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Variational Iteration Method for Solving Legendre Differential Equations

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Abstract: *In this study, variational iteration method (VIM) is performed to elucidate the Legendre ordinary differential equations (LODEs). Before this, we apply a substitution which modifies the LODEs to a standard ordinary differential equations (SODEs). Using this substitution, this approach will be very simple and execute the results straight forward. Hence, the excellent results reveal that it is feasible to obtain an analytical or approximate solution of any problem by VIM. Some illustrations are specified, which reduces the calculation and overcomes the hurdle of nonlinear terms to demonstrate the competence of this method.*

Key Word*:VIM, Lagrange multiplier, Correction functional, Legendre equation*.

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I. Introduction

Differential equations have wide applications in various engineering and science disciplines. In general, modeling of the variation of physical quantity, such as temperature, pressure, velocity etc. with the change of time or location or both would result in differential equation. In fact, many engineering subjects such as, mechanical variation or structural dynamics, heat transfer and theory of electric circuits are founded on the theory of differential equation. In 1999, the variational iteration method was developed by He [1, 2, 3]. This method [4, 5, 6] now used by many researchers for handling a large number of linear and nonlinear differential equations. Linear and nonlinear ordinary differential equations with variable coefficients play a significant role in applied mathematics, physics and engineering [7, 8]. It was shown by many authors [9, 10, 11, 12, 13,14,15] that this method is more powerful tool than existing techniques such as the Adomian method [16, 17]. A deep research work has been invested in the study of the linear and nonlinear differential equations. In this paper, we take second order and third order Legendre differentia equation with different initial conditions. We see that the results are good agreement with variational iteration method as the results are very close to the exact solution using the variational iteration method.

This paper is organized as follow: Section 2 recall the basic definition of variational iteration method. In section 3, the method of variational iteration method is explained that how to apply this method on second order and third order Legendre differential equations. In section 4, we illustrate two relevant examples to solve them via variational iteration method. Conclusion is explained in the last section.

II. He's Variational Iteration Method

Consider the following nonlinear differential equation

$$L[u(t)] + N[u(t)] = g(t) \qquad (1)$$

Where L and N represent, respectively, the linear and nonlinear term, while g(t) is the source term. The basic characteristic of He's method is to construct a correction functional.

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\xi)[Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)]d\xi \qquad (2)$$

Where "λ" is general Langrange multiplier which can be identified optimally via variational theory. $u_0(t)$ is an initial approximation with possible unknowns. " $\tilde{u}_n$" is considered as restricted variation i.e.$\delta\tilde{u}_n = 0$. Since the Langrange Multiplier λ is a crucial and critical in the method as it may be a constant or a function. Therefore, first we will find out the Langrange Multiplier λ, which can be found using integration by parts easily, furthermore by the successive approximation upon $u_{n+1}(t), n \geq 0$ for the solution of $u_n(t)$, will be readily obtained by using the obtained values of Langrange Multiplier and by using selective function u_0. Consequently, the exact solution can be obtained by using $u = \lim_{n\to\infty} u_n$.

III. Analysis of theMethod

Second order:

We now extend our analysis to the second order linear ODE with constant coefficients given by

$$u^{''}(x) + au^{'}(x) + bu(x) = g(x) \tag{3}$$

With initial conditions

$$u(0) = \alpha\,, \qquad u^{'}(0) = \beta \tag{4}$$

The VIM admits the use of correction functional for the equation is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x [\lambda(\eta)\{u^{''}{}_n(\eta) + a\tilde{u}^{'}{}_n(\eta) + b\tilde{u}^{'}{}_n(\eta) - g(\eta)\}] \tag{5}$$

Where $\lambda(\eta)$ is Lagrange multiplier and $\tilde{u}$is a restricted value, where,

$$\delta\tilde{u}_n = 0$$

Taking the variation on both sides of (5) with respect to the independent variable u_n, we can find,

$$\frac{\delta u_{n+1}(x)}{\delta u_n(x)} = 1 + \frac{\delta}{\delta u_n(x)}\left(\int_0^x [\lambda(\eta)\,\{u^{''}{}_n(\eta) + a\tilde{u}^{'}{}_n(\eta) + b\tilde{u}^{'}{}_n(\eta) - g(\eta)\}]\right)$$

Or equivalently

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta\left(\int_0^x [\lambda(\eta)\,\{u^{''}{}_n(\eta) + a\tilde{u}^{'}{}_n(\eta) + b\tilde{u}^{'}{}_n(\eta) - g(\eta)\}]\right)$$

That gives,

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta\left(\int_0^x \lambda(\eta)(u^{''}{}_n(\eta)d\eta)\right) \tag{6}$$

Obtained by using $\delta\tilde{u}_n = 0$ and $\delta\widetilde{u^{'}}_n(\eta) = 0$

Integrating equation (6) by parts twice, which gives the result as follow:

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta\,\lambda(\eta)u^{'}{}_n(x) - \delta\lambda'(\eta)u_n(x) + \delta\int_0^x \lambda''(\eta)(u_n(\eta)d\eta)$$

Or equivalently,

$$\delta u_{n+1} = \delta(1-\lambda')u_n + \delta\,\lambda u^{'}{}_n + \delta\int_0^x \lambda'' u_n d\eta \tag{7}$$

The extreme condition of u_{n+1} requires that $\delta u_{n+1} = 0$. This means that the left-hand side of equation (7) is zero and as a result the of right-hand side must be zero as well. This yields the stationary conditions, i.e

$$1 - \lambda^{'}|_{\eta=x} = 0 \quad , \quad \lambda|_{\eta=x} = 0 \quad , \quad \lambda''|_{\eta=x} = 0$$

This in turn gives

$$\lambda = \eta - x \tag{8}$$

Substituting this value of the Lagrange multiplier into equation (5), gives the iteration formula

$$u_{n+1}(x) = u_n(x) + \int_0^x [(\eta - x)\{u^{''}{}_n(\eta) + au^{'}{}_n(\eta) + bu_n(\eta) - g(\eta)\}] \tag{9}$$

Obtained upon deleting the restriction of u_n and u'_n that was used for determination of λ. Considering the given condition $u(0) = \alpha$, $u^{'}(0) = \beta$, we can select the zeroth approximation as $u_0 = \alpha + \beta x$. Using this selection into equation (9), we obtain the following successive approximations

$$u_0(x) = \alpha + \beta x$$

$$u_1(x) = u_0(x) + \int_0^x [(\eta - x)\{u_0''(\eta) + au_0'(\eta) + bu_0(\eta) - g(\eta)\}]$$

$$u_2(x) = u_1(x) + \int_0^x [(\eta - x)\{u_1''(\eta) + au_1'(\eta) + bu_1(\eta) - g(\eta)\}]$$

$$u_3(x) = u_2(x) + \int_0^x [(\eta - x)\{u_2''(\eta) + au_2'(\eta) + bu_2(\eta) - g(\eta)\}]$$

$$\vdots$$

$$\vdots$$

$$u_{n+1}(x) = u_n(x) + \int_0^x [(\eta - x)\{u_n''(\eta) + au_n'(\eta) + bu_n(\eta) - g(\eta)\}]$$

Recall that

$$u(x) = \lim_{n\to\infty} u_{n+1}(x)$$

That may give the exact solution if a closed form solution exits or we can use the (n+1)th approximation for the numerical purpose.

Third Order:

We now extend our analysis to the third order linear ODE with constant coefficients given by

$$u^{'''}(x) + au^{''}(x) + bu'(x) + cu(x) = g(x) \tag{10}$$

With initial conditions

$$u(0) = \alpha\,, \qquad u^{'}(0) = \beta\,, \quad u^{''}(0) = \gamma \tag{11}$$

The VIM admits the use of correction functional for the equation is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x [\lambda(\eta)\{u^{'''}{}_n(\eta) + a\tilde{u}^{''}{}_n(\eta) + b\tilde{u}^{'}{}_n(\eta) + c\tilde{u}^{'}{}_n(\eta) - g(\eta)\}]\, d\eta \tag{12}$$

Where $\lambda(\eta)$ is Lagrange multiplier and $\tilde{u}$is a restricted value, where,

$$\delta\tilde{u}_n = \delta\tilde{u}^{'}{}_n = \delta\tilde{u}^{''}{}_n = 0$$

Taking the variation on both sides of (12) with respect to the independent variable $u_n(x)$, we can find,

$$\frac{\delta u_{n+1}(x)}{\delta u_n(x)} = 1 + \frac{\delta}{\delta u_n(x)}\left(\int_0^x [\lambda(\eta)\{u^{'''}{}_n(\eta) + a\tilde{u}^{''}{}_n(\eta) + b\tilde{u}^{'}{}_n(\eta) + c\tilde{u}^{'}{}_n(\eta) - g(\eta)\}]d\eta\right)$$

Or equivalently,

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta\left(\int_0^x [\lambda(\eta)\{u^{'''}{}_n(\eta) + a\tilde{u}^{''}{}_n(\eta) + b\tilde{u}^{'}{}_n(\eta) + c\tilde{u}^{'}{}_n(\eta) - g(\eta)\}]d\eta\right)$$

That gives

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta\left(\int_0^x \lambda(\eta)(u^{'''}{}_n(\eta))d\eta\right) \tag{13}$$

Obtained by using $\delta\tilde{u}_n = 0$, $\delta\tilde{u}'_n(\eta) = 0$ and $\delta\tilde{u}''_n(\eta) = 0$

Integrating equation (13) by parts three times, which gives the result as follow:

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta\,\lambda(\eta)u^{'\,'}{}_n(x) - \delta\lambda'(\eta)u^{'}{}_n(x) + \delta\lambda''(\eta)u_n(x) + \delta\int_0^x \lambda'''(\eta)(u_n(\eta)d\eta)$$

Or equivalently,

$$\delta u_{n+1} = \delta(1+\lambda'')u_n + \delta\,\lambda u'^{'}{}_n - \delta\,\lambda' u'_n + \delta\int_0^x \lambda'' u_n\, d\eta \tag{14}$$

The extreme condition of u_{n+1} requires that $\delta u_{n+1} = 0$. This means that the left-hand side of equation (14) is zero and as a result the, right hand side must be zero as well. This yields the stationary conditions, i.e.

$$1+\lambda^{''}|_{\eta=x} = 0 \quad , \quad \lambda|_{\eta=x} = 0 \quad , \quad \lambda'|_{\eta=x} = 0 \quad , \quad \lambda'''|_{\eta=x} = 0$$

This in turn gives,

$$\lambda = -\frac{1}{2!}(\eta - x)^2 \tag{15}$$

Substituting this value of the Lagrange multiplier into equation (12), gives the iteration formula,

$$u_{n+1}(x) = u_n(x) - \frac{1}{2!}\int_0^x [(\eta - x)^2\{u^{'''}{}_n(\eta) + a\tilde{u}^{''}{}_n(\eta) + b\tilde{u}^{'}{}_n(\eta) + c\tilde{u}^{'}{}_n(\eta) - g(\eta)\}]\, d\eta \tag{16}$$

Obtained upon deleting the restriction of u_n and u'_n that was used for determination of λ. Considering the given condition $u(0) = \alpha\,,\ u^{'}(0) = \beta\,,\ u^{''}(0) = \gamma$, we can select the zeroth approximation as$u_0 = \alpha + \beta x + \frac{1}{2}\gamma x^2$. Using this selection into equation (16), we obtain the following successive approximations,

$$u_0 = \alpha + \beta x + \frac{1}{2}\gamma x^2$$

$$u_1(x) = u_0(x) - \frac{1}{2!}\int_0^x [(\eta - x)^2\{u_0{'''}(\eta) + a u_0{''}(\eta) + b u_0{'}(\eta) + c u_0(\eta) - g(\eta)\}]\, d\eta$$

$$u_2(x) = u_1(x) - \frac{1}{2!}\int_0^x [(\eta - x)^2\{u_1{'''}(\eta) + a u_1{''}(\eta) + b u_1{'}(\eta) + c u_1(\eta) - g(\eta)\}]\, d\eta$$

$$u_3(x) = u_2(x) - \frac{1}{2!}\int_0^x [(\eta - x)^2\{u_2{'''}(\eta) + a u_2{''}(\eta) + b u_2{'}(\eta) + c u_2(\eta) - g(\eta)\}]\, d\eta$$

$$\vdots$$

$$\vdots$$

$$u_{n+1}(x) = u_n(x) - \frac{1}{2!}\int_0^x [(\eta - x)^2\{u_n{'''}(\eta) + a u_n{''}(\eta) + b u_n{'}(\eta) + c u_n(\eta) - g(\eta)\}]\, d\eta$$

Recall that

$$u(x) = \lim_{n\to\infty} u_{n+1}(x)$$

That may give the exact solution if a closed form solution exits or we can use the (n+1)th approximation for the numerical purpose.

IV. Illustrated Problems

Example 4.1

Consider the second order Legendre differential equation

$$(1+x)^2\frac{d^2u}{dx^2} + (1+x)\frac{du}{dx} + u = 0 \tag{17}$$

With the following conditions

$$u(0) = 1, u^{'}(0) = 0 \tag{18}$$

Which has the exact solution

$$y(x) = -x^2 + 2x^{-3} + 2x^2 lnx. \tag{19}$$

The VIM can be employed in the same manner as described above for second order equation. Based on this we can easily find that

$$\lambda = t - z$$

Using this value of Lagrange multiplier" λ " gives the iteration formula as follow,

$$u_{n+1}(z) = u_n(z) + \int_0^z (t-z)[u''_n(t) + u_n(t)]dt \qquad (20)$$

The zeroth approximation is defined as follow,

$$u_0(z) = \alpha + \beta z, where\ \alpha = 1, \beta = 0$$

therefore,$u_0(z) = 1$, Using this zeroth approximation into (20), we obtained
the following iteration formula

$$u_{n+1}(z) = 1 + \int_0^z (t-z)[u''_n(t) + u_n(t)]dt$$

This in turns gives the successive approximation

$$u_0(z) = 1$$

$$u_1(z) = 1 + \int_0^z (t-z)[0+1]dt$$

$$= 1 - \frac{z^2}{2}$$

similarly

$$u_2(z) = 1 - \frac{z^2}{2} + \int_0^z (t-z)\left[-1 + 1 - \frac{t^2}{2}\right]dt$$

$$u_2(z) = 1 - \frac{z^2}{2} + \frac{z^4}{24}$$

By continuing this process, we get

$$u_3(z) = 1 - \frac{z^2}{2} + \frac{z^4}{24}$$

$$u_4(z) = 1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots$$

the other approximations are computed but not listed. The last approximation converges to the exact solution i.e.

$$u(z) = cosz \qquad (21)$$

but as we have in the start that, $z = \ln(1 + x)$

so the equation (21) becomes as follow

$$u(x) = \cos[ln(1 + x)] \ (22)$$

which is the exact solution.

Example 4.2
Consider the 3rd order Legendre differential equation

$$(1+x)^3\frac{d^3u}{dx^3}-3(1+x)^2\frac{d^2u}{dx^2}+6(1+x)\frac{du}{dx}-6u=0 \qquad (23)$$

with initial conditions

$$u(0)=\ 1, u^{'}(0)=0, u^{''}(0)=1 \qquad (24)$$

Let the following substitution

$\ln(1+x)=z\ \Longrightarrow (1+x)=e^x$

$(1+x)\frac{du}{dx}=\frac{du}{dz}$

$(1+x)^2\frac{d^2u}{dx^2}=\frac{d^2u}{dz^2}-\frac{du}{dz}$

$(1+x)^3\frac{d^3u}{dx^3}=\frac{d^3u}{dx^3}-3\frac{d^2u}{dz^2}+2\frac{du}{dz}$

putting the above substitution in equation (23), we get

$$\frac{d^3u}{dx^3}-6\frac{d^2u}{dz^2}+11\frac{du}{dz}-6u=0 \qquad (25)$$

with the conditions

$$u(z=0)=1, u^{'}(z=0)=0, u^{''}(z=0)=1 \qquad (26)$$

The VIM can be employed in the same manner as described above for third order equation. Based on this we can easily find that

$$\lambda=-\frac{1}{2}(t-z)^2$$

Using this value of Lagrange multiplier"λ" gives the iteration formula as follow,

$$u_{n+1}(z)=\ u_n(z)+\int_0^z-\frac{1}{2!}(t-z)^2[u^{'}{''}_n(t)-6u^{'}{'}_n(t)+11u'_n(t)-6u_n(t)]dt \quad (27)$$

The zeroth approximation is defined as follow,

$u_0(z)=\ \alpha+\beta z+\frac{1}{2}\gamma z^2, where\ \alpha=1,\beta=0,\gamma=1$

therefore,$u_0(z)=1$, Using this zeroth approximation into (9), we obtained the following iteration formula

$$u_{n+1}(z)=\ 1+\frac{1}{2}z^2+\int_0^z-\frac{1}{2!}(t-z)^2\left[u^{'''}{}_0(t)-6u^{'}{'}_0(t)+11u^{'}{}_0(t)-6u_0(t)\right]dt$$

This in turns gives the successive approximation

$u_0(z)=1+\frac{1}{2}z^2$

$u_1(z)=\ 1+\frac{1}{2}z^2-\frac{1}{2}\int_0^z(t-z)^2\left[0-6(1)+11(t)-6(1+\frac{1}{2}t^2)\right]dt$

$u_1(z)=1+\frac{1}{2}z^2+\frac{5}{2}z^3-\frac{11}{24}z^4+\ \frac{1}{20}z^5$

similarly,

$u_2(z)=1+\frac{1}{2}z^2+2z^3+\frac{79}{24}z^4-\ \frac{45}{24}z^5$

$$u_3(z) = 1 + \frac{1}{2}z^2 + 2z^3 + \frac{61}{24}z^4 - \frac{87}{30}z^5 + \cdots$$
$$u_4(z) = 1 + \frac{1}{2}z^2 + 2z^3 + \frac{61}{24}z^4 + 2z^5 + \cdots$$

while the other approximations are computed but not listed. The last approximation converges to the exact solution i.e

$$\mathrm{u(z)} = \frac{7}{2}\mathrm{e^z} - 4\mathrm{e^{2z}} + \frac{3}{2}\mathrm{e^{3z}} \qquad (29)$$

but as we have in the start that,

$$z = \ln(1+x) \Rightarrow e^z = (1+x), e^{2z} = (1+x)^2, e^{3z} = (1+x)^3$$

so the equation (29) becomes as follow

$$u(x) = \frac{7}{2}(1+x) - 4(1+x)^2 + \frac{3}{2}(1+x)^3 \qquad (30)$$

which is the exact solution.

V. Conclusion

In this paper, variational iteration method is proposed for solving of 2nd order and 3rd order Legendre equations. He's variational iteration method is an alternative approach to solve the Legendre equations. Moreover, the VIMreduces the size of calculation, hence the iterations are direct and straight forward. The variational iteration method facilitates the computational work andgives the solution rapidly than other methods. The obtained results show thatvariational iteration method is a powerful tool.

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