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Conjugacy Classes and Action of $\Delta(3,4,k)$ *on* $PL(F_q)$

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Abstract: *The triangle group $\Delta(3,4,k)$ can be defined as $< r,s: r^3 = s^4 = (rs)^k = 1 >$, where r,s are the generators of the group. In this paper, we have discussed conjugacy classes that arises from the actions of $\Delta(3,4,k)$ on $PL(F_q)$. Here, F_q is a finite field for any prime q and $PL(F_q) = F_q \cup \infty$. A relation between conjugacy classes of a homomorphism and parameters of F_q has also drawn by using computer coding scheme.*

Keywords: *Conjugacy classes, Linear-fractional transformations, Parameterization and Non-degenerate homomorphism.*

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I. Introduction

It is well known [2, 3] that $\Gamma = G^{3,4}(2,Z)$ is the group of linear-fractional transformations of the form $z \to \frac{az+b}{cz+d}$, where $a,b,c,d \in Z$, $ad - bc \neq 0$. This group is generated by r, s satisfying the relations

$$r^3 = s^4 = 1. \qquad (1.1)$$

It is also proved in [2, 3] that if a linear-fractional transformation t inverts both r and s, that is, $t^2 = (rt)^2 = (st)^2 = 1$, then we get an extended group $\Gamma^* = G^{*3,4}(2,Z)$ which is again a group of transformations having form

$$z \to \frac{az+b}{cz+d}; a,b,c,d \in Z$$

The defining relations of this extended group are:

$$\Gamma^* =< r,s,t: r^3 = s^4 = t^2 = (rt)^2 = (st)^2 = 1 >. \qquad (1.2)$$

Thus we can define the group $G^{*3,4}(2,q)$ as the group of linear-fractional transformations of the form $z \to \frac{az+b}{cz+d}$, where $a,b,c,d \in F_q$ and $ad - bc \neq 0$. We can also define a group $G^{3,4}(2,q)$ as a subgroup of $G^{*3,4}(2,q)$ such that $ad - bc$ is a non-zero square in F_q [5]. It is well known in [7, 8] that triangle group $\Delta(k,l,m)$ is finite precisely when $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} > 1$, and infinite in case of $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} \leq 1$. $\Delta(2,4,k)$ is infinite for $k \geq 4$, whereas for $k = 1,2,3$ triangle group $\Delta(2,4,k)$ is C_2, D_8, S_4 respectively [8, 9]. A general description of triangle group $\Delta(3,4,k)$ having representation $< r,s: r^3 = s^4 = (rs)^k = 1 >$ can be found in [1, 4, 6]. It is also known that by adjoining an involution t, which inverts both r and s, the groups $\Delta(3,4,k)$ can be extended to the triangle groups $\Delta^*(3,4,k) = < r,s,t: r^3 = s^4 = (rs)^k = t^2 = (rt)^2 = (st)^2 = 1 >$. The triangle group $\Delta(3,4,k)$ is of index 2 in $\Delta^*(3,4,k)$ and so is normal in $\Delta^*(3,4,k)$.

II. Parameters of Conjugacy Classes for $\Gamma^* = G^{*3,4}(2,Z)$

Let $\alpha: G^*(2,Z) \to G^*(2,q)$ be a homomorphism. Choose $\underline{r} = r\alpha, \underline{s} = s\alpha$ and $\underline{t} = t\alpha$, in $G^*(2,q)$ satisfying

$$\underline{r}^3 = \underline{s}^n = \underline{t}^2 = (\underline{rt})^2 = (\underline{st})^2 = 1. \quad (2.1)$$

This homomorphism α is termed as 'non-degenerate' if r and s have same orders as that of $(r)\alpha$ and $(s)\alpha$ respectively. It means none of the generators r, s lies in kernel of α so that their images $\underline{r} = r\alpha$, $\underline{s} = s\alpha$ are of orders 3 and n respectively.

If a natural map $GL(2,q) \to G^*(2,q)$ maps matrix M to an element g of $G^*(2,q)$, then $\theta = (trace(M))^2/det(M)$ is called invariant of conjugacy class of g. It can be pertained as parameter of element g or of conjugacy class. Actions of $G(2,Z)$ on $PL(F_q)$, via α, will be considered so that g be taken as $(rs)\alpha = \underline{r}\,\underline{s}$. Hence, θ is the parameter of the class containing $\underline{r}\,\underline{s}$. We can also establish a relation between α and $\theta \in F_q$. It can be proved very easily that if R and S are two non-singular 2×2 matrices corresponding to the generators $\underline{r}$ and $\underline{s}$ of Γ^* with $det(RS) = I$ and $trace(RS) = m_2$, then RS satisfy the following characteristic equation:

$$(RS)^2 - m_2 RS + I = 0$$

$(RS)^2 = m_2 RS - I$ (2.2)

Multiplying both sides of this equation by S , we get:

$(RS)^3 = m_2 (RS)^2 - (RS)I$ (2.3)

By putting equation (2.2) in equation (2.3), we obtain

$$(RS)^3 = (m_2^2 - 1) - m_2 I$$

On recursion, we get

$(RS)^k =$
$\{(k-1\ 0)m_2^{k-1} - (k-2\ 1)m_2^{k-3}\ldots\}RS - \{(k-2\ 0)m_2^{k-2} - (k-3\ 1)m_2^{k-4} + \ldots\}I$ (2.4)

Furthermore, if

$f(m_2) = \{(k-1\ 0)m_2^{k-1} - (k-2\ 1)m_2^{k-3}\ldots\}RS - \{(k-2\ 0)m_2^{k-2} - (k-$
3 1 m2k−4+...} (2.5)

and substituting $m_2^2 = \theta$ in the polynomial $f(m_2)$ if k is odd and $m_2 = \sqrt{\theta}$ otherwise, we obtain a polynomial $f(\theta)$. We can find a minimal polynomial for positive integer k by using equation (2.5).

III. Main Results

Following important result is necessary to prove Theorem 3.2.

Lemma 3.1: For a non-singular 2×2 matrix, if its trace is zero then it represents an involution provided its entries are from F_q.

Theorem 3.2: Let $\underline{r}, \underline{s}$ be any two elements of $G^{*3,4}(2,q)$ and R, S be their corresponding matrices respectively, then $m_2^2 - \sqrt{2}m_2 - 1 = 0$, where m_2 is the trace of matrix RS.

Proof: Consider two elements $\underline{r}, \underline{s}$ of $G^{*3,4}(2,q)$, such that order of $\underline{r}$ is 3 whereas that of $\underline{s}$ is 4.Let $R = [r_1\ r_2\ r_3\ r_4\ ]$ and $S = [s_1\ s_2\ s_3\ s_4\ ]$ be their corresponding matrices and are the elements of $GL(2,q)$. Since $\underline{r}^3 = 1$, so R^3 will be a scalar matrix and its determinant will be a square in F_q. Since, for any matrix M, $M^3 = \lambda I$ if and only if $(trace(M))^2 = det(M)$, so we may assume that $trace(R) = r_1 + r_4 = -1$. Replacing R by a suitable scalar, we can also assume that $det(R) = 1$. Thus $R = [r_1\ r_2\ r_3\ \ -r_1 - 1\ ]$. Therefore we have $det(R) = -r_1^2 - r_1 - kr_3^2$. Since $det(R) = 1$, so

$$1 + r_1^2 + r_1 + kr_3^2 = 0 \quad (3.1)$$

As $\underline{r}^3 = 1$ and $trace(R) = -1$, so every element of $GL(2,q)$ with trace equal to -1 has up to scalar multiplication, a conjugate of the form $[0\ k\ 1\ \ -1\ ]$. Therefore, we can assume that R has the form $[0\ k\ 1\ \ -$ *1* . Similarly , *S= s1 ks3 s3 −s1−2* giving *det(S)=−s12−2s1−ks32=1*, so that

$$1 + s_1^2 + \sqrt{2}s_1 + ks_3^2 = 0. \quad (3.2)$$

Consider an invertible element $\underline{t}$ in $G^{*3,4}(2,q)$ such that it satisfies the relation:

$$\underline{t}^2 = (\underline{rt})^2 = (\underline{st})^2 = 1. \quad (3.3)$$

Let $T = [t_1\ t_2\ t_3\ t_4\ ]$ be a matrix representing $\underline{t}$. Then, since $\underline{t}$ is an involution, therefore $t_4 = -t_1$ yields $T = [t_1\ t_2\ t_3\ \ -t_1\ ]$. Let RT be the matrix representing $\underline{rt}$ of $G^{*3,4}(2,q)$. Then $RT = [kt_3\ \ -kt_1\ t_1 - t_3\ t_1 + t_2\ ]$, which again by lemma 3.1, and $(\underline{rt})^2 = 1$, implies that

$$t_1 + t_2 = -kt_3. \quad (3.4)$$

Similarly, if ST is a matrix that represents an element $\underline{st}$ of $G^{*3,4}(2,q)$, then we get
$ST = [s_1t_1 + s_2t_3\ s_1t_2 - s_2t_2\ s_3t_1 + t_3(\sqrt{2} - s_1)\ s_3t_2 - t_1(\sqrt{2} - s_1)\ ]$. Since $\underline{st}$ is also an involution therefore by the arguments given above, we have $s_1t_1 + s_2t_3 + s_3t_2 - t_1(\sqrt{2} - s_1) = 0$, which together with equation (3.4) yields $2s_1t_1 + s_2t_3 - s_3t_1 - ks_3t_3 - \sqrt{2}t_1 = 0$. That is,

$$t_1(2s_1 - s_3 + \sqrt{2}) + t_3(s_2 - ks_3) = 0. \quad (3.5)$$

Now for a non-singular matrix T, we must have $det(T) \neq 0$, that is

$$-t_1^2 + t_1t_3 + kt_3^2 \neq 0. \quad (3.6)$$

Therefore, necessary and sufficient conditions for the existence of $\underline{t}$ in $G^{*3,4}(2,q)$ are the equations (3.4) , (3.5) and (3.6). Hence $\underline{t}$ exists in $G^{*3,4}(2,q)$ unless $kt_3^2 - t_1^2 + t_1t_3 = 0$. If both $2s_1 - s_3 + \sqrt{2}$ and $s_2 - ks_3$ are equal to zero, then the existence of $\underline{t}$ is trivial. If not, then $t_1/t_3 = -(s_2 - ks_3)/(2s_1 - s_3 - \sqrt{2})$, and so equation (3.6) is equivalent to $(s_2 - ks_3)^2 - (2s_1 - s_3 + \sqrt{2})(2ks_1 + \sqrt{2}k - s_2) \neq 0$. Thus $\underline{t}$ exists in $G^{*3,4}(2,q)$ satisfying equation (3.3) unless $(s_2 - ks_3)^2 = (2s_1 - s_3 + \sqrt{2})(2ks_1 + \sqrt{2}k - s_2)$. Which after simplification gives

$$(s_2 - ks_3)(s_2 - ks_3 + 2s_1 + \sqrt{2}) = -4k + s_2s_3 - 2. \quad (3.7)$$

Now $RS = \left[ks_3\ k(\sqrt{2}-s_1)\ s_1 - s_3\ s_2 - \sqrt{2} + s_1\right]$, this implies that the $tr(RS) = s_1 + s_2 + ks_3 - \sqrt{2}$. Let $tr(RS) = m_2$. Also, using equation (3.7),we have $det(RS) = k(s_2s_3 - \sqrt{2}s_1 + s_1^2)$. Since $det(RS) = 1$. So $k = -1$. Hence we have

$$1 = \sqrt{2}s_1 - s_1^2 - s_2s_3 \quad (3.8)$$

Also, we have

$$m_2 = s_1 + s_2 - s_3 - \sqrt{2} \qquad (3.9)$$

Substituting $k = -1$ and values from equations (3.8) and (3.9) in equation (3.7), we get,

$$m_2^2 - \sqrt{2}m_2 + 2 = 3$$

$$m_2^2 - \sqrt{2}m_2 - 1 = 0. \quad (3.10)$$

Theorem 3.3: Let $\underline{g}$ be any non-trivial element of $G^{*3,4}(2,q)$, such that order of both $\underline{g}$ and its dual not equal to 2, then $\underline{g}$ is the image of rs under some non-degenerate homomorphism of Γ into $G^{*3,4}(2,q)$.

Proof: To prove this result, we show by using theorem 3.2, that every non-trivial element of $G^{*3,4}(2,q)$ is the product of two elements, one having order 3 whereas other of order 4. In fact we must find elements $\underline{r}, \underline{s}$ and $\underline{t}$ belong to $G^{*3,4}(2,q)$ and satisfy the relations (2.1), too.

For this, consider the elements $\underline{r}, \underline{s}$ and $\underline{t}$ of $G^{*3,4}(2,q)$ represented by the matrices $R = [r_1\ kr_3\ r_3\ \ -r_1 - 1\ ]$, $S = \left[s_1\ ks_3\ s_3\ \ -\sqrt{2} - s_1\ \right]$ and $T = [0\ \ -k\ 1\ 0\ ]$, where r_1, r_3, s_1, s_3, k are in F_q, with $k \neq 0$, so that

$$1 + r_1 + r_1^2 + kr_3^2 = 0. (3.11)$$

Further, let assume the determinant of S be equal to 1, we have

$$1 + ks_3^2 + s_1^2 + \sqrt{2}s_1 = 0. \qquad (3.12)$$

We take $\underline{r}\ \underline{s}$ in a given conjugacy class. A matrix representing $\underline{r}\ \underline{s}$ is given by

$$RS = \left[r_1s_1 + kr_3s_3\ kr_1s_3 + kr_3(-\sqrt{2} - s_1)\ r_3s_1 - s_3(r_1 + 1)\ kr_3s_3 - r_1(-\sqrt{2} - s_1) + \sqrt{2} + s_1\ \right]$$

Its trace, which we denote by m_2, is given by

$$m_2 = trace(RS) = 2kr_3s_3 + r_1(2s_1 + \sqrt{2}) + (s_1 + \sqrt{2}). \ (3.13)$$

As determinant of R and S is 1, therefore $det(RS) = det(R)det(S) = 1$,. Hence, we have

$$RST = \left[kr_1s_3 - \sqrt{2}kr_3 - kr_3s_1\ \ -kr_1s_1 - k^2r_3s_3\ kr_3s_3 + \sqrt{2}r_1 + r_1s_1 + \sqrt{2} + s_1\ \ -kr_3s_1 + kr_1s_3 + ks_3\ \right].$$

So, $trace(RST) = k(2r_1s_3 - 2r_3s_1 + s_3 - \sqrt{2}r_3)$. Let $trace(RST) = km_3$, then

$$m_3 = 2r_1s_3 - r_3(2s_1 + \sqrt{2}) + s_3. \qquad (3.14)$$

Hence, we have

$$m_2^2 + km_3^2 - \sqrt{2}m_2 - 1 = 0. \qquad (3.15)$$

Since $\underline{g} = \underline{r}\ \underline{s}$ (or its dual $\underline{r}\ \underline{st}$) are not of order 2, so we must have $(\underline{r}\ \underline{s})^2 \neq 1$ and $(\underline{r}\ \underline{st})^2 \neq 1$. Thus by lemma 3.1, the traces of the matrices RS and RST are not equal to zero. Hence $m_2 \neq 0$, and $m_3 \neq 0$, so that $\theta = m_2^2 \neq 0$; and it is sufficient to show that we can choose r_1, r_3, s_1, s_3, k in F_q so that m_2^2 is indeed equal to θ

From equation (3.15), we have $km_3^2 = 1 - m_2^2 + \sqrt{2}m_2$. If $m_2^2 - \sqrt{2}m_2 \neq 1$, we can select the value of k as per same argument.

Theorem 3.4: For any non-degenerate homomorphism α and its dual α',

$$\theta + \phi = 1 + \sqrt{2}m_2,$$

where θ and ϕ are the parameters of α and α' respectively.

Proof: Consider a non-degenerate homomorphism $\alpha: \Gamma \to G^{*3,4}(2,q)$ satisfies the relations $r\alpha = \underline{r}, s\alpha = \underline{s}$ and $t\alpha = \underline{t}$ and α' is its dual. Consider the matrices $R = [r_1\ kr_3\ r_3\ \ -r_1 - 1\ ]$, $S = \left[s_1\ ks_3\ s_3\ \ -\sqrt{2} - s_1\ \right]$ and $T = [0\ \ -k\ 1\ 0\ ]$, representing the elements $\underline{r}, \underline{s}$ and $\underline{t}$, of $G^{*3,4}(2,q)$ respectively. By lemma 3.1, $trace(RS) = trace(RST) = 0$ if and only if $(\underline{r}\ \underline{s})^2 = (\underline{r}\ \underline{st})^2 = 1$. As $det(RS) = 1$, so we can assume that parameter θ(say) of $\underline{r}\ \underline{s}$ equals to m_2^2. Also since $trace(RST) = km_3$ and $det(RST) = k$ (since $det(R) = 1$, $det(S) = 1$ and $det(T) = k$), we get the parameter ϕ of $\underline{r}\ \underline{st}$ equals to km_3^2. Therefore, we have $\theta + \phi = m_2^2 + km_3^2$.

Substituting the value of m_2^2 from equation (3.15), we get $\theta + \phi = 1 + \sqrt{2}m_2$. Hence if θ is the parameter of the non-degenerate homomorphism α, then $\phi = 1 + \sqrt{2}m_2 - \theta$ is the parameter of the dual α' of α.

Corollary 3.5: If $\underline{t}$ inverts both $\underline{r}$ and $\underline{s}$ then order of $\underline{rs}$ is 12.
Proof: From theorem 3.2, we have $m_2^2 = 1 + \sqrt{2}m_2$. After rearranging this result, we get

$$m_2^2 - 1 = \sqrt{2}m_2 \quad (3.16)$$

Taking square on both sides of equation (3.16), we get

$$m_2^4 - 2m_2^2 + 1 = 2m_2^2 \quad (3.17)$$

Replacing m_2^2 by θ in equation (3.17) , we get

$$\theta^2 - 4\theta + 1 = 0 \quad (3.18)$$

From table 1 given below, it is evident that this is the corresponding equation for $k = 12$. Hence order of $\underline{rs}$ is 12.

Table 1: Minimal Equations satisfied by θ

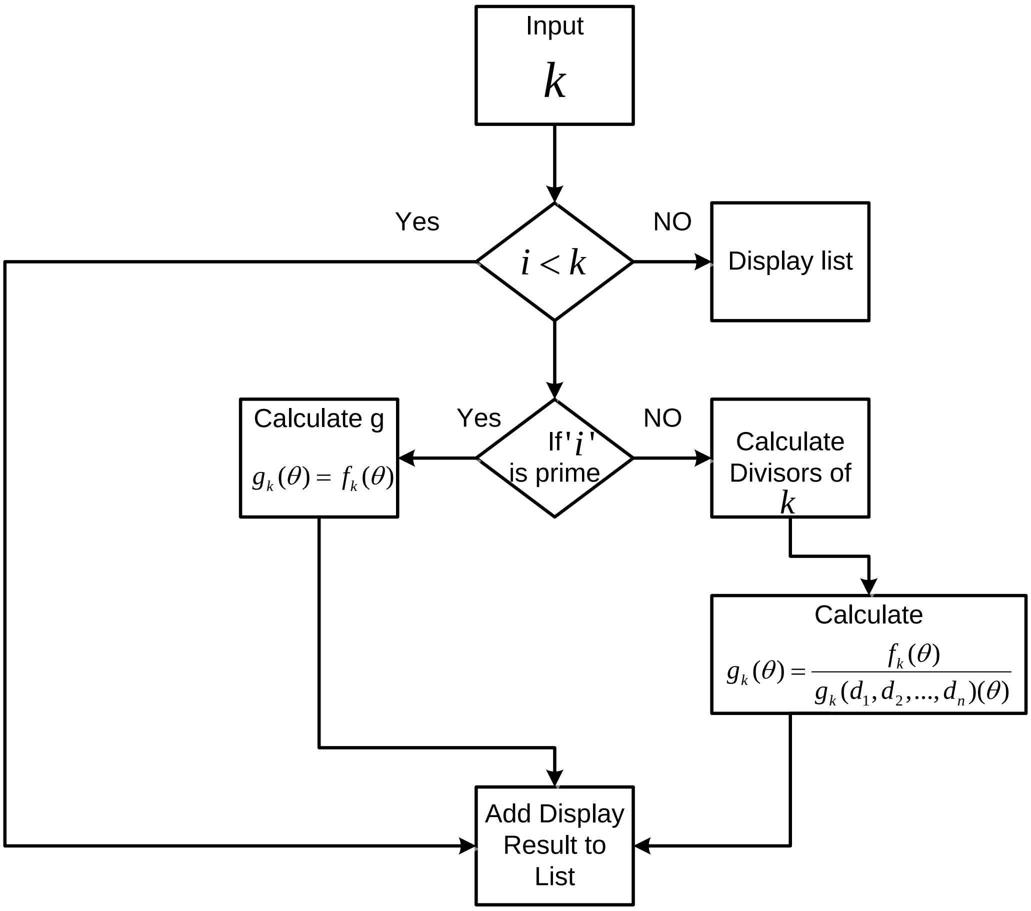
Triangle Group $\Delta(3,4,k)$	Minimal Equation satisfied by θ
$\Delta(3,4,1)$	$\theta - 4 = 0$
$\Delta(3,4,2)$	$\theta = 0$
$\Delta(3,4,3)$	$\theta - 1 = 0$
$\Delta(3,4,4)$	$\theta - 2 = 0$
$\Delta(3,4,5)$	$\theta^2 - 3\theta + 1 = 0$
$\Delta(3,4,6)$	$\theta - 3 = 0$
$\Delta(3,4,7)$	$\theta^3 - 5\theta^2 + 6\theta - 1 = 0$
$\Delta(3,4,8)$	$\theta^2 - 4\theta + 2 = 0$
$\Delta(3,4,9)$	$\theta^3 - 6\theta^2 + 9\theta - 1 = 0$
$\Delta(3,4,10)$	$\theta^2 - 5\theta + 5 = 0$
$\Delta(3,4,11)$	$\theta^5 - 9\theta^4 + 28\theta^3 - 35\theta^2 + 15\theta - 1 = 0$
$\Delta(3,4,12)$	$\theta^2 - 4\theta + 1 = 0$
$\Delta(3,4,13)$	$\theta^6 - 11\theta^5 + 45\theta^4 - 84\theta^3 + 70\theta^2 - 21\theta + 1 = 0$
$\Delta(3,4,14)$	$\theta^3 - 7\theta^2 + 14\theta - 7 = 0$
$\Delta(3,4,15)$	$\theta^4 - 9\theta^3 + 26\theta^2 - 24\theta + 1 = 0$

IV. Computational Approach to Calculate Conjugacy Classes

Flowchart and Algorithm

Following flowchart and algorithm help us to develop a computer coding scheme for drawing relation between homomorphism and parameters of conjugacy classes.

Figure 1: Flow Chart



1. Input integer values k, set $i = 0$.
2. For $i < k$. If i is prime, calculate $g_k(\theta) = f(\theta)$
3. Otherwise calculate divisors for k
4. Calculate $g_k(\theta) = \frac{f(\theta)}{g_k(d_1, d_2, ..., d_n)(\theta)}$.
5. Add $g_k(\theta)$) to the list.
6. Display list in table form.

Coding Scheme

Following code written in Java programming language will generate the conditions in form of equations $f(\theta) = 0$ for the existence of triangle groups $\Delta(3,4,k)$ for $1 \le k \le n$ as shown in table 1 for $1 \le k \le 15$.

$$(* \; Get \; Input \; from \; user \; *)$$

$$k = Input[Enter \quad the \quad value \quad of \quad K];$$

$$(* \; InitializedenominatortobeusedwhenKisnoprime \; *)$$

$$mylist = Range[k];$$

$$resultlist = List[];$$

$$denom = 1;$$

$$finalResult = 1;$$

$$r = 2;$$

$$(* \; Functionthatimplementstheformula \; *)$$

$$r = \sqrt{\theta};$$

$$\mathcal{S}olver[k\_]: \sum_{n=1}^{\frac{(k+1)}{2}} (-1)^{n+1}\left(\frac{(k-n)!}{\big((k-n)-(n-1)\big)!\,(n-1)!}\right)(r)^{k-(2n-1)};$$

$$(* \; Loopfrom1toinputRange \; *)$$

$$For[i = 1, i \le k, i++,$$

$$(* \; checkkforprimecondition. *)$$

$$If[i == 1, finalResult = \theta - 4,$$

$$If[PrimeQ[i],$$

$$(* \; IfKisPrime \; *)$$

$$finalResult = solver[i], (* \; g_k(\theta) = f_k(\theta) \; *)$$

$$divofK = Divisors[i]; (* \; IfKisNotPrime \; *)$$

$$length = Length[divofK];$$

$$newlist = Delete\big[divofK, \{\{1\}, \{-1\}\}\big]; (* \; GetDivisorsofK \; *)$$

$$length2 = Length[newlist];$$

$$Do\big[denom = denom * solver\big[Part[newlist, n]\big], \{n, 1, length2, 1\}\big]; (* \; g_k(\theta) = f_k \frac{(\theta)}{g_{k|d1,d2,d3,\ldots|}}(\theta) \; *)$$

$$finalResult = \frac{solver[i]}{denom};]]$$

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