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Functional Equalities Induced By Mean Of Arbitrary Degrees To Mean Of Arbitrary Degrees And Apply To Solve Some Mathematical Problem

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Abstract:*Functional equalities are very difficult and are frequently exploited in regional, national and international olympiads (of high-school or university level). The development used to application in several areas - not only in mathematics but also in other disciplines. Functional equalities are applied computers, economics, polynomials, engineering, information theory, reproducing scoring system, taxation, etc. In recent years, many author studied functional equations. In this paper, we establish some functional equalities induced bymean of arbitrary degrees tomean of arbitrary degrees and apply to solve some mathematical problems .*

Keywords: *Functional, functional equalities, arithmetic mean, quadratic mean, mean of arbitrary degrees.*

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I. Preliminaries

In this paper, we derive some mathematical problems via functional equalities
Arithmetic mean of argument

$$\frac{x+y}{2};x,y\ \in\mathbb{R};$$

Quadratic mean of argument

$$\sqrt{\frac{x^2+y^2}{2}};x,y\ \in\mathbb{R}^+;$$

Mean of argument with t (t>1)

$$\sqrt[t]{\frac{x^t+y^t}{2}};x,y\ \in\mathbb{R}^+;$$

and
Arithmetic mean of functions

$$\frac{f(x)+f(y)}{2};$$

Quadratic mean of functions

$$\sqrt{\frac{[f(x)]^2+[f(y)]^2}{2}};$$

Mean of functions with t (t>1)

$$\sqrt[t]{\frac{[f(x)]^t+[f(y)]^t}{2}};$$

To solve some mathematical problems via functional equalities, we usemethod of induction or method of substitution...

Remark 1. To use method of substitution, we often change special values
+) Example, let $x=a$ such that $f(a)$ appears much in the equation.
+) $x=a,y=b$ interchange to refer$f(a)$ and $f(b)$.
+) Let $f(0)\ =b,f(1)=b,$
+) If f is surjection, exist a: $f(a)=0$. Choicex,y to destroy $f(g(x,y))$ in the equation. The function has x, we show that it isinjective or surjection.
+) To occur $f(x)$.
+) $f(x)\ =f(y)$ for all x, y ∈ A. Hence$f(x)\ =const$ for all $x\ \in\ A$.

II. Functional Equalities Induced By Mean Arbitrary Degrees To Mean Of Arbitrary Degrees

In section, we consider the problems with Functional equalities induced by mean arbitrary degrees to mean of arbitrary degrees.

Problem 1 (Cauchy functional equalities). Determine all functions$f(x)$continuous on$\mathbb{R}$such that

$$f(x+y) = f(x) + f(y), \forall\, x, y \in \mathbb{R}. \qquad (1)$$

Solution. By (1), we have

$$f(0) = 0, f(-x) = -f(x).$$

Setting$y = x$, we get

$$f(2x) = 2f(x), \forall\, x \in \mathbb{R}. \quad (2)$$

Suppose$k$ positive integer, we have

$$f(kx) = kf(x), x \in \mathbb{R}.$$

Hence

$$f((k+1)x) = f(kx + x) = f(kx) + f(x) = kf(x) + f(x) = (k+1)f(x), \qquad \forall x \in \mathbb{R}, n \in \mathbb{N}.$$

Then,

$$f(nx) = nf(x), \forall x \in \mathbb{R}.$$

Since f$(-x) = -f(x)$, we get

$$f(mx) = mf(x), \forall\, m \in \mathbb{Z}, \forall\, x \in \mathbb{R}.$$

In (2), we have

$$f(x) = 2f\left(\frac{x}{2}\right) = 2^2 f\left(\frac{x}{2^2}\right) = \ldots = 2^n f\left(\frac{x}{2^n}\right). \quad (3)$$

Then

$$f\left(\frac{m}{2^n}\right) = \frac{m}{2^n} f(1), \forall\, m \in \mathbb{Z}, \forall\, n \in \mathbb{N}^+. \qquad (4)$$

In view of the continuity of $f(x)$, we have

$$f(x) = ax, \forall\, x \in \mathbb{R}, a = f(1).$$

We can check directly$f(x) = ax$satisfies (1).
Hence,

$$f(x) = ax, with\ for\ arbitrary\ \ a \in \mathbb{R}.$$

Problem 2 (From arithmetic mean to arithmetic mean). Determine all functions$f(x)$continuous on $\mathbb{R}$ such that:

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}, \forall x, y \in \mathbb{R}. \quad (5)$$

Solution. Let$f(0) = b$and$f(x) = b + g(x)$. Then$g(0) = 0$. Replacing(5), we get

$$b + g\left(\frac{x+y}{2}\right) = \frac{2b + g(x) + g(y)}{2}, \forall x, y \in \mathbb{R},$$

$$\Leftrightarrow g\left(\frac{x+y}{2}\right) = \frac{g(x) + g(y)}{2}, \forall x, y \in \mathbb{R}, \qquad (6)$$

with$g(0) = 0$.
Let$y = 0$. Then (6), we have

$$g\left(\frac{x}{2}\right) = \frac{g(x)}{2},$$

or

$$\text{g}\left(\frac{x+y}{2}\right) = \frac{g(x)+g(y)}{2}, \forall x, y \in \mathbb{R}\ .$$

Replacing (6), we have

$$\text{g}\left(\frac{x+y}{2}\right) = \frac{g(x)+g(y)}{2}, \forall x, y \in \mathbb{R}\ .$$

$$g(x+y) = g(x) + g(y), \forall x, y \in \mathbb{R}\,. \quad (7)$$

Since$g(x)$ is continuous on $\mathbb{R}$. Then (7) is Cauchy function. Then$g(x) = ax$.
We get$f(x) = ax + b, a, b \in \mathbb{R}$.
We can check directly$f(x) = ax + b$satisfies (5).
Hence,

$$f(x) = ax + b, with\ for\ arbitrary\ \ a, b \in \mathbb{R}.$$

Problem 3 (From quadratic mean to quadratic mean). Determine all functions$f(t)$ continuous on $\mathbb{R}^+$ such that:

$$f\left(\frac{\sqrt{x^2+y^2}}{2}\right) = \sqrt{\frac{[f(x)]^2 + [f(y)]^2}{2}}; \forall x, y \in \mathbb{R}^+ \ (8)$$

Solution. By assumption, we have$f(x) \geq 0, x \in \mathbb{R}^+$. Then

$$(8) \Leftrightarrow \left[f\left(\frac{\sqrt{x^2+y^2}}{2}\right)\right]^2 = \frac{[f(x)]^2 + [f(y)]^2}{2}, x, y \in \mathbb{R}^+.$$

or

$$g\left(\sqrt{\frac{u\ +\ v}{2}}\right)=\frac{g(\sqrt{u})+g(\sqrt{v})}{2},\forall u,v>0,$$

with $g(u)=[f(u)]^2\geq\ 0,\forall\ u\ >0.$
Then,

$$h\left(\frac{u+v}{2}\right)=\frac{h(u)+h(v)}{2},\ \forall\ u,v\ >0,$$

with $h(u)=g(\sqrt{u})$. In Problem 2, we have$h(u)=au+b,\forall u\ >0$. Then $g(x)=ax^2+b$. With $g(x)\geq 0,\forall\ x>0$, we choice$a\geq 0$and$b\geq 0$.
Thus

$$f(t)=\sqrt{at^2+b}with\ for\ arbitrary\ a,b\ \geq\ 0.$$

Problem 4 (From mean of arbitrary degrees to mean of arbitrary degrees). Determine all functions$f(x)$continuous on$\mathbb{R}^+$such that

$$f\left(\sqrt[t]{\frac{x^t+y^t}{2}}\right)=\sqrt[t]{\frac{[f(x)]^t+[f(y)^t]}{2}};\ \forall \mathrm{x,y}\ \in\mathbb{R}^+,\forall \mathrm{t}\geq 1. \qquad (9)$$

Solution. By$f(x)\geq\ 0,\forall\ x\in\mathbb{R}^+$. We have

$$9\Leftrightarrow\left[f\left(\sqrt[t]{\frac{x^t+y^t}{2}}\right)\right]^t=\frac{[f(x)]^t+[f(y)^t]}{2};\ \forall \mathrm{x,y}\ \in\mathbb{R}^+.$$

or

$$g\left(\sqrt[t]{\frac{u+v}{2}}\right)=\frac{g(\sqrt[t]{u})+g(\sqrt[t]{v})}{2},\forall \mathrm{u,v}\ >0,$$

with$g(u)=[f(u)]^t,\forall\ u>0.$

Then,

$$h\left(\frac{u+v}{2}\right)=\ \frac{h(u)\ \ +\ h(v)}{2},\forall\ u,v\ >0,$$

with$h(u)=g(\sqrt[t]{u})$. In Problem 2, we have$h(u)=au+b,\forall u\ >0$. Then $g(x)=ax^t+b$. Withg(x)≥ 0,∀ x>0, we choice$a\geq 0$and$b\geq\ 0$.
Thus,

$$f(x)=\sqrt[t]{ax^t+b}\ with for\ arbitrary\ a,b\ \geq\ 0.$$

III. Apply To Solve Some Mathematical Problems

In section, we would like to look at some functional equality problemsand apply to solve some mathematical problems.

Problem 5. Let$\alpha,\beta\ \in\mathbb{R}\backslash\{0\}$. Determine all functions $f:\mathbb{R}^+\to\mathbb{R}^+$continuous on$\mathbb{R}^+$such that

$$f\left(\left(\frac{x^\alpha+y^\alpha}{2}\right)^{1/\alpha}\right)=\left[\frac{f(x)^\beta+f(y)^\beta}{2}\right]^{1/\beta},\forall x,y\ \in\mathbb{R}^+. \qquad (10)$$

Solution. Let $[f(x)]^\beta=g(x)$. By (10), we get

$$g\left(\left(\frac{x^\alpha+y^\alpha}{2}\right)^{1/\alpha}\right)=\frac{g(x)+g(y)}{2},\forall x,y\ \in\mathbb{R}^+.$$

Next, let$x=u^{1/\alpha}\ y=v^{1/\alpha}$, we have

$$h\left(\frac{u+v}{2}\right)=\ \frac{h(u)\ \ +\ h(v)}{2},\forall\ u,v\ \in\mathbb{R}^+,$$

with

$$h(u)=g(u^{1/\alpha}).$$

By Problem 2, we have

$$h(u)=au+b,\forall\ a,b\geq\ 0,a+b>0,$$

and

$$f(x)=(ax^\alpha+b)^{1/\beta},a,b\geq 0,\qquad a+b>0.$$

Thus

$$f(x) = (ax^{\alpha} + b)^{1/\beta}, a, b \geq 0, \qquad a + b > 0.$$

Problem 6. Determine all functions $f(t)$continuous on $\mathbb{R}$such that

$$f\left(\frac{x+y}{2}\right) - \frac{f(x) - f(y)}{2} = (x-y)^2, \forall x, y \in \mathbb{R}^+. \qquad (11)$$

Solution. Using inequation, we get

$$\left(\frac{x+y}{2}\right)^2 - \frac{x^2}{2} - \frac{y^2}{2} = -\frac{1}{4}(x-y)^2$$

Next, by (11), we have

$$f\left(\frac{x+y}{2}\right) + 4\left(\frac{x+y}{2}\right)^2 - \frac{f(x) + 4x^2 + f(y) + 4y^2}{2} = 0, \forall x, y \in \mathbb{R}.$$

or

$$g\left(\frac{x+y}{2}\right) = \frac{g(x) \quad + g(y)}{2}, \qquad \forall x, y \in \mathbb{R},$$

with $g(x) = f(x) + 4x^2$.

By Problem 2, then

$$g(x) = ax + b, \forall a, b \in \mathbb{R},$$

and

$$f(x) = ax + b + 4x^2, \forall a, b \in \mathbb{R}.$$

Hence.

$$f(t) = at + b + 4t^2, with\ for\ arbitrary\ a, b \in \mathbb{R}.$$

Problem 7. Determine all functions$f(t)$, $g(t)$ on $\mathbb{R}$, such that$f(t)$continuous on$\mathbb{R}$and

$$f(x) + f(y) + 2xy = (x+y)g(x+y), \forall x, y \in \mathbb{R}. \quad (12)$$

Solution. Let$x = 0, y = 0$in(12), we have$f(0) = 0$. Let$f(x) - x^2 = h(x)$. Then$h(x)$is continuous on $\mathbb{R}$ and $h(0) = 0$. By (12), we have

$$(x+y)^2 + h(x) + h(y) = (x+y)g(x+y). \qquad (13)$$

Let $x = y = t/2$. By (13), we get

$$g(t) = \begin{cases} t + \frac{2}{t}h\left(\frac{t}{2}\right), \forall t \neq 0; \\ c, \qquad with\ t = 0 \end{cases}.$$

Let$g(x)$in (12), we get

$$(x+y)^2 + h(x) + h(y) = \begin{cases} (x+y)^2 + 2h\left(\frac{x+y}{2}\right); \quad x + y \neq 0 \\ 0;\ x + y = 0 \end{cases}$$

for all$x, y \in \mathbb{R}$.

Hence,

$$h\left(\frac{x+y}{2}\right) = \frac{h(x) \quad + \ h(y)}{2}, \forall\ x, y \in \mathbb{R}.$$

Since $h(x)$is continuous on $\mathbb{R}$ and$h(0) = 0$. Then$h(x) = ax$.

Hence,

$$f(x) = x^2 + ax, \qquad g(x) = \begin{cases} x + a; \forall x \neq 0 \\ c; \qquad x = 0 \end{cases}$$

with for arbitrary$a, c \in \mathbb{R}$. We can check directly $f(x), g(x)$satisfies problem.

Hence,

$$f(t) = t^2 + at, g(t) = \begin{cases} t + a; \forall t \neq 0 \\ c; \qquad t = 0 \end{cases}$$

with for arbitrary$a, c \in \mathbb{R}$.

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