of treating the heat equation with Dirchlet boundary condition. In this work the IADM is used to deal with linear and nonlinear STPBVPs with Dirchlet boundary conditions. The improvement is based on the ADM and Lesnic's work later developed by Ebaid (2010). Lesnic (2001) proposed the inverse operators:

$$L^{-1}{}_{xx}(.) = \int_{x_0}^{x}\int_{x_0}^{x}(.)dxdx - \frac{x-x_0}{1-x_0}\int_{x_0}^{1}\int_{x_0}^{x}(.)dxdx \ , L_t^{-1} = \int_{0}^{t}(.)dt \quad (3.27)$$

to solve the Dirchlet BVP for the heat equation

$$u_t = u_{xx}, x_0 < x < 1, t > 0 \quad (3.28)$$

under the boundary conditions $u(x_0,t) = f_0(t)$, $u(1,t) = f_1(t)$ and the initial condition $u(x,0) = p(x)$. Using the definition in (3.27) it is observed that

$$L^{-1}{}_{xx}(u_{xx}) = u(x,t) - u(x_0,t) - \frac{x-x_0}{1-x_0}[u(1,t) - u(x_0,t)] \quad (3.29)$$

i.e., the boundary conditions can be used directly. However, from (3.27) again one can see that the lower bound of all integrations is restricted to the initial point x_0.
In fact, this restriction can be avoided by using a new definition of $L^{-1}{}_{xx}$ which gives the same result as in (3.29) and given by:

$$L^{-1}{}_{xx}(.) = \int_{x_0}^{x}\int_{c}^{x}(.)dxdx - \frac{x-x_0}{1-x_0}\int_{x_0}^{1}\int_{c}^{x}(.)dxdx \quad (3.30)$$

where, c is free lower point. This free lower point plays an important role if the equation being solved has a singular point.So the desired operator originally designed by the author mentioned earlieris derived as follows.
$L^{-1}{}_{xx}(.)$ is defined as:

$$L^{-1}{}_{xx}(.) = \int_{a}^{x}\int_{c}^{x}(.)dxdx - z(x)\int_{d}^{b}\int_{e}^{x}(.)dxdx \quad (3.31)$$

where $z(x)$ is to be determined such that $L^{-1}{}_{xx}(u''(x))$ can be expressed only in terms of the boundary conditions given inequation:
$u''(x) + p(x)u'(x) + q(x)f(u(x)) = r(x), x \in (a,b)$with the respective boundary conditions.
Using this definition, thus:
$L^{-1}{}_{xx}(u''(x)) = u(x) - u(a) - (x-a)u'(c) - z(x)[u(b) - u(d) - (b-d)u'(e)]$.
$= u(x) - u(a) - z(x)[u(b) - u(d)] - (x-a)u'(c) + z(x)[(b-d)]u'(e)$.
Setting $d = a$ and $e = c$,

$$L^{-1}{}_{xx}(u''(x)) = u(x) - u(a) - z(x)[u(b) - u(a)] - (x-a)u'(c) + z(x)(b-a)u'(c). \quad (3.32)$$

In order to express $L^{-1}{}_{xx}(u''(x))$ in terms of the two boundary conditions only, the coefficient $u'(c)$ has to be eliminated by setting
$-(x-a)u'(c) + z(x)[(b-a)u'(c)] = 0$ assuming $u'(c) \neq 0$.

$$\Rightarrow \ z(x) = \frac{x-a}{b-a} \quad (3.33)$$

Using (3.33) in (3.31), the operator below proposed by Ebaid (2010) is obtained to solve the singular two-point Dirchlet BVPs.

$$L^{-1}{}_{xx}(.) = \int_{a}^{x}\int_{c}^{x}(.)dxdx - \frac{x-a}{b-a}\int_{a}^{b}\int_{c}^{x}(.)dxdx \, , a \neq b, c - \text{constant} \quad (3.34)$$

Thus (3.32) is reduced to:

$$L^{-1}{}_{xx}(u''(x)) = u(x) - u(a) - \frac{x-a}{b-a}[u(b) - u(a)] \quad (3.35)$$

From (3.34) it is noted that $L^{-1}{}_{xx}(u''(x))$is already expressed in terms of the given boundary conditions without any restrictions on c. So, the choice of the value that c can take depend properly on the singular point of the equation under consideration. For example, if the equation has a singular point say at $x = x_0$, c will be chosento be any real value except the value of x_0. Moreover, if the equation has two singular points at $x = x_1$and $x = x_2$, then c willbeconsidered any real value except these values of $x_1$and x_2. In general, if the equation has $n$singular points x_1, x_2,...,x_n, then c takes any real value except the values of these singular points.
For solving linear and non-linear singular two-point boundary value problems under the Dirchlet boundary condition, IADM is established using (3.34) together with the standard ADM.
Consider $u''(x) + p(x)u'(x) + q(x)f(u(x)) = r(x), x \in (a,b)$ in the form:

$u''(x) = r(x) - p(x)u'(x) - q(x)f(u(x))$

Applying the operator $L^{-1}{}_{xx}(.)$ given by (3.34) on both sides of this it is observed that:

$$u(x) = u(a) + \frac{x-a}{b-a}[u(b) - u(a)] + L^{-1}{}_{xx}[r(x)] - L^{-1}{}_{xx}[p(x)u'(x)] - L^{-1}{}_{xx}[q(x)f(u(x))]. \quad (3.36)$$

The Adomian decomposition method introduces the solution $u(x)$ and the nonlinear function $f(u)$ by infinite series as in (3.2) and (3.4), respectively.

Substituting the results into (3.36) and according to the ADM, the solution $u(x)$ can be smartly computed by using the recurrence relations constructed based on the following cases.

Case 1:-If $f(u) = u$, i.e., linear function, then the solution $u(x)$can be computed by using the recurrence relation:

$$u_0(x) = u(a) + \frac{x-a}{b-a}[u(b) - u(a)] + L^{-1}{}_{xx}[r(x)]$$
$$u_{n+1}(x) = -L^{-1}{}_{xx}[p(x)u_n'(x) + q(x)u_n(x)],\ n \geq 0 \quad (3.37)$$

Case 2:- If $f(u)$ is nonlinear function, then the recurrence relation required to compute the solution $u(x)$ is:

$$u_0(x) = u(a) + \frac{x-a}{b-a}[u(b) - u(a)] + L^{-1}{}_{xx}[r(x)]$$
$$u_{n+1}(x) = -L^{-1}{}_{xx}[p(x)u_n'(x) + q(x)A_n(x)],\ n \geq 0 \quad (3.38)$$

where (3.37) and (3.34), (3.38) and (3.34) improve the standard ADM, (IADM) and can be used to solve linear and nonlinear singular two-point boundary value problems subject toDirchlet boundary conditions.Hence, using the recurrence relation (3.37) or(3.38) depending on linearity behavior of the boundary value problem, the $n-$ termtruncated approximate solution can be computed easily.

3.2 Numerical and Analytical Illustrations

Example 3.1. Consider the inhomogeneous singular Bessel equation

$$u''(\mathrm{x}) + \frac{1}{x}u'(\mathrm{x}) + u(\mathrm{x}) = 4 - 9\mathrm{x} + x^2 - x^3 \quad (3.39)$$

subject to the boundary conditions $u(0) = 0$ and $u(1) = 0$.

I. IADM solution

The equation considered has a singular point at $x = 0$. So, by IADM the free lower point $c$can be chosen to be any real value except zero. The appropriate recurrence relation to determine solution of the inhomogeneous singular BVP would be:

$$u_0(x) = u(a) + \frac{x-a}{b-a}[u(b) - u(a)] + L^{-1}{}_{xx}[r(x)]$$
$$u_{n+1}(x) = -L^{-1}{}_{xx}[p(x)u_n'(x) + q(x)u_n(x)], n \geq 0 \quad (3.40)$$

where $a = 0, b = 1, u(a) = 0 = u(b)$ and $r(x) = 4 - 9\mathrm{x} + x^2 - x^3, p(x) = \frac{1}{x}, q(x) = 1$.

Putting the given information above, the simplified u_0 looks like:

$u_0 = L^{-1}{}_{xx}[r(x)]$

But the operator $L^{-1}{}_{xx}$which is designed so as to treat the singularity at $x = 0$is defined as:

$$L^{-1}{}_{xx}(.) = \int_a^x \int_c^x (.)dxdx - \frac{x-a}{b-a}\int_a^b \int_c^x (.)dxdx \quad (3.41)$$

where a and b are the two-points of the BVP, c is the free constant which plays a great role in treating the problem with singularity feature.Hence using$r(x)$ in (3.41)results in:

$$u_0 = -\frac{8x}{15} + 2x^2 - \frac{3x^3}{2} + \frac{x^4}{12} - \frac{x^5}{20}$$

which is the first term of the decomposition.

For $n = 0$in the second equation of the recurrence relation(3.41), $u_1$can be determined to be:

$$u_1 = \frac{711x}{560} - 2x^2 + \frac{151x^3}{180} - \frac{7x^4}{80} + \frac{7x^5}{80} - \frac{x^6}{360} + \frac{x^7}{840} + \frac{8}{15}x\ln x$$

This way MATHEMATICA is used to determine the rest of the required components of the solution to get better result. To demonstrate, the next four approximate solutions are found to be the following:

$$u_2 = -\frac{488519x}{302400} + 2x^2 - \frac{16843x^3}{30240} + \frac{25x^4}{108} - \frac{919x^5}{14400} + \frac{19x^6}{2700} - \frac{23x^7}{10080} + \frac{x^8}{20160} - \frac{x^9}{60480} - \frac{711}{560}x\ln x - \frac{4}{45}x^3\ln x - \frac{4}{15}x(\ln x)^2$$

$$u_3 = -\frac{943433x}{67060224} + \frac{194849x^2}{604800} - \frac{117037x^3}{604800} - \frac{9041x^4}{72576} + \frac{14389x^5}{1008000} - \frac{7243x^6}{1296000} + \frac{2453x^7}{1814400} - \frac{719x^8}{8467200} + \frac{x^9}{32256} - \frac{x^{10}}{2721600} + \frac{x^{11}}{6652800} + \frac{263x^2\ln x}{1120} + \frac{4159x^3\ln x}{30240} + \frac{1}{135}x^4\ln x + \frac{1}{225}x^5\ln x + \frac{2}{15}x^2(\ln x)^2 + \frac{2}{45}x^3(\ln x)^2$$

$$u_4 = \frac{54021054343189x}{274611617280000} - \frac{30587x^2}{86400} + \frac{1230960917x^3}{10059033600} + \frac{19237x^4}{907200} + \frac{532313x^5}{60480000} + \frac{1458971x^6}{272160000} - \frac{50719x^7}{95256000} + \frac{398047x^8}{3556224000} - \frac{1691x^9}{74649600} + \frac{6751x^{10}}{6858432000} - \frac{79x^{11}}{266112000} + \frac{x^{12}}{359251200} - \frac{x^{13}}{1037836800} + \frac{943433x\ln x}{67060224} + \frac{107x^2\ln x}{3360} - \frac{563x^3\ln x}{12096} - \frac{3293x^4\ln x}{362880} - \frac{18107x^5\ln x}{3024000} - \frac{x^6\ln x}{4050} - \frac{x^7\ln x}{9450} - \frac{2}{15}x^2(\ln x)^2 - \frac{1}{45}x^3(\ln x)^2 - \frac{1}{90}x^4(\ln x)^2 - \frac{1}{450}x^5(\ln x)^2$$

$$u_5 = -\frac{4358100181606400153x}{7612234031001600000} + \frac{394649x^2}{604800} - \frac{161635981016989x^3}{1647669703680000} + \frac{68831x^4}{2332800} - \frac{9387291997x^5}{1005903360000} - \frac{14899621x^6}{8164800000} - \frac{25647329x^7}{160030080000} - \frac{84312989x^8}{746807040000} + \frac{1445527x^9}{146313216000} - \frac{11692409x^{10}}{8641624320000} + \frac{67717x^{11}}{287400960000} - \frac{76781x^{12}}{9958443264000} + \frac{19x^{13}}{9580032000} - \frac{x^{14}}{65383718400} + \frac{x^{15}}{217945728000} - \frac{54021054343189x\ln x}{274611617280000} - \frac{1003x^2\ln x}{3360} + \frac{19748507x^3\ln x}{2011806720} - \frac{4099x^4\ln x}{272160} + \frac{10267x^5\ln x}{4032000} + \frac{4369x^6\ln x}{54432000} + \frac{16187x^7\ln x}{127008000} + \frac{x^8\ln x}{226800} + \frac{x^9\ln x}{680400} - \frac{943433x(\ln x)^2}{134120448} + \frac{2}{15}x^2(\ln x)^2 + \frac{1}{90}x^3(\ln x)^2 + \frac{2}{135}x^4(\ln x)^2 + \frac{1}{600}x^5(\ln x)^2 + \frac{x^6(\ln x)^2}{2700} + \frac{x^7(\ln x)^2}{18900}$$

The first ten terms of the decomposition were used to get the IADM solution of the problem as:

$$\Phi_{10} = \sum_{i=0}^{9} u_i$$

II. MADM solution

Based on MADM by Wazwaz (1999) the singular BVP can be treated as follows.

$(x^\alpha u')' = F(x,u)$

$u'' + \frac{1}{x}u' + u = 4 - 9\mathrm{x} + x^2 - x^3$

Multiplying both sides by $x$the above equation becomes

$xu'' + u' = -\mathrm{x}u + 4\mathrm{x} - 9x^2 + x^3 - x^4$

Writing the left-hand side in its compact form becomes:

$(xu')' = -xu + 4x - 9x^2 + x^3 - x^4$

Now use the standard ADM recursive relation. In addition, using MADM by Wazwaz (1999) minimizes the size of computations and results in the exact solution the following way.

$$u_0 = \int_0^x \left(x^{-1} \int_0^x [4x - 9x^2 + x^3 - x^4] dx \right) dx$$

$$\Rightarrow u_0 = x^2 - x^3$$

$$u_1 = \frac{x^4}{16} - \frac{x^5}{25} - \int_0^x \left(x^{-1} \int_0^x [xu_0] dx \right) dx \quad \Rightarrow u_1 = 0 \quad \Rightarrow u_{n+1} = 0, n \geq 1.$$

The exact solution becomes:

$$u(x) = x^2 - x^3.$$

MATLAB is used to plot the graphs of the IADM and MADM solutions togetherin figure below considering numerical results of the first ten components of the solutions as follows.

x =[0 0.05 0.1 0.15 0.2 0.25 0.3 0.35 0.4 0.45 0.5 0.55 0.6 0.65 0.7 0.75 0.8 0.85 0.9 0.95 1];

Φ_{IADM} =[0 0.004511209 0.009022418 0.020523838 0.032025259 0.047528057 0.063030855 0.079527513 0.096024171 0.110520476 0.125016781 0.134513954 0.144011127 0.145509085 0.147007044 0.13850554 0.128004037 0.104502898 0.081001759 0.04500879 0];
$u =$ [0 0.002375000 0.009000000 0.019125000 0.032000000 0.046875000 0.063000000 0.0796250000.096000000 0.111375000 0.125000000 0.136125000 0.144000000 0.147875000 0.147000000 0.140625000 0.128000000 0.108375000 0.081000000 0.045125000 0];

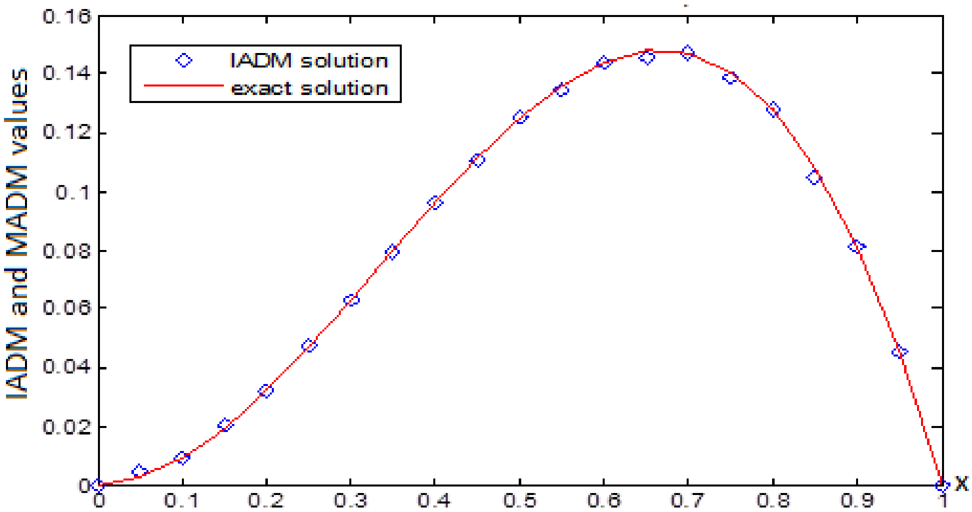

Figure 3.1. Comparison of IADM and MADM solutions of example 3.1.

Numerical illustrations of the problem by the methods used in this research and other related methods are shown in table 3.1.

Table 3.1. Numerical results for example 3.1.

x	Approximate Solution	Exact Solution	$u(x) - u^*_{26}$	Error $/u(x) - \Phi_{IADM}/$
0	0	0	0	0
0.1	0.009022418	0.009000000	2.3E−05	2.2418E−05
0.2	0.032025259	0.032000000	1.1E−05	2.5259E−05
0.3	0.063030855	0.063000000	5.5E−05	3.0855E−05
0.4	0.096024171	0.096000000	2.3E−04	2.4171E−05
0.5	0.125016781	0.125000000	1.1E−04	1.6781E−05
0.6	0.144011127	0.144000000	1.2E−04	1.1127E−05
0.7	0.147007044	0.147000000	1.6E−04	7.0442E−06
0.8	0.128004037	0.128000000	1.5E−04	4.0377E−06
0.9	0.081001759	0.081000000	4.1E−05	1.7590E−06
1	0	0	0	0

u^*_{26} is the approximate solution obtained at the 26th iteration by Cui and Geng (2007).
In order to verify numerically whether the approach IADM leads to accurate solutions, the symbolic programs, MATLAB and MATHEMATICA are used to evaluate the decomposition series solutions using the *n*-terms approximation. The approximate solution Φ_{10} is compared with the exact solution$u(x) = x^2 - x^3$obtained by MADM both in figure3.1 and table3.1. The numerical results showthat a very good approximation is achieved using small values of n-terms of the decomposition series solution. It is also important to note that the approach by Lesnic (2001) fails to overcome the singularity at $x = 0$ for this singular problem. Moreover, a comparison of the numerical results for the absolute errors$/u(x) - \Phi_{IADM}/$with thatof Reproducing Kernel method byCui and Geng (2007) are shown in table 3.1.In addition, almost the same result as the result at the 26th iteration is obtained by the method mentioned after 51 iterations. This shows that the approaches used in this project are not only easier and confidential but also by far more accurate.

Example 3.2. Consider the nonlinear singular BVP

$$u''(x) + \frac{1}{2x}u'(x) = e^u\left(\frac{1}{2} - e^u\right), x \in (0,1)$$

subject to the boundary conditions $u(0) = ln(2)$ and $u(1) = 0$.

I. IADM solution

As can be expected, it is necessary to represent the nonlinear part by the Adomian polynomials. Here the nonlinear term is

$$p(u) = e^{u}(0.5 - e^{u})$$

The required recurrence relation is (5.61) which is given by:

$$u_0(x) = u(a) + \frac{x-a}{b-a}[u(b) - u(a)] + L^{-1}{}_{xx}[r(x)]$$
$$u_{n+1}(x) = -L^{-1}{}_{xx}[p(x)u_n'(x) + q(x)A_n(x)],\ n \geq 0$$
$$\Rightarrow u_0 = ln2 + x(-ln2) = (1-x)ln2$$

But for fast convergence to the exact solution, MADM by Wazwaz (1999) helps to rearrange the result obtained above so thatu_0 assumes to be zero and all the existing terms obtained to be added to u_1 the following way:

For $n = 0$, $u_1 = -L^{-1}{}_{xx}[\frac{1}{2x}u_n' + A_0]$

The Adomian polynomial A_0 can be determined either using the Adomian formula (3.4) or the results provided by MATHEMATICA at appendix C to be:

$$A_0 = p(u_0) = e^{u_0}(0.5 - e^{u_0})$$

$$\Rightarrow L^{-1}{}_{xx}[-0.5] = \frac{1}{4}x - \frac{1}{4}x^2 \qquad (3.42)$$

Adding the previous result of u_0 i.e. $(1-x)ln2$ to (3.42) one can get u_1 as:

$$u_1 = (1-x)\ln 2 + \frac{1}{4}x - \frac{1}{4}x^2$$

Recall that $A_1 = u_1 p'(u_0)$

$$A_2 = u_2 p'(u_0) + \frac{1}{2}u_1^2 p''(u_0)$$

$$A_3 = u_3 p'(u_0) + u_1 u_2 p''(u_0) + \frac{1}{6}u_1^3 p'''(u_0)$$

$$\text{i.e.}, A_1 = \frac{1}{2}(1 - 4e^{u_0})u_1$$

$$A_2 = \frac{1}{4}e^{u_0}[u_1^2 + 2u_2 - 8e^{u_0}(u_1^2 + u_2)]$$

$$A_3 = \frac{1}{12}e^{u_0}[u_1^3 + 6u_1u_2 + 6u_3 - 8e^{u_0}(2u_1^3 + 6u_1u_2 + 3u_3)]$$

For $n = 1$,

u_2 from the second equation of the recurrence relation would be:

$$u_2 = -L^{-1}{}_{xx}[\frac{1}{2x}\left(\frac{1}{4}ln2 - \frac{1}{2}x\right) + \frac{3}{2}((1-x)ln2 + \frac{1}{4}x - \frac{1}{4}x^2)]$$

$$\Rightarrow u_2 = \frac{x}{8} + \frac{x^2}{8} - \frac{x^3}{16} + \frac{x^4}{32} - \frac{3}{4}x^2\ln 2 + x\left(-\frac{7}{32} + \frac{3\ln 2}{4} + \frac{\ln 16}{16}\right) - \frac{1}{8}x\ln 16 + \frac{1}{16}x^3\ln 16 - \frac{1}{8}x\ln x + \frac{1}{8}x\ln 16\ln x$$

The terms of the decomposition components are getting too vast to solve by hand but MATHEMATICA facilitates computing. So it can be used to list as many decomposition terms as desired. Though more than ten terms of the decomposition are used in the IADM solution, it is believed not economical to write the next iterative results, it is found important to list only up to the fifth iteration below.

$$\begin{aligned} u_3 = & -\frac{3x}{64} - \frac{x^2}{16} + \frac{5x^3}{384} - \frac{23x^4}{768} + \frac{x^5}{64} - \frac{x^6}{192} + \frac{3}{8}x\ln 2 + \frac{3}{8}x^2\ln 2 - \frac{1}{3}x^3\ln 2 + \frac{23}{96}x^4\ln 2 - \frac{7}{160}x^5\ln 2 \\ & - \frac{7}{8}x^2(\ln 2)^2 + \frac{7}{12}x^3(\ln 2)^2 - \frac{7}{48}x^4(\ln 2)^2 - \frac{1}{16}x\ln 16 + \frac{5}{192}x^3\ln 16 - \frac{3}{640}x^5\ln 16 \\ & + x(\frac{89}{768} - \frac{49\ln 2}{80} + \frac{7(\ln 2)^2}{16} + \frac{79\ln 16}{1920} - \frac{\ln 256}{64}) + \frac{1}{64}x\ln 256 + \frac{3}{64}x\ln x + \frac{1}{32}x^3\ln x \\ & - \frac{3}{8}x\ln 2\ln x + \frac{1}{16}x\ln 16\ln x - \frac{1}{32}x^3\ln 16\ln x - \frac{1}{64}x\ln 256\ln x + \frac{1}{32}x(\ln x)^2 \\ & - \frac{1}{32}x\ln 16\,(\ln x)^2 \end{aligned}$$

$$u_4 = \frac{1}{7680}x(-345 + 912\ln 2 - 1976(\ln 2)^2 + \frac{1}{7}(210 - 524\ln 2 - 79\ln 16) - \frac{3}{2}(-51 - 1616(\ln 2)^2 + 320(\ln 2)^3 - 56\ln 2(-15 + \ln 16) - 39\ln 16) + 96\ln 16 - \ln 2(-524 + 56\ln 16) + \frac{1}{6}(695 + 39120(\ln 2)^2 - 28800(\ln 2)^3 - 446\ln 16 - 8\ln 2(337 + 560\ln 16)) + 240(-1 - 14(\ln 2)^2 + 20(\ln 2)^3 + \ln 64) + \frac{5}{6}(-132 - 7424(\ln 2)^2 + 2880(\ln 2)^3 - 140\ln 16 + 49(\ln 16)^2 + 56\ln 2(37 + \ln 4096))) + \frac{1}{7680}(\frac{15x^8}{2} + x(265 - 912\ln 2 + 1680(\ln 2)^2 - 82\ln 16) + \frac{3}{2}x^5(-51 - 1616(\ln 2)^2 + 320(\ln 2)^3 - 56\ln 2(-15 + \ln 16) - 39\ln 16) + \frac{1}{7}x^7(-210 + 524\ln 2 + 79\ln 16) + x^6(\frac{145}{2} + 296(\ln 2)^2 - 14\ln 16 + \ln 2(-524 + 56\ln 16)) + \frac{1}{6}x^3(-695 - 39120(\ln 2)^2 + 28800(\ln 2)^3 + 446\ln 16 + 8\ln 2(337 + 560\ln 16)) - 240x^2(-1 - 14(\ln 2)^2 + 20(\ln 2)^3 + \ln 64) - \frac{5}{6}x^4(-132 - 7424(\ln 2)^2 + 2880(\ln 2)^3 - 140\ln 16 + 49(\ln 16)^2 + 56\ln 2(37 + \ln 4096)) + x(-265 + 912\ln 2 - 1680(\ln 2)^2 + 60x^4(-1 + \ln 16) + 70x^3(-1 + \ln 16)^2 + 82\ln 16 - 10x^2(5 + 10\ln 16 + 8\ln 2(-16 + 7\ln 16)))\ln x + 30x(-3 + x^2(-2 + \ln 256) + \ln 65536)(\ln x)^2 + 40x(-1 + \ln 16)(\ln x)^3)$$

Hence, the result obtained is used to show the numerical illustrations in table 3.1.

II. Exact solution

Consider the given equation

$u''(x) + \frac{1}{2x}u'(x) = e^u\left(\frac{1}{2} - e^u\right), x \in (0,1)$

Re writing, one can get:

$\left(x^{1/2}u'\right)' = x^{1/2}e^u\left(\frac{1}{2} - e^u\right).$

By the recursive for ADM,

$u_0 = u(0)$

$$u_{n+1} = \int_0^x \left(x^{-\alpha}\int_0^x [r(x)A_n]dx\right)dx, n \geq 0 \qquad (3.43)$$

Here $\alpha = \frac{1}{2}$ and $r(x) = x^{1/2}$ but the Adomian polynomials are already determined in IADM solution above.

From the recursive relation above it can be obtained that

$u_0 = ln(2)$

For $n = 0$, (3.43)reduces to

$u_1 = \int_0^x \left(x^{-1/2}\int_0^x [x^{1/2}A_0]dx\right)dx$

$= \int_0^x \left(x^{-1/2}\int_0^x [x^{1/2}e^{u_0}(0.5 - e^{u_0})]dx\right)dx$

$= -x^2$

Similarlyusing (3.43)one can easily observe that

$u_2 = \frac{x^4}{2}$

$u_3 = -\frac{x^6}{3}$

$u_4 = \frac{x^8}{4}$

$u_5 = -\frac{x^{10}}{5}$

.

.

.

$u_n = (-1)^n \frac{x^{2n}}{n}, n \geq 1.$

Now these results by the ADM can be used to get the series expansion of $\ln(1+x^2)$. The solution is then given by

$$u = \sum_{n=0}^{\infty} u_n$$

$$= ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n}$$

$$= ln(2) - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x^2)^n}{n}$$

$$= ln(2) - \ln(1+x^2)$$

$$\Longrightarrow u(x) = ln\left(\frac{2}{1+x^2}\right)$$

which is the exact solution.

MATLAB is used to plot the IADM and exact solutions in the figure 3.2.

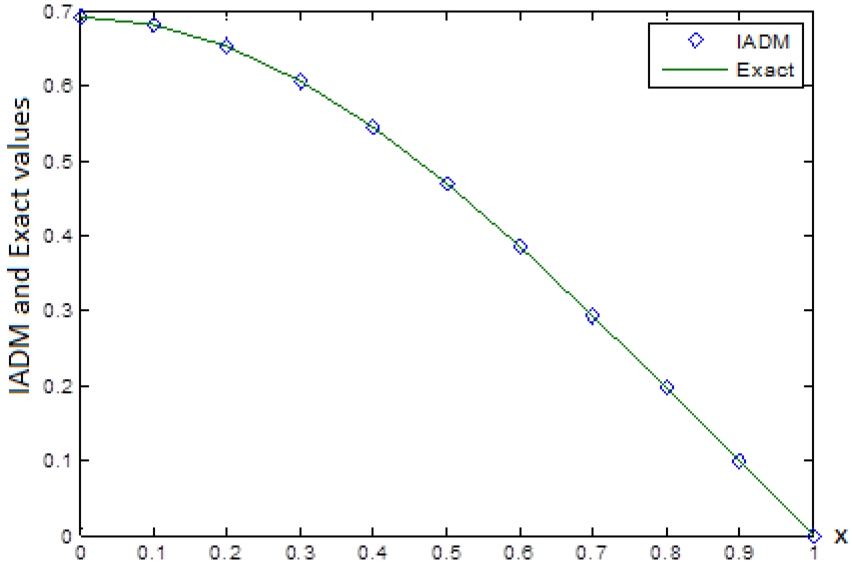

Figure 3.2. Comparison of the IADM and exact solutions for example 3.2.

The numerical illustrations of the problem considered are shown in table below.

Table 3.2. Numerical results for example 3.2.

x	Approximate solutionΦ_{IADM}	Exact solution$u(x)$	Error $/u(x) - \Phi_{IADM}/$
0	0.693147181	0.693147181	0
0.1	0.683195177	0.683196850	1.67E-06
0.2	0.653924627	0.653926467	1.84E-06
0.3	0.606967226	0.606969484	2.26E-06
0.4	0.544725404	0.544727175	1.77E-06
0.5	0.470002399	0.470003629	1.23E-06
0.6	0.385661666	0.385662481	8.15E-07
0.7	0.294370545	0.294371061	5.16E-07
0.8	0.198450643	0.198450939	2.96E-07
0.9	0.099820206	0.099820335	1.29E-07

1	0	0	0

In the overlapping plots shown above, one can easily observe that the IADM and the exact solutions are nearly identical.Furthermore, numerical results are shown in table 3.1 in which an absolute error $\leq 10^{-6}$ is obtained. This shows that the IADM converges faster to the exact solution.Unlike that of the DTM, in this example it can easily be observed that it is not difficult to obtain exact solutionsfor nonlinear inhomogeneous BVPs using the standrad ADM.

IV. Conclusion and Recommendations

4.1. Conclusion

Classes of linear and nonlinear singular boundary value problems can be treated through the standard Adomian decomposition method and its modifications analytically and numerically. All the ADM and the modifications require no perturbation, discretization and linearization to treat two-point BVPs both numerically and analytically.

Although the classical ADM is very powerful, it fails to treat some singular boundary value problems due to the existence of singular point at $x = 0$.So this difficulty is alleviated by the modifications MADM and IADM; and shown by treating STPBVPs with Dirchlet and mixed boundary conditions holding singular feature both numerically and analytically.

It is demonstrated that the modifications can be well suited to attain an accurate solution to the second-order singular boundary value problems, linear and nonlinear as well. The difficulty of those singular problems, due to the existence of the singular point at $x = 0$, is overcome in these contributions. The illustrative examples show that the standard ADM and its modifications are very effective in providing promising results.

But the standard ADM has drawbacks in that it fails to provide the self-cancelling noise terms for homogeneous cases where sum of the noise terms vanishes in the limit for inhomogeneous case which makes it inconsistent in the area. In addition, the operators of the standard Adomian decomposition method support intervals of only the form $(0, c)$ in which the independent variable is defined where c is positive real number.To the contrary, it was noted that in addition to their consistency, the improved decomposition methods IADM and MADM are effective in solving both linear and nonlinear two-point boundary value problems with singular feature.

Though the methods are very convenient for software treatments, it is noted that the modified decomposition methods (especially IADM) may encounter difficulties only in obtaining each component for some complex nonlinear problems even if symbolic packages are used since each component is obtained by cumbersome definite integrals.

4.2. Recommendations

It would be worthwhile to expand application of the operators so that the method could be used to treat two-point BVPs with multiple singularities.

Though series of solutions found by ADM can rapidly be convergent in a very small region, it has very slow convergence rate in wider regions and the truncated series solution is an inaccurate solution in that region which will seriously restrict the application area of the method. An investigation into this claim would greatly benefit the scientific community.

Acknowledgement

I am deeply beholden to Dr. Seleshi Demie next to Almighty God for his profound commitments and encouragements he has shown throughout my study period. I pay respect and express indebtedness to him for his inspiring guidance, and his suggestions in selection of the right tools.
My gratitude goes to Gambella Regional State Ethics and Anti-Corruption Commission, GTEHSC, the University, IOSR and all the authors whose works played an important role in the success of this research work.

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