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Differential geometry of Curvature

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Abstract: *In the present paper some aspects of surface, curvature, Dupins indicatrix, Weingarten Equations are studied. The importance of this study lies in the fact that Mean and Gaussian curvature is established with the help of the coefficient of first and second fundamental form.*

Keywords: *Curvature, Curvature Directions and Quadratic form of curvature directions, Mean & Gaussian curvature, Dupin Indicatrix and Asymptotic Direction, Weingarten Equations.*

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I. Introduction

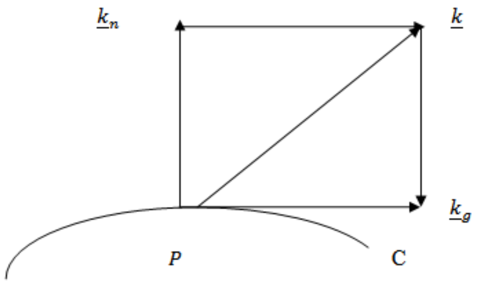
Differential geometry is a branch of mathematics using calculus to study the geometric properties of curves and surfaces. It arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces [5]. The theory developed in this study originates from mathematicians of the 18th and 19th centuries, mainly; Euler (1707-1783), Monge (1746-1818) and Gauss (1777-1855).Mathematical study of curves and surfaces has been developed to answer some of the nagging and unanswered questions that appeared in calculus, such as the reasons for relationships between complex shapes and curves, series and analytic functions. Study of curvatures [2] is an important part of differential geometry. During study we have derived the equations of some topics such as surface, first and second fundamental forms, curvature, normal and principle curvature, Dupins indicatrix, asymptotic directions and Weingarten equations. This paper also deals with some important theorems and examples.

II. Curvature

Definition: The locus of a point whose Cartesian coordinates (x, y, z) are functions of a single parameter is called curve and the locus of a point whose Cartesian coordinates (x, y, z) are functions of two independent parameters u, v (say) is defined as a surface.

Definition: Let C be a curve on the surface passing through a point P and $\underline{t}$ is the unit tangent of C. Let $\frac{d\underline{t}}{ds} = \underline{k}$. We now decompose $\underline{k}$ into a component $\underline{k_n}$ in the direction of the normal $\underline{N}$ and a component $\underline{k_g}$ in the tangential direction to the surface.

The vector $\underline{k_n}$ is called the normal curvature vector and can be expressed in terms of the unit surface normal vector $\underline{N}$.



$$\underline{k_n} = k_n.\underline{N}$$

where k_n is called the normal curvature.

Definition: The *normal* to the surface at the point P is a line passing through P and perpendicular to the tangent line at P.

Theorem (Meusnier): If φ is the angle between the principal normal and the surface normal to a curve C at any point P, then $R = R_n \cos\varphi$, where

$$R = \frac{1}{k}, \quad R_n = \frac{1}{k_n}.$$

Proof: Let P be a point (u, v) on the surface $\underline{X} = \underline{X}(u, v)$. Suppose $\underline{n}$ and $\underline{N}$ denote the principal normal to the curve on the surface and surface normal [1] at P respectively. Since φ is the angle between $\underline{n}$ and $\underline{N}$, therefore,

$$\cos\varphi = \underline{n}.\underline{N}$$

Now, suppose $\underline{k}$ is the curvature vector of the given curve at P. Then

$$\underline{k} = k\underline{n}$$
$$\underline{k} = \underline{k}_n + \underline{k}_g$$
$$k\underline{n} = k_n.\underline{N} + \underline{k}_g \quad [\underline{k}_n = k_n.\underline{N}]$$
$$(k\underline{n}).\underline{N} = (k_n.\underline{N} + \underline{k}_g).\underline{N}$$
$$k\underline{n}.\underline{N} = k_n.\underline{N}.\underline{N} + \underline{k}_g.\underline{N}$$
$$k\cos\varphi = k_n + 0$$
$$\frac{1}{R}\cos\varphi = \frac{1}{R_n}$$
$$k\cos\varphi = k_n$$
$$R = R_n\cos\varphi$$

which is called the Meusnier's theorem.

Remark We have $k_n = k$ if and only if $\varphi = 0$. Thus the necessary and sufficient condition for the curvature of a curve at P to be equal to the normal curvature at P in the direction of that curve is that the principal normal to the curve is along the surface normal at the point.

Definition: Let $\underline{x} = \underline{x}(u, v)$ be the equation of the surface. The quadratic differential equation of the form $ds^2 = E(du)^2 + 2\,Fdu\,dv + G(dv)^2$ where

$$E = \underline{x}_u.\underline{x}_u, \quad F = \underline{x}_v.\underline{x}_v, \quad G = \underline{x}_u.\underline{x}_v$$

is called first fundamental form.

Definition: Let $\underline{x} = \underline{x}(u, v)$ be the equation of the surface and $\underline{N} = \underline{N}(u, v)$ be unit surface normal. Then the second fundamental form II is defined by

$$\text{II}-d\underline{x}.d\underline{N} = e(du)^2 + 2fdudv + g(dv)^2$$

is called the second fundamental form.

III. Curvature Directions and Quadratic form of curvature directions

We have $\underline{N}.\underline{t} = 0$, we obtain by differentiation,

$$\frac{d\underline{N}}{ds}.\underline{t} + \underline{N}.\frac{d\underline{t}}{ds} = 0$$
$$\underline{N}.\frac{d\underline{t}}{ds} = -\frac{d\underline{N}}{ds}.\underline{t} = \frac{d\underline{N}}{ds}.\frac{d\underline{x}}{ds} \quad [\because \underline{t} = \frac{d\underline{x}}{ds}]$$
$$k.\underline{N} = -\frac{d\underline{x}.d\underline{N}}{(ds)^2} = -\frac{d\underline{x}.d\underline{N}}{d\underline{x}.d\underline{x}} \; [\because (ds)^2 = d\underline{x}.d\underline{x}]$$
$$(k_n.\underline{N} + \underline{k}_g).\underline{N} = -\frac{d\underline{x}.d\underline{N}}{d\underline{x}.d\underline{x}} \, [\because \underline{k} = \underline{k}_n + \underline{k}_g \; and \; \underline{k}_n = k_n.\underline{N}]$$
$$k_n.\underline{N}.\underline{N} + 0 = -\frac{d\underline{x}.d\underline{N}}{d\underline{x}.d\underline{x}} = \frac{II}{I}$$
$$k_n = \frac{e(du)^2 + 2fdudv + g(dv)^2}{E(du)^2 + 2Fdudv + G(dv)^2} \quad \text{[Using first and second fundamental theorem]}$$

$$= \frac{e + 2f\frac{dv}{du} + g\left(\frac{dv}{du}\right)^2}{E + 2F\frac{dv}{du} + G\left(\frac{dv}{du}\right)^2}$$

$$= \frac{e + 2f\lambda + g\lambda^2}{E + 2F\lambda + G\lambda^2} \quad [\text{where } \lambda = \frac{dv}{du}]$$

here, $\frac{dv}{du}$ gives the directions, i.e. λ determines the direction.

Definition: The directions for which k_n is either maximum or minimum are called curvature directions.

$k_n = k_n(\lambda)$ is a function of λ,

Where $k_n = \frac{e + 2f\lambda + g\lambda^2}{E + 2F\lambda + G\lambda^2}$

$$\therefore \; \frac{dk_n}{d\lambda} = \frac{(E + 2F\lambda + G\lambda^2)(2f + 2g\lambda) - (e + 2f\lambda + g\lambda^2)(2F + 2G\lambda)}{(E + 2F\lambda + G\lambda^2)^2}$$

For extreme values (maximum or minimum) of k_n ,

$\frac{dk_n}{d\lambda}$ should be zero

$\frac{dk_n}{d\lambda} = 0$

Therefore, extreme values of k_n , we have,

$(E + 2F\lambda + G\lambda^2)(2f + 2g\lambda) - (e + 2f\lambda + g\lambda^2)(2F + 2G\lambda) = 0$

$(E + 2F\lambda + G\lambda^2)(f + g\lambda) = (e + 2f\lambda + g\lambda^2)(F + G\lambda)$

$\frac{f+g\lambda}{F+G\lambda} = \frac{e+2f\lambda+g\lambda^2}{E+2F\lambda+G\lambda^2}$

$k_n = \frac{e+f\lambda+\lambda(f+g\lambda)}{E+F\lambda+\lambda(F+G\lambda)} = \frac{f+g\lambda}{F+G\lambda}$

$\frac{f+g\lambda}{F+G\lambda} = \frac{-\lambda(f+g\lambda)}{-\lambda(F+G\lambda)} = \frac{e+f\lambda+\lambda(f+g\lambda)}{E+F\lambda+\lambda(F+G\lambda)}$

$= \frac{-\lambda(f+g\lambda)+e+f\lambda+\lambda(f+g\lambda)}{-\lambda(F+G\lambda)+E+F\lambda+\lambda(F+G\lambda)}$

$\frac{f+g\lambda}{F+G\lambda} = \frac{e+f\lambda}{E+F\lambda}$

Since $k_n = \frac{e+f\lambda+\lambda(f+g\lambda)}{E+F\lambda+\lambda(F+G\lambda)} = \frac{f+g\lambda}{F+G\lambda}$ and considering (3.01) & (3.02), we have

$k_n = \frac{f+g\lambda}{F+G\lambda} = \frac{e+f\lambda}{E+F\lambda}$

$\therefore \quad eF + eG\lambda + fF\lambda + fG\lambda^2 = fE + gE\lambda + fF\lambda + gf\lambda^2$

Or, $(gF - FG)\lambda^2 + (gE - eG)\lambda + (fE - eF) = 0$

This is a quadratic equation in λ.

Definition: The normal curvature of a surface which have greatest and least curvatures are called principal curvatures.

Definition: The two perpendicular directions for which the values of k_n take on maximum or minimum values are called principal directions.

Definition: The normal curvatures in the curvature directions are called principal curvature.

Definition: The lines in the curvature directions are called lines of curvature [8].

Theorem (Rodrigues Formula): In the direction of a principal direction, the vector $d\underline{N}$ is parallel to $d\underline{x}$ and is given by

$$d\underline{N} = -k_n d\underline{x}$$

Where k_n is the principal curvature in that direction.

Proof: We know that the principal curvatures are given by

$k_n = \frac{f+g\lambda}{F+G\lambda} = \frac{e+f\lambda}{E+F\lambda}$, $\lambda = \frac{dv}{du}$

Or, $Fk_n + Gk_n\lambda = f + g\lambda$ and $Ek_n + Fk_n\lambda = e + f\lambda$

Or, $(Ek_n - e) + (Fk_n - f)\lambda = 0$ and $(Fk_n - f) + (Gk_n - g)\lambda = 0$

Or, $(Ek_n - e) + (Fk_n - f)\frac{dv}{du} = 0$ and $(Fk_n - f) + (Gk_n - g)\frac{dv}{du} = 0$

Or, $(Ek_n - e)du + (Fk_n - f)dv = 0$ and $(Fk_n - f)du + (Gk_n - g)dv = 0$

Or, $(\underline{x}_u.\underline{x}_u k_n + \underline{x}_u.\underline{N}_u)du + (\underline{x}_u.\underline{x}_v k_n + \underline{x}_u.\underline{N}_v)dv = 0$ and

$(\underline{x}_u.\underline{x}_v k_n + \underline{x}_v.\underline{N}_u)du + (\underline{x}_v.\underline{x}_v k_n + \underline{x}_v.\underline{N}_v)dv = 0$

Or, $[(\underline{x}_u k_n + \underline{N}_u)du + (\underline{x}_v k_n + \underline{N}_v)dv].\underline{x}_u = 0$ and

$[(\underline{x}_u k_n + \underline{N}_u)du + (\underline{x}_v k_n + \underline{N}_v)dv].\underline{x}_v = 0$

Or, $\left[(\underline{N}_u du + \underline{N}_v dv) + k_n(\underline{x}_u du + \underline{x}_v dv)\right].\underline{x}_u = 0$ and

$\left[(\underline{N}_u du + \underline{N}_v dv) + k_n(\underline{x}_u du + \underline{x}_v dv)\right].\underline{x}_v = 0$

Or, $\left[d\underline{N} + k_n d\underline{x}\right].\underline{x}_u = 0$ and $\left[d\underline{N} + k_n d\underline{x}\right].\underline{x}_v = 0$

Since, $\underline{x}_u$ and $\underline{x}_v$ are linearly independent [i.e. $d\underline{N} + k_n d\underline{x}$ is the tangent plane], we must have,

$d\underline{N} + k_n d\underline{x} = 0$

Or, $d\underline{N} = -k_n d\underline{x}$

$\therefore\ d\underline{N} = -k_n d\underline{x}$

which is called the Rodrigues formula.

IV. Mean & Gaussian curvature

We know that the roots of $(EG - F^2){k_n}^2 + (eG + gE - 2fF)k_n + eg - f^2 = 0$ are called principal curvatures.

Definition: The arithmetic mean of principal curvatures k_{n1} and k_{n2} is called the mean curvature [3] and is defined as

$$M = \frac{1}{2}(k_{n1} + k_{n2}) = \frac{eG + gE - 2fF}{2(EG - F^2)}$$

Definition: The product of principal curvatures k_{n1} and k_{n2} is called the Gaussian curvature and is defined as

$$K = k_{n1}k_{n2} = \frac{eg - f^2}{EG - F^2}$$

Theorem (Eulers): The normal curvature k_n in any direction can be expressed as,

$$k_n = k_{n1}\cos^2\alpha + k_{n2}\sin^2\alpha$$

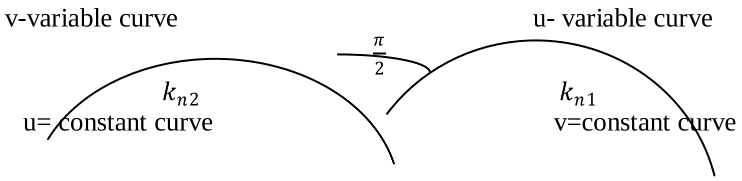
Where k_{n1} and k_{n2} are the principal curvature and α is the angle between the u parametric curve and arbitrary directions.

Proof : We know that the normal curvature is given by

$$k_n = \frac{e(du)^2 + 2fdudv + g(dv)^2}{E(du)^2 + 2Fdudv + G(dv)^2}$$

When the parametric lines coincide with curvature directions then we have $f = 0, F = 0$.In this case

$$k_n = \frac{e(du)^2 + g(dv)^2}{E(du)^2 + G(dv)^2}$$



Let us suppose that for u- variable curve the principal curvature is k_{n1} and for v- variable curve the principal curvature is k_{n2} . Therefore k_{n1} is obtained from

$k_n = \frac{e(du)^2+g(dv)^2}{E(du)^2+G(dv)^2}$

By setting $dv = 0$, we get,

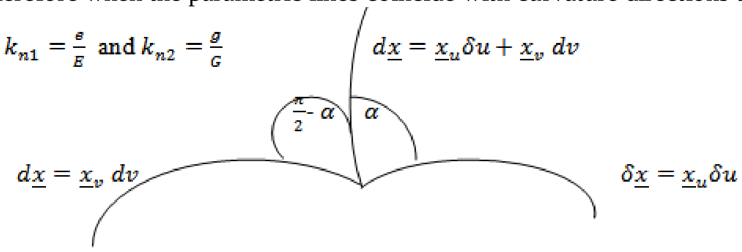
$k_{n1} = \frac{e(du)^2}{E(du)^2} = \frac{e}{E}$

And k_{n2} is obtained by setting $du = 0$

$k_{n2} = \frac{g(dv)^2}{G(dv)^2} = \frac{g}{G}$

Therefore when the parametric lines coincide with curvature directions then we have

$k_{n1} = \frac{e}{E}$ and $k_{n2} = \frac{g}{G}$



Let us suppose that any arbitrary direction makes angle α with the parametric curve v=constant.

$\cos\alpha = \frac{d\underline{x}.\delta\underline{x}}{|d\underline{x}.\delta\underline{x}|} = \frac{Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v}{\sqrt{E(du)^2+2Fdudv+G(dv)^2}.\sqrt{E(\delta u)^2+2F\delta u\delta v+G(\delta v)^2}}$

Since, the parametric curves coincide with the curvature directions, we have $f = 0, F = 0$ and

$\cos\alpha = \frac{Edu\delta u + Gdv\delta v}{\sqrt{E(du)^2+G(dv)^2}.\sqrt{E(\delta u)^2+G(\delta v)^2}}$

Here, α is the angle between $v =$ constant curve and any arbitrary direction. So, we set $\delta v = 0$ and $\cos\alpha = \frac{Edu\delta u}{\sqrt{E(du)^2+G(dv)^2}.\sqrt{E(\delta u)^2}} = \frac{\sqrt{E}du}{\sqrt{E(du)^2+G(dv)^2}}$

For $u =$ constant curve

$\cos(\frac{\pi}{2} - \alpha) = \frac{EduDu + GdvDv}{\sqrt{E(du)^2+G(dv)^2}\,.\,\sqrt{E(Du)^2+G(Dv)^2}}$

Since $Du = 0$

$\sin\alpha = \frac{GdvDv}{\sqrt{E(du)^2+G(dv)^2}\,.\,\sqrt{G(Dv)^2}} = \frac{\sqrt{G}dv}{\sqrt{E(du)^2+G(dv)^2}}$

Now,

$k_{n1}\cos^2\alpha = \frac{e}{E}.\frac{E(du)^2}{E(du)^2+G(dv)^2} = \frac{e(du)^2}{E(du)^2+G(dv)^2}$

$k_{n2}\sin^2\alpha = \frac{g}{G}.\frac{G(dv)^2}{E(du)^2+G(dv)^2} = \frac{g(dv)^2}{E(du)^2+G(dv)^2}$

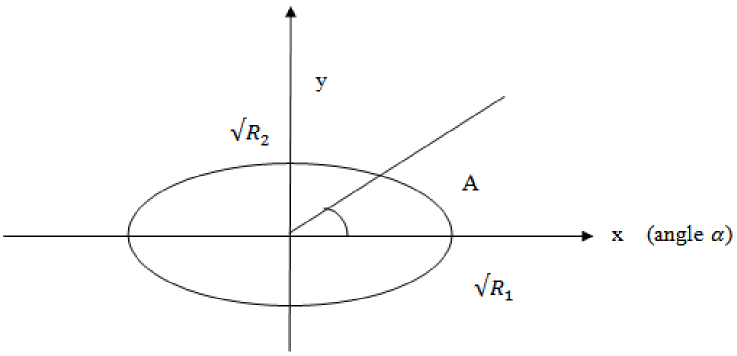
$k_{n1}\cos^2\alpha + k_{n2}\sin^2\alpha = \frac{e(du)^2+g(dv)^2}{E(du)^2+G(dv)^2} = k_n$

$k_n = k_{n1}\cos^2\alpha + k_{n2}\sin^2\alpha$

where α is the angle between any curvature direction and any arbitrary direction. This is called Euler's theorem.
Definition: The points where $eg - f^2 > 0$ are called elliptic points. Here either $k_{n1} < 0$ and $k_{n2} < 0$ or $k_{n1} > 0$ and $k_{n2} > 0$.
Definition: The points where $eg - f^2 < 0$ are called hyperbolic points. Here either $k_{n1} < 0$ or $k_{n2} < 0$ i.e. k_{n1} and k_{n2} have opposite sign.
Definition: The points where $eg - f^2 = 0$ are called parabolic points. Here either $k_{n1} = 0$ or $k_{n2} = 0$ also, k_{n1} and k_{n2} are both zero.

V. Dupin Indicatrix and Asymptotic Direction

If the x-axis represents one of the curvature of the directions of curvature, then the distance of any point on the ellipse to the curve is the square root of the radius of the curvature in the corresponding direction on the surface.
The Form of the Dupin Indicatrix for ellipse**:** Let us draw an ellipse where principal semi axes are
$\sqrt{R_{n1}} = \frac{1}{\sqrt{K_{n1}}}$ and $\sqrt{R_{n2}} = \frac{1}{\sqrt{K_{n2}}}$
$\therefore$ The equation to the ellipse will be $k_1 x^2 + k_2 y^2 = 1$



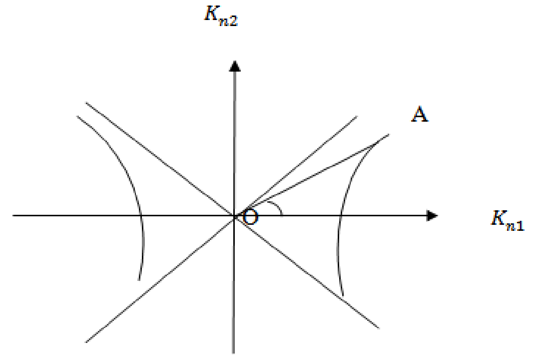
Let A is at (x_1, y_1) and $OA = \sqrt{{x_1}^2 + {y_1}^2}$
$= \sqrt{{x_1}^2 + {x_1}^2 \tan^2 \alpha}$
$= x_1 \sec \alpha$
Again, $k_1 {x_1}^2 + k_2 {y_2}^2 = 1$
Or, $k_1 {x_1}^2 + k_2 {x_1}^2 \tan^2 \alpha = 1$
Or, $k_1 {x_1}^2 \cos^2 \alpha + k_2 {x_1}^2 \sin^2 \alpha = \cos^2 \alpha$
Or, ${x_1}^2 = \frac{\cos^2 \alpha}{k_1 \cos^2 \alpha + k_2 \sin^2 \alpha}$
Or, $OA = x_1 \sec \alpha = \frac{\cos \alpha}{\sqrt{k_1 \cos^2 \alpha + k_2 \sin^2 \alpha}} . \sec \alpha$
$= \frac{1}{\sqrt{k_1 \cos^2 \alpha + k_2 \sin^2 \alpha}}$

But according to Euler's theorem,
$k_n = k_1 \cos^2 \alpha + k_2 \sin^2 \alpha$
From above,
$OA = \frac{1}{\sqrt{k_n}} = \sqrt{R_n}$
Hence, OA gives the normal curvature in the direction of OA, where the curvature directions are in the direction of the coordinate axes.
In this case $k_{n1} > 0$ and $k_{n2} > 0$, i.e. we have considered the case for ellipse points.
The ellipse $k_{n1} x^2 + k_{n2} y^2 = 1$ is called Dupin Indicatrix.
At the point where, $k_{n1} < 0$ and $k_{n2} < 0$, i.e. we have considered the case for hyperbolic points and we get the equation of hyperbola.
At the parabolic points, say $k_{n2} = 0$, we get $k_{n1} x^2 = 1$,
$x = \pm \frac{1}{\sqrt{k_{n1}}} = \pm \sqrt{R_{n1}}$
$k_n = k_{n1} \cos^2 \alpha$
The asymptotes of Dupin Indicatrix corresponds to the asymptotic directions on the surface, the axes of indicatrix to the directions of the curvature.
Definition: A direction at a point on a surface for which $II = edu^2 + 2fdudv + gdv^2 = 0$
is called an asymptotic direction.

Since, $k_n = \frac{II}{I}$ and I is positive definite, the asymptotic directions are also the directions in which $k_n = 0$. At an elliptic point there are no asymptotic directions, at a hyperbolic point there are two distinct asymptotic directions, at a parabolic point there is one asymptotic direction.

Theorem : The curvature directions bisect the asymptotic directions.

Proof. Consider the Dupin indicatrix at the parabolic points. We have $= \sqrt{R_n}$. If the straight line OA be the asymptote of the hyperbola then A is at infinity and $OA = \infty$.



$$k_n = \frac{1}{\sqrt{OA}} = 0$$

∴ The asymptotes of this hyperbola give the asymptote directions. But the coordinate axes directions are in the direction of the coordinate axes.

Therefore, the curvature directions bisect the asymptotic directions.

Theorem: Binormal of asymptotic line is the normal to the surface.

Proof. We know,

$k_n = -\frac{d\underline{x}.d\underline{N}}{d\underline{x}.d\underline{x}}$

$= -\frac{d\underline{x}.d\underline{N}}{(ds)^2}$

$= -\frac{d\underline{x}}{ds}.\frac{d\underline{N}}{ds}$

$= -\underline{t}.\frac{d\underline{N}}{ds}$

But for asymptotic curve, $k_n = 0$

∴ for asymptotic curve,

$-\underline{t}.\frac{d\underline{N}}{ds} = 0$

Again, $\underline{t}.\underline{N} = 0$

$\frac{d\underline{t}}{ds}.\underline{N} + \underline{t}.\frac{d\underline{N}}{ds} = 0$

Or, $k_n.\underline{N} + \underline{t}.\frac{d\underline{N}}{ds} = 0$

Since, $\underline{t}.\frac{d\underline{N}}{ds} = 0$, we have for asymptotic curve, $k\underline{n}.\underline{N} = 0$

Therefore, for asymptotic curve either $k = 0$ or, $\underline{n}.\underline{N} = 0$.

∴ The straight line on a surface are asymptotic lines.

Since $\underline{t}.\underline{N} = 0$ and $\underline{n}.\underline{N} = 0$.

$\underline{N}$ is perpendicular to $\underline{t}$ and $\underline{n}$,so $\underline{N}$ is parallel to $\underline{b}$.

Hence, $\underline{N} = \underline{b}$

Hence, for curved asymptotic lines, surface normal coincides with the binormal of the asymptotic lines.

VI. Weingarten Equations

These equations were first investigated by J. Weingarten. The Weingarten equations express the derivatives of the vectors $\underline{x}_u$, $\underline{x}_v$ and $\underline{N}$ as linear combinations of these vectors with coefficients which are functions of the first and second fundamental coefficients.

Theorem: On a surface $x = x(u, v)$, the vectors $\underline{x}_u$, $\underline{x}_v$, $\underline{N}_u$ and $\underline{N}_v$ lie on the tangent plane, then the Weingarten equations will be of the form

$\underline{N}_u = \frac{fE-eG}{EG-F^2}\ \underline{x}_u + \frac{eF-fE}{EG-F^2}\ \underline{x}_v$

$\underline{N}_v = \frac{gF-fG}{EG-F^2}\ \underline{x}_u + \frac{fF-gE}{EG-F^2}\ \underline{x}_v$

Proof: We know $\underline{x}_u$, $\underline{x}_v$ are linearly independent and lie on the tangential plane. Since, $\underline{N}_u$ and $\underline{N}_v$ lie on the tangential plane, then $\underline{N}_u$ and $\underline{N}_v$ can be written as a linear combination of $\underline{x}_u$, $\underline{x}_v$.

$\underline{N}_u = a_1\underline{x}_u + a_2\underline{x}_v$; $\underline{N}_v = b_1\underline{x}_u + b_2\underline{x}_v$

$\therefore\ \underline{N}_u . \underline{x}_u = a_1\underline{x}_u . \underline{x}_u + a_2\underline{x}_v . \underline{x}_u$

Or, $-e = a_1 E + a_2 F$

$\therefore\ \underline{N}_u . \underline{x}_v = a_1\underline{x}_u . \underline{x}_v + a_2\underline{x}_v . \underline{x}_v$

Or, $-f = a_1 F + a_2 G$

Or, $a_1 E + a_2 F + e = 0$

$a_1 F + a_2 G + f = 0$

$\frac{a_1}{fE-eG} = \frac{a_2}{eF-fE} = \frac{1}{EG-F^2}$

Or, $a_1 = \frac{fE-eG}{EG-F^2}$, $a_2 = \frac{eF-fE}{EG-F^2}$

Again, $\underline{N}_v . \underline{x}_u = b_1\underline{x}_u . \underline{x}_u + b_2\underline{x}_v . \underline{x}_u$

Or, $-f = b_1 E + b_2 F$

$\therefore\ \underline{N}_v . \underline{x}_v = b_1\underline{x}_u . \underline{x}_v + b_2\underline{x}_v . \underline{x}_v$

Or, $-g = b_1 F + b_2 G$

Or, $b_1 E + b_2 F + f = 0$

$b_1 F + b_2 G + g = 0$

$\frac{b_1}{gF-fG} = \frac{b_2}{fF-gE} = \frac{1}{EG-F^2}$

$b_1 = \frac{gF-fG}{EG-F^2}$, $b_2 = \frac{fF-gE}{EG-F^2}$

Therefore, $\underline{N}_u = \frac{fE-eG}{EG-F^2}\ \underline{x}_u + \frac{eF-fE}{EG-F^2}\ \underline{x}_v$ (4.01)

$\underline{N}_v = \frac{gF-fG}{EG-F^2}\ \underline{x}_u + \frac{fF-gE}{EG-F^2}\ \underline{x}_v$ (4.02)

The equations (4.01) and (4.02) are called Weingarten equations.

Theorem: (Beltrami Enneper) The torsion of the asymptotic lines through point of surface [7] is

$$\tau = \pm\sqrt{-k}\ .$$

Where k is the Gaussian curvature at the point.

Proof: We know, for asymptotic curve,

$k_n = -\frac{d\underline{x}.d\underline{N}}{d\underline{x}.d\underline{x}} = \frac{II}{I} = 0$

Hence, for asymptotic curve, $II = 0$ and $\underline{N} = \underline{b}$.

Again, we have, $d\underline{N}.d\underline{N} - 2MII + kI = 0$

Where, I, II and $III =$ first, second and third fundamental form, $M =$ Mean curvature & $k =$ Gaussian curvature.

Or, $d\underline{N}.d\underline{N} + kI = 0$ $[\because II = 0]$

Or, $d\underline{N}.d\underline{N} + k\ d\underline{x}.d\underline{x} = 0$

Or, $d\underline{b}.d\underline{b} + k(ds)^2 = 0$ $[\because d\underline{x}.d\underline{x} = (ds)^2]$

Or, $\frac{d\underline{b}}{ds}.\frac{d\underline{b}}{ds} + k = 0$

Or, $\tau^2 + k = 0$ $[\because \frac{d\underline{b}}{ds} = -\tau\underline{n}, \frac{d\underline{b}}{ds}.\frac{d\underline{b}}{ds} = \tau^2\ as\ \underline{n}.\underline{n} = 1]$

Or, $\tau^2 = -k$

$\therefore\ \tau = \pm\sqrt{-k}$

VII. Conclusion

Here we have discussed curvature elaborately. We have solved some examples and also problems related to this topic.

References

[1]. Boothby, W. An Introduction to Differentiable Manifolds and Differential Geometry, Academic Press, NewYork, 1975.

[2]. M. Deserno, Notes on Difierential Geometry with special emphasis on surfaces in R3, Los Angeles, USA, 2004.

[3]. Ahmed, K. M., Md. Showkat Ali, 2008. A study of spaces of constant curvature on Riemannian manifolds, Dhaka Uni. J. Sci. **56(1)**: 65-67.

[4]. M. P. do-Carmo, Differential Geometry of curves and surfaces, Prentice-Hall, Inc., Englewood Cliffs, New Zealand, USA, 1976.

[5]. M. Raussen, Elementary Differential Geometry: Curves and Surfaces, Aalborg University, Denmark, 2008.

[6]. M. Spivak, A Comprehensive Introduction to Differential Geometry, Vol. 1, Third Edition, Publish or Perish Inc., Houston, USA, 1999.

[7]. T. Shifrin, Differential Geometry: A First Course in Curves and Surfaces, Preliminary Version, University of Georgia, 2016.

[8]. W. Zhang, Geometry of Curves and Surfaces, Mathematic Institute, University of Warwick, 2014.